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A NEW ITERATIVE ALGORITHM FOR GENERALIZED SPLIT EQUILIBRIUM PROBLEM IN HILBERT SPACES

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Abstract. In this paper, we introduce a new non-combination iterative algorithm to find a solution of a generalized split equilibrium problem in Hilbert spaces. In our algorithm, the parameter γ is chosen from $(0, \frac{2}{M^2})$, where M is an arbitrary boundedness above of the norm $\|A\|$ of the operator A which is such that the parameter γ is easier to chose.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* for F is to find $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of all solutions of (1.1) is denoted by $EP(F)$, i.e.,

$$EP(F) = \{z \in C : F(z, y) \geq 0, \ \forall y \in C\}.$$

Let $A : C \to H$ be a nonlinear operator. The *generalized equilibrium problem* for F and A is to find $v \in C$ such that

$$F(v, y) + \langle Av, y - v\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

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The set of all solutions of (1.2) is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Av, y - v\rangle \geq 0,\ \forall y \in C\}.$$

In (1.2), if $F = 0$, then (1.2) is deduced to the following *variational inequality problem*: to find $v \in C$ such that

$$\langle Av, y - v\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $VI(C, A)$.

Many problems in physics, optimization and economics can be reduced to find the solution of equilibrium problems (see [3, 11, 16, 18]). Also, many iterative algorithms are considered to find the solutions of variational inequality problems, equilibrium problems and generalized equilibrium problems (see [5, 16, 18, 21, 22, 23, 30, 31, 32]).

In 2012, He [8] introduced a new equilibrium problem called a split equilibrium problem which is also mentioned in [15].

Let H_1, H_2 be two real Hilbert spaces and C, Q be the nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions. The *split equilibrium problem* is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \text{ sovles } F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.5}$$

Let Ω denote the set of solutions of (1.4) and (1.5), that is,

$$\Omega = \{z \in C : z \in EP(F_1),\ Az \in EP(F_2)\}.$$

In [8], the author introduced some strong and weak iterative algorithms to solve the split equilibrium problem and some examples to illustrate the iterative algorithms.

In 2013, Kazmi and Rizvi [12] gave the following iterative algorithm to solve the split equilibrium problem and fixed point problem for nonexpansive mapping in Hilbert spaces:

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n D u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n \end{cases} \tag{1.6}$$

for each $n \geq 1$, where $D : C \to H_1$ is a τ-inverse strongly monotone mapping and $S : C \to C$ is a nonexpansive mapping. Under some certain assumptions on the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{r_n\}$, the authors proved that the sequence $\{x_n\}$ generated by (1.6) strongly converges to a point $z = P_\Theta v$, where $\Theta = Fix(S) \cap \Omega \cap VI(C, D)$.

Recently, Wang et al. [25] introduced a strong convergence algorithm to solve the split equilibrium problem and fixed point problem for asymptotically nonexpansive mappings in Hilbert spaces as follows:

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{s_n}^{F_2})A)x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S^n y_n \end{cases} \tag{1.7}$$

for each $n \geq 1$, where $B : C \to H_1$ is a β-inverse strongly monotone mapping and $S : C \to C$ is an asyptotically nonexpansive mapping. They proved that the sequence $\{x_n\}$ defined by (1.7) strongly converges to the point $z = P_\Theta z$, where $\Theta = Fix(S) \cap \Omega \cap VI(C, B)$.

Very recently, Xu et al. [29] considered a cloud hybrid method to solve the split equilibrium problems and fixed point problems for a family of quasi-Lipschitz mappings in Hilbert spaces. For more details on the split equilibrium problem, refer to [1, 26].

