Nonlinear Functional Analysis and Applications
Vol. 22, No. 5 (2017), pp. 925-934
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

ON THE n-KKM PRINCIPLE FOR ABSTRACT CONVEX SPACES

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. In this paper, we review some works of Kulpa *et al.* closely related to our KKM theory. In fact, we show that their L^*-spaces are particular to our partial KKM spaces. We introduce new n-KKM spaces and show that L^*_n-spaces of Kulpa et al. are particular to our partial n-KKM spaces. Some related results are also given.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptiness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or other solutions of various equilibrium problems. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [8], which concerns with certain types of multimaps called the KKM maps later.

The KKM theory, first named by the author [14], is the study of applications of equivalent formulations or generalizations of the KKM theorem. From 1961,

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[0]Received October 3, 2017. Revised November 27, 2017.
[0]2010 Mathematics Subject Classification: 47H04, 47H10, 49J27, 49J35, 49J53, 54H25, 55M20, 91B50.
[0]Keywords: Abstract convex space, n-KKM space, n-KKM principle, L^*-operator, L^*_n-operator.

Ky Fan showed that the KKM theorem provides foundations for many of the modern essential results in diverse areas of mathematical sciences. He extended the KKM theorem to arbitrary topological vector spaces and applied it to various problems; see [14,15]. Fan's works were expanded systematically by Granas [5] to new topological methods in convex analysis on convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [12], and to c-spaces (or H-spaces) by Horvath [6,7], and others. Since the last decade of the last century, the KKM theory had been extended to generalized convex (G-convex) spaces in a sequence of papers of the author and others; for details, see [15,16] and references therein.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [17]-[21]. The partial KKM principle for an abstract convex space is an abstract form of the KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its *open* version. In our recent works [19]-[21], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are closely related to KKM spaces or partial KKM spaces, that is, spaces satisfying the partial KKM principle. Moreover, a number of such results are equivalent to each other.

On the other hand, it is well-known that any family of closed balls in a hyperconvex metric space has a nonempty intersection whenever each two members of the family intersects. Motivated by this fact, T.-H. Chang *et al.* [2]-[4] introduced 2-KKM maps and generalized 2-KKM maps on metric spaces, and then obtained a 2-KKM theorem, a fixed point theorem without compactness condition, some minimax inequalities, and other results for hyperconvex metric spaces.

Recently, Kulpa, Szymanski and Turzansk [11] defined an L_n^*-operator. Motivated by this, we define n-KKM maps on abstract convex spaces and show that L_n^*-spaces are particular to our partial KKM spaces.

In Section 2, we give only a small portion of basic concepts in our KKM theory on abstract convex spaces as a preliminary. Section 3 is devoted to review some works of Kulpa *et al.* [9]-[11] closely related to our KKM theory. In fact, we show their L^*-spaces are particular forms of our partial KKM spaces. Section 4 deals with L_n^*-spaces of Kulpa *et al.* [11]. We define n-KKM spaces and show that L_n^*-spaces are particular to our partial KKM spaces. Finally, in Section 5, we introduce abstracts of three articles containing materials closely related to the contents in this paper. Each of them can be used to obtain some new generalized results if possible.

2. Abstract convex spaces

In this section, we mainly follow [23] with some modifications and the corrections in [25].

Multimaps are also called simply maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Example 2.4. We gave lots of examples of abstract convex spaces in [18]-[21].

Theorem 2.5. ([23]) *For an abstract convex space $E, D; \Gamma$, the following two statements are equivalent:*

(1) **The KKM principle.** *For any closed-valued [resp., open-valued] KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property.*
(2) **The Fan matching property.** *Let $S : D \multimap E$ be a map satisfying*

(i) *$S(z)$ is open [resp., closed] for each $z \in D$; and*
(ii) *$E = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$.*
Then there exists an $N \in \langle M\rangle$ such that