In this paper, we consider the generalized split equilibrium problem in Hilbert spaces. For each $i = 1, 2$, let H_i be a real Hilbert space and C_i be a nonempty closed and convex subset of H_i, let $F_i : C_i \times C_i \to \mathbb{R}$ be a bifunction and $A_i : C_i \to H_i$ be a nonlinear operator. Let $A : H_1 \to H_2$ be a bounded linear operator. The *generalized split equilibrium problem* is to find a point $z \in C_1$ such that

$$F_1(z, y) + \langle A_1 z, y - z \rangle \geq 0, \quad \forall y \in C_1 \tag{1.8}$$

and $v = Az$ such that

$$F_2(v, y) + \langle A_2 v, y - v \rangle \geq 0, \quad \forall y \in C_2. \tag{1.9}$$

In the main results of this paper, the operators A_1 and A_2 are two monotone operators which are more general than inverse strongly monotone operators or strongly monotone operators. On the other hand, the bifunctions F_1 and F_2 in [1, 8, 12, 25, 26, 29] are required to be satisfy the following four conditions:

- $(A1)$ $F(x, x) = 0$ for all $x \in C$;
- $(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- $(A3)$ for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
- $(A4)$ for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

However, F_1 and F_2 in our result are required to satisfy the different conditions with (A1)-(A4).

In this paper, we introduce a new non-convex combination iterative algorithm which is neither viscosity approximation algorithm nor CQ hybrid algorithm to solve the generalized split equilibrium problem (1.8) and (1.9).

In our results, the conditions (A3) and (A4) on the bifunctions F_1 and F_2 are relaxed. On the other hand, the parameter γ can be obtained by an arbitrary boundedness above of $\|A\|$. In the similar results to others (see, for example, [1, 12, 25, 26, 29]), the parameter γ is from $(0, \frac{1}{L^2})$, where L the spectral radius of the operator A^*A. In our algorithm, the parameter γ is chosen by the boundedness above of $\|A\|$ not the spectral radius of the operator A^*A. Obviously, the parameter γ in our algorithms is easier to chose. The strong convergence of the proposed algorithm is proved.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed subset of H. For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. Such a P_C is called the *metric projection* from H onto C. It is well known that P_C is a firmly nonexpansive mapping from H onto C, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C.$$

Definition 2.1. A mapping $A : C \to H$ is said to be:

(1) *monotone* if

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall x, y \in C.$$

(2) *strongly monotone* if there exists $\delta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \delta\|x - y\|^2, \quad \forall x, y \in C.$$

(3) *inverse strongly monotone* if there exists $\lambda > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \lambda\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Lemma 2.2. ([24, Lemmas 2.5, 2.6]) Let C be a bounded nonempty closed convex subset of a real Hilbert H. Let $A : C \to H$ be a continuous and monotone operator and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

$(B1)$ $F(x, x) = 0$ for all $x \in C$;
$(B2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
$(B3)$ for each $x \in C$, $y \mapsto F(x, y)$ is convex.

Let $r > 0$ and define the mapping $T_r^F : H \to C$ as follows:

$$T_r^F(x) = \{z \in C : F(z, y) + \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ for\ all\ y \in C\}$$

for all $x \in H$. Then the following hold:

(1) T_r^F is single-valued.

(2) T_r^F is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle.$$

(3) $F(T_r^F) = EP$.

(4) EP is closed and convex.

Remark 2.3. In the proof process of [24, Lemma 2.5, 2.6], the condition that $F(x, y)$ is lower semi-continuous in the second argument is not used. Hence we omit the restriction in (B3). The continuous and monotone operator A in Lemma 2.1 is the special case of T in [24, Lemma 2.5,2.6].

Since T_r^F is firmly nonexpansive and also is 1-inverse strongly monotone, $I - T_r^F : H \to H$ is 1-inverse strongly monotone. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}
&\|(I - T_r^F)x - (I - T_r^F)y\|^2 \\
&= \|x - y\|^2 - 2\langle x - y, T_r^F x - T_r^F y\rangle + \|T_r^F x - T_r^F y\|^2 \\
&\leq \|x - y\|^2 - 2\langle x - y, T_r^F x - T_r^F y\rangle + \|T_r^F x - T_r^F y\|^2 \\
&\leq \|x - y\|^2 - \langle x - y, T_r^F x - T_r^F y\rangle \\
&= \langle x - y, (I - T_r^F)x - (I - T_r^F)y\rangle.
\end{aligned}$$

Hence $I - T_r^F$ is a 1-inverse strongly monotone mapping.

Lemma 2.4. (Demiclosed Principle) Let C be a nonempty closed convex subset of H and $T : C \to C$ be a nonexpansive mapping. Suppose that $\{x_n\} \subset C$ weakly converges to a point $x' \in C$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Then $x' = Tx'$.

Lemma 2.5. *Let H be a real Hilbert space. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$$

for all $x, y \in H$.

Lemma 2.6. ([14]) Let $\{s_n\}$, $\{c_n\}$ be the sequences of nonnegative real numbers and $\{a_n\}$ be a sequence in $(0, 1)$. Suppose that $\{b_n\}$ is a real number sequence such that

$$s_{n+1} \leq (1 - a_n)s_n + b_n + c_n$$

for all $n \geq 0$. Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(1) If $b_n \leq \beta a_n$ for all $n \geq 0$, where $\beta \geq 0$, then $\{s_n\}$ is a bounded sequence.
(2) If we have
$$\sum_{n=0}^{\infty} a_n = \infty, \quad \limsup_{n\to\infty} \frac{b_n}{a_n} \leq 0,$$
then $\lim_{n\to\infty} s_n = 0$.

3. Main Results

In this section, let H_1, H_2 be two real Hilbert spaces and $C_1 \subset H_1, C_2 \subset H_2$ be the bounded nonempty closed and convex subsets. Let $A : H_1 \to H_2$ be a linear bounded operator. Let $F_1 : C_1 \times C_1 \to \mathbb{R}, F_2 : C_2 \times C_2 \to \mathbb{R}$ be the bifunctions satisfying the conditions (B1)-(B3). Let $A_1 : C_1 \to H_1$ and $A_2 : C_2 \to H_2$ be the continuous and monotone operators. Let
$$EP_i = \{x \in C_i : F_i(x, y) + \langle A_i x, y - x\rangle \geq 0,\ \forall y \in C_i\}, \quad i = 1, 2.$$
Let $\Omega = \{z \in C_1 : z \in EP_1,\ Az \in EP_2\}$ and assume that $\Omega \neq \emptyset$.

Algorithm A. We consider the following algorithm.

Step 1. Choose the control sequences $\{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and set $\gamma, r, s > 0$. Take the initial point $x_0 \in C_1$ arbitrarily.

Step 2. Find $u_n \in C_2$ such that
$$F_2(u_n, y) + \langle A_2 u_n, y - u_n\rangle + \frac{1}{s}\langle y - u_n, u_n - Ax_n\rangle \geq 0, \quad \forall y \in C_2.$$
Set $v_n = x_n - \gamma A^*(Ax_n - u_n)$ and find $w_n \in C_1$ such that
$$F_1(w_n, y) + \langle A_1 v_n, y - w_n\rangle + \frac{1}{r}\langle y - w_n, w_n - v_n\rangle \geq 0, \quad \forall y \in C_1.$$

Step 3. If $u_n = Ax_n$ and $w_n = x_n$, stop and $x_n \in \Omega$; otherwise, go to the next step.

Step 4. Generate x_{n+1} by
$$x_{n+1} = \beta_n(1 - \gamma_n)x_n + (1 - \beta_n)w_n$$
for each $n \geq 1$ and $n = n + 1$ and then go to Step 2.

Obviously, by Lemma 2.2, it follows that
$$u_n = T_s^{F_2} Ax_n \in C_2, \quad w_n = T_r^{F_1} v_n \in C_1 \tag{3.1}$$
for each $n \geq 0$. Hence, if the stop criterion is satisfied for some $n \geq 1$, then we can get
$$Ax_n = T_s^{F_2} Ax_n, \quad x_n = T_r^{F_1} x_n.$$

Therefore, $x_n \in \Omega$. For showing the convergence of Algorithm A, we assume that the stop criterion can not be satisfied for all $n \geq 1$. Let $M > 0$ be the arbitrary boundedness above of $\|A\|$.