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

Proof. (1) $\Rightarrow$ (2): Let $G : D \multimap E$ be a map given by $G(z) := E \setminus S(z)$ for $z \in D$. Then G has closed [resp., open] values. Suppose that, on the contrary to the conclusion, for any $N \in \langle M\rangle$, we have $\Gamma_N \cap \bigcap_{z\in N} S(z) = \emptyset$; that is,

$$\Gamma_N \subset E \setminus \bigcap_{z\in N} S(z) = \bigcup_{z\in N} (E \setminus S(z)) = G(N).$$

Then $G|_M : M \multimap E$ is a KKM map. It is easily checked that $(E, M; \Gamma|_{\langle M\rangle})$ also satisfies the KKM principle (1) [12, Lemma 2]; and hence there exists a

$$\widehat{y} \in \bigcap_{z\in N} G(z) = \bigcap_{z\in N} (E \setminus S(z)).$$

Hence $\widehat{y} \notin S(z)$ for all $z \in N$. Since N is arbitrary, this violates condition (ii).
(2) $\Rightarrow$ (1): Let $G : D \multimap E$ be a closed [resp., open] valued KKM map. Define a map $S : D \multimap E$ by $S(z) := E \setminus G(z)$ for $z \in D$. Then S is open [resp., closed] valued. Hence (i) holds. Suppose that there exists an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} G(z) = \emptyset$ on the contrary to the conclusion of (1). Then

$$E = \bigcup_{z\in M} (E \setminus G(z)) = \bigcup_{z\in M} S(z),$$

that is, (ii) holds. Therefore, by (2), there exists an $N \in \langle M\rangle$ such that

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

Hence there exists an $x_0 \in \Gamma_N$ such that $x_0 \in \bigcap_{z\in N} S(z) = \bigcap_{z\in N}(E \setminus G(z))$ or $x_0 \notin G(N)$. Therefore, $\Gamma_N \not\subset G(N)$ and G is not a KKM map. A contradiction. □

Note that a KKM space is an abstract convex space satisfying the statements (1) and (2) in Theorem 2.5, and a partial KKM space is the one satisfying the closed valued version of (1) and the open case of (2).

3. L^*-spaces of Kulpa *et al.*

In 2008, Kulpa and Szymanski [9] introduced a series of theorems called *Infimum Principles* and applied them to some classical results. Note that most of results in [9] originate from the *Theorem of Indexed Families* (Theorem 3) and that this is an easy consequence of a Fan type matching theorem

([13] Theorem 1). Their study is based on and utilizes the techniques of simplicial structures and CO families (equivalently, multimaps with nonempty convex values and open fibers). Here a simplicial space is a topological space having a certain collection of singular simplices. As applications, they derived fixed point theorems due to Schauder, Tychonoff, Kakutani, and Fan-Browder; minimax theorem; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Fan minimax inequality. Finally, the authors of [9] suggested a way of extending their results to a wider class of topological spaces called *L^*-spaces.*

In 2012 [24], we recall that for any abstract convex spaces satisfying abstract KKM principle, we can deduce such classical theorems without using any Infimum Principles. In fact, such theorems are consequences of some equivalents of the KKM theorem on a simplicial space and hence are typical particular results of the KKM theory for abstract convex spaces. Moreover, we show that simplicial spaces and L^*-spaces are of particular type of KKM spaces due to ourselves and that some of main results in [9] are consequences of corresponding ones in KKM spaces.

In 2014, Kulpa and Szymanski [10] discussed Park's abstract convex spaces and their relevance to classical convexities and L^*-operators. They constructed an example of a space satisfying the partial KKM principle that is not a KKM space. The existence of such a space solves a problem raised by Park.

In 2015, the present author [26] traces out the history of [10] and responds to some remarks given there. One of the objections given in [10] is the use of D in our abstract convex spaces $(E, D; \Gamma)$. Recently, another example of the importance of D is given in [1].