Lemma 3.1. *If $\gamma \in (0, \frac{2}{M^2})$, then the sequence $\{x_n\}$ generated by Algorithm A is bounded.*

Proof. We first show that $\{x_n\}$ is bounded. Let $p \in \Omega$ arbitrarily. Since $\gamma < \frac{2}{M^2}$ and $I - T_s^{F_2}$ is 1-inverse strongly monotone, it follows from (3.1) that

$$\begin{aligned}\|w_n - p\|^2 &= \|T_r^{F_1} v_n - T_r^{F_1} p\|^2 \leq \|v_n - p\|^2 \\ &= \|(I - \gamma A^*(I - T_s^{F_2})A)x_n - (I - \gamma A^*(I - T_s^{F_2})A)p\|^2 \\ &= \|x_n - p\|^2 - 2\gamma\langle x_n - p, A^*(I - T_s^{F_2})Ax_n - A^*(I - T_s^{F_2})Ap\rangle \\ &\quad + \gamma^2\|A^*(I - T_s^{F_2})Ax_n - A^*(I - T_s^{F_2})Ap\|^2 \\ &= \|x_n - p\|^2 - 2\gamma\langle Ax_n - Ap, (I - T_s^{F_2})Ax_n - (I - T_s^{F_2})Ap\rangle \\ &\quad + \gamma^2\|A^*(I - T_s^{F_2})Ax_n - A^*(I - T_s^{F_2})Ap\|^2 \\ &\leq \|x_n - p\|^2 + \gamma(\gamma\|A\|^2 - 2)\|(I - T_s^{F_2})Ax_n\|^2 \\ &\leq \|x_n - p\|^2 + \gamma(\gamma M^2 - 2)\|(I - T_s^{F_2})Ax_n\|^2 \\ &\leq \|x_n - p\|^2 \end{aligned} \tag{3.2}$$

for each $n \geq 0$. Thus it follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n(1 - \gamma_n)x_n + (1 - \beta_n)w_n - p\| \\ &= \|\beta_n(1 - \gamma_n)(x_n - p) + (1 - \beta_n)(w_n - p) - \beta_n\gamma_n p\| \\ &\leq \beta_n(1 - \gamma_n)\|x_n - p\| + (1 - \beta_n)\|x_n - p\| + \beta_n\gamma_n\|p\| \\ &= (1 - \beta_n\gamma_n)\|x_n - p\| + \beta_n\gamma_n\|p\| \\ &\leq \max\{\|x_n - p\|, \|p\|\}\end{aligned}$$

for each $n \geq 0$. Hence $\{x_n\}$ is bounded. Further $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$ and $\{w_n\}$ are also bounded. □

Lemma 3.2. *Suppose $\gamma \in (0, \frac{2}{M^2})$. If the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(a) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(b) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(c) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \|u_n - Ax_n\| = 0.$$

Proof. Since $(I - T_s^{F_2})$ is 1-inverse strongly monotone, by (3.1), we have

$$\begin{aligned}
&\|w_{n+1} - w_n\|^2 \\
&= \|T_r^{F_1} v_{n+1} - T_r^{F_1} v_n\|^2 \\
&\leq \|v_{n+1} - v_n\|^2 \\
&= \|(I - \gamma A^*(I - T_s^{F_2})A)x_{n+1} - (I - \gamma A^*(I - T_s^{F_2})A)x_n\|^2 \\
&= \|x_{n+1} - x_n\|^2 - 2\gamma\langle x_{n+1} - x_n, A^*(I - T_s^{F_2})Ax_{n+1} - A^*(I - T_s^{F_2})Ax_n\rangle \\
&\quad + \gamma^2\|A^*(I - T_s^{F_2})Ax_{n+1} - A^*(I - T_s^{F_2})Ax_n\|^2 \\
&= \|x_{n+1} - x_n\|^2 - 2\gamma\langle Ax_{n+1} - Ax_n, (I - T_s^{F_2})Ax_{n+1} - (I - T_s^{F_2})Ax_n\rangle \\
&\quad + \gamma^2\|A^*(I - T_s^{F_2})Ax_{n+1} - A^*(I - T_s^{F_2})Ax_n\|^2 \\
&\leq \|x_{n+1} - x_n\|^2 - 2\gamma\langle Ax_{n+1} - Ax_n, (I - T_s^{F_2})Ax_{n+1} - (I - T_s^{F_2})Ax_n\rangle \\
&\quad + \gamma^2\|A^*\|^2\|(I - T_s^{F_2})Ax_{n+1} - (I - T_s^{F_2})Ax_n\|^2 \\
&\leq \|x_{n+1} - x_n\|^2 + \gamma(\gamma\|A\|^2 - 2)\|(I - T_s^{F_2})Ax_{n+1} - (I - T_s^{F_2})Ax_n\|^2 \\
&\leq \|x_{n+1} - x_n\|^2
\end{aligned}$$

for each $n \geq 0$ and hence

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\beta_n(1 - \gamma_n)x_n + (1 - \beta_n)w_n - \beta_{n-1}(1 - \gamma_{n-1})x_{n-1} - (1 - \beta_{n-1})w_{n-1}\| \\
&= \|\beta_n(1 - \gamma_n)(x_n - x_{n-1}) + \beta_n(1 - \gamma_n)x_{n-1} + (1 - \beta_n)(w_n - w_{n-1}) \\
&\quad + (1 - \beta_n)w_{n-1} - \beta_{n-1}(1 - \gamma_{n-1})x_{n-1} - (1 - \beta_{n-1})w_{n-1}\| \\
&= \|\beta_n(1 - \gamma_n)(x_n - x_{n-1}) + \big(\beta_n(1 - \gamma_n) - \beta_{n-1}(1 - \gamma_{n-1})\big)x_{n-1} \\
&\quad + (1 - \beta_n)(w_n - w_{n-1}) + (\beta_{n-1} - \beta_n)w_{n-1}\| \\
&\leq \beta_n(1 - \gamma_n)\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\gamma_{n-1} - \gamma_n|)\|x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n|\|w_{n-1}\| + (1 - \beta_n)\|x_n - x_{n-1}\| \\
&\leq (1 - \beta_n\gamma_n)\|x_n - x_{n-1}\| + (2|\beta_n - \beta_{n-1}| + |\gamma_{n-1} - \gamma_n|)M_0
\end{aligned}$$

for each $n \geq 0$, where $M_0 = \max\{\sup_{n\in\mathbb{N}} \|w_n\|, \sup_{n\in\mathbb{N}} \|x_n\|\}$. Thus, by Lemma 2.6 and the conditions (a)-(c), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

On the other hand, from (3.3) and (b) it follows that

$$\begin{aligned}
(1 - \beta_n)\|w_n - x_n\| &= \|x_{n+1} - x_n + \beta_n\gamma_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + \beta_n\gamma_n\|x_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. This with (a) yields that

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.4}$$

Let

$$z_n = \beta_n x_n + (1-\beta_n) w_n. \tag{3.5}$$

By (3.4), we have

$$\|z_n - x_n\| = (1-\beta_n)\|w_n - x_n\| \to 0 \tag{3.6}$$

as $n \to \infty$. Since $T_r^{F_1}$ is firmly nonexpansive, it follows from (3.1) that

$$\begin{aligned}\|w_n - p\|^2 &= \|T_r^{F_1} v_n - T_r^{F_1} p\|^2 \\ &\le \langle v_n - p, w_n - p\rangle \\ &= \frac{1}{2}(\|w_n - p\|^2 + \|v_n - p\|^2 - \|v_n - w_n\|^2)\end{aligned}$$

and hence

$$\|w_n - p\|^2 \le \|v_n - p\|^2 - \|v_n - w_n\|^2 \tag{3.7}$$

for each $n \ge 1$. By (3.5) and (3.7), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\|v_n - p\|^2 - \|w_n - v_n\|^2) \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\|x_n - p\|^2 - \|w_n - v_n\|^2) \\ &= \|x_n - p\|^2 - (1-\beta_n)\|w_n - v_n\|^2.\end{aligned} \tag{3.8}$$