Kulpa, Szymanski and Turzanski [11] called again an L^*-operator $\Lambda : \langle X \rangle \multimap X$ on a topological space X whenever it satisfies the following condition:

$$\begin{aligned}&\text{If } A \in \langle X \rangle \text{ and } \{U_x \mid x \in A\} \text{ is an open cover of } X, \text{ then there exists}\\ &B \subset A \text{ such that } \Lambda(B) \cap \bigcap \{U_x \mid x \in B\} \neq \emptyset. \qquad (3.1)\end{aligned}$$

Note that considering a map $S : X \multimap X$ defined by $S(x) := U_x$ for $x \in X$, $E = D = X$, and $\Lambda = \Gamma$, (3.1) is a particular form of the open case of Statement (2) in Theorem 2.5.

Kulpa *et al.* [11] defined that an L^*-space is a topological space endowed with an L^*-operator. Therefore, an L^*-space is a particular one of partial KKM spaces.

4. n-KKM spaces

First of all, we recall the L_n^*-spaces due to Kulpa *et al.* [11] as follows:

Definition 4.1. An operator $\Lambda : \langle X \rangle \multimap X$ on a topological space X is called an L_n^*-*operator* if it satisfies the following condition (4.1):

$$\begin{aligned} &\text{If } A \in \langle X \rangle \text{ and } \{U_x \mid x \in A\} \text{ is an open cover of } X, \text{ then there exists} \\ &B \subset A \text{ such that } |B| \le n+1 \text{ and } \Lambda(B) \cap \bigcap \{U_x \mid x \in B\} \ne \emptyset. \end{aligned} \tag{4.1}$$

In this case, the pair $(X;\Lambda)$ is called an L_n^*-*space*. The condition defining L_n^*-operators is the original condition (3.1) with an additional requirement imposed on the size of the set B.

Motivated by this definition, we define the following new one:

Definition 4.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ and a natural $n \in \mathbb{N}$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y)$$

for all $A \in \langle D \rangle$ with the cardinal $|A| \le n+1$, then G is called a *n-KKM map* with respect to F. A *n-KKM map* $G : D \multimap E$ is a n-KKM map with respect to the identity map 1_E.

Definition 4.3. The *partial n-KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued n-KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *n-KKM principle* is the statement that the same property also holds for any open-valued n-KKM map.

An abstract convex space is called a (*partial*) *n-KKM space* if it satisfies the (partial) n-KKM principle, respectively.

Example 4.4. (1) For each $n \in \mathbb{N}$, the (partial) KKM principle is the (partial) n-KKM principle; every (partial) KKM space is a (partial) n-KKM space; and every KKM map is an n-KKM map.
(2) The original KKM theorem [8] is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation. The (partial) KKM principle for this case is the (partial) n-KKM principle from the beginning.
(3) The 2-KKM spaces in [22] are n-KKM spaces for n=2.
(4) The L_n^*-spaces of Kulpa *et al.* [11] can be shown to be n-KKM spaces as follows.

For an abstract convex space $(E, D; \Gamma)$, let us consider the following:

Theorem 4.5. (Characterizations of the n-KKM spaces) *For an n-KKM space $(E, D; \Gamma)$, the following are equivalent:*

(1) **The n-KKM principle.** *Let $(E, D; \Gamma)$ be an n-KKM space, that is, for any closed-valued [resp., open-valued] n-KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property.*

(2) **The Fan matching property.** *Let $S : D \multimap E$ be a map satisfying*
  (i) *$S(z)$ is open [resp., closed] for each $z \in D$; and*
  (ii) *$E = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$.*
  Then there exists an $N \in \langle M\rangle$ with $|N| \leq n + 1$ such that

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

Proof. (1) ⇒ (2): Let $G : D \multimap E$ be a map given by $G(z) := E \setminus S(z)$ for $z \in D$. Then G has closed [resp., open] values. Suppose, on the contrary to the conclusion, that for any $N \in \langle M\rangle$ with $|N| \leq n+1$, we have $\Gamma_N \cap \bigcap_{z\in N} S(z) = \emptyset$; that is,

$$\Gamma_N \subset E \setminus \bigcap_{z\in N} S(z) = \bigcup_{z\in N} (E \setminus