Combing (3.6) and (3.8), we obtain

$$\begin{aligned}(1-\beta_n)\|w_n - v_n\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) \\ &\to 0\end{aligned}$$

as $n \to \infty$. This with (a) imply that

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.9}$$

Now by (3.2), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\beta_n((1-\gamma_n)x_n-p)+(1-\beta_n)(w_n-p)\|^2\\
&\le \beta_n\|(1-\gamma_n)x_n-p\|^2+(1-\beta_n)\|w_n-p\|^2\\
&\le \beta_n\|(1-\gamma_n)x_n-p\|^2+(1-\beta_n)[\|x_n-p\|^2\\
&\quad+\gamma(\gamma\|A\|^2-2)\|(I-T_s^{F_2})Ax_n\|^2]\\
&=\beta_n[\|x_n-p\|^2-2\gamma_n\langle x_n-p,x_n\rangle+\gamma_n^2\|x_n\|^2]\\
&\quad+(1-\beta_n)\left[\|x_n-p\|^2+\gamma(\gamma\|A\|^2-2)\|(I-T_s^{F_2})Ax_n\|^2\right]\\
&=\|x_n-p\|^2-2\beta_n\gamma_n\langle x_n-p,x_n\rangle+\beta_n\gamma_n^2\|x_n\|^2\\
&\quad+\gamma(1-\beta_n)(\gamma\|A\|^2-2)\|(I-T_s^{F_2})Ax_n\|^2\\
&\le \|x_n-p\|^2-2\beta_n\gamma_n\langle x_n-p,x_n\rangle+\beta_n\gamma_n^2\|x_n\|^2\\
&\quad+\gamma(1-\beta_n)(\gamma M^2-2)\|(I-T_s^{F_2})Ax_n\|^2.
\end{aligned}\tag{3.10}$$

From (3.3), (3.10) and (b), it follows that

$$\begin{aligned}
&\gamma(1-\beta_n)(2-\gamma M^2)\|(I-T_s^{F_2})Ax_n\|^2\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)-2\beta_n\gamma_n\langle x_n-p,x_n\rangle+\beta_n\gamma_n^2\|x_n\|^2\\
&\to 0
\end{aligned}$$

as $n\to\infty$. Since $2-\gamma M^2>0$ and $\limsup_{n\to\infty}\beta_n<1$, we have

$$\lim_{n\to\infty}\|(I-T_s^{F_2})Ax_n\|=\lim_{n\to\infty}\|u_n-Ax_n\|=0. \tag{3.11}$$

This completes the proof. □

Now, we give the main results in this paper.

Theorem 3.3. *Suppose that the sequences $\{\gamma_n\}$ and $\{\beta_n\}$ satisfy the conditions in Lemma* 3.2. *Then the sequence $\{x_n\}$ generated by Algorithm A strongly converges to the point $x^*=P_\Omega\theta$, where θ denotes the zero element of H_1.*

Proof. Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle -x^*,x_n-x^*\rangle=\lim_{j\to\infty}\langle -x^*,x_{n_j}-x^*\rangle.$$

Since$\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ converging weakly to a point $v\in H_1$. Without loss of generality, we can assume that $x_{n_j}\rightharpoonup v$ as $j\to\infty$. From (3.3), (3.4), (3.9) and (3.11), it follows that

$$w_{n_j}\rightharpoonup v,\quad v_{n_j}\rightharpoonup v,\quad u_{n_j}\rightharpoonup Av$$

as $j \to \infty$. Since $\{w_{n_j}\} \subset C_1$ and C_1 is closed, it follows that $v \in C_1$. Similarly, $Av \in C_2$ since $\{u_{n_j}\} \subset C_2$.

Now, we show that $v \in EP_1$. In fact, since $T_r^{F_1}$ is nonexpansive, by (3.1), (3.9) and Lemma 2.4, we can conclude that $v \in Fix(T_r^{F_1})$. Further, from Lemma 2.2, it follows that $v \in EP_1$. Similarly, by (3.1), (3.11) and Lemma 2.2 we have $Av \in EP_2$. Therefore, we have $v \in \Omega$. By Lemma 2.4, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle &= \lim_{j\to\infty}\langle -x^*, x_{n_j} - x^*\rangle \\ &= \langle -x^*, v - x^*\rangle \\ &\leq 0.\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}x_{n+1} &= z_n - \beta_n\gamma_n x_n \\ &= (1-\beta_n\gamma_n)z_n + \beta_n\gamma_n(z_n - x_n) \\ &= (1-\beta_n\gamma_n)z_n + \beta_n\gamma_n(1-\beta_n)(w_n - x_n),\end{aligned}$$

by (3.8) and Lemma 2.4, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \|(1-\beta_n\gamma_n)(z_n - x^*) + \beta_n\gamma_n[(1-\beta_n)(w_n - x_n) - x^*]\|^2 \\ &\leq (1-\beta_n\gamma_n)\|z_n - x^*\|^2 + 2\beta_n\gamma_n(1-\beta_n)\langle w_n - x_n, x_{n+1} - x^*\rangle \\ &\quad + 2\beta_n\gamma_n\langle -x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\beta_n\gamma_n)\|x_n - x^*\|^2 + 2\beta_n\gamma_n(1-\beta_n)\langle w_n - x_n, x_{n+1} - x^*\rangle \\ &\quad + 2\beta_n\gamma_n\langle -x^*, x_{n+1} - x^*\rangle\end{aligned} \tag{3.13}$$

for each $n \geq 0$. By (a), (b), (3.4), (3.12), (3.13) and Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

This completes the proof. □

Since each inverse strongly monotone mapping is monotone and continuous, we have the following:

Corollary 3.4. *Let H_i be a real Hilbert space and C_i be a nonempty bounded closed convex subset of H_i, $A_i : C_i \to H_i$ be a λ_i-inverse strongly monotone mapping, and $F_i : C_i \times C_i \to \mathbb{R}$ satisfying the conditions* (B1)-(B3), *for $i = 1, 2$. Let $A : H_1 \to H_2$ be a bounded linear operator. If the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(a) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(b) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(c) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
then the sequence $\{x_n\}$ *generated by Algorithm A strongly converges to the element* $x^* = P_\Omega \theta$, *where* θ *denotes the zero element of* H_1.

Since each strongly monotone mapping is monotone, we have the following:

Corollary 3.5. *Let* H_i *be a real Hilbert space and* C_i *be a nonempty bounded closed convex subset of* H_i. *Let* $A_i : C_i \to H_i$ *be a strongly monotone and continuous mapping with the parameter* $\lambda_i > 0$, $F_i : C_i \times C_i \to \mathbb{R}$ *satisfying the conditions* (B1)-(B3), *for* $i = 1, 2$ *and* $A : H_1 \to H_2$ *be a bounded linear operator. If the sequences* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*
(a) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(b) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(c) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
then the sequence $\{x_n\}$ *generated by Algorithm A strongly converges to a point* $x^* = P_\Omega \theta$, *where* θ *denotes the zero element of* H_1.

Remark 3.6. In [1, 12, 25, 26, 29], the parameter is chosen from $(0, \frac{2}{L^2})$, where L is the spectral radius of the operator A^*A. In this paper, the parameter γ is chosen by the boundedness above of $\|A\|$. Obviously, the parameter γ is easier to chose. On the other hand, at each step, x_{n+1} is computed by a non-convex combination of x_n and w_n, which is very different with the similar ones of others in the literature.

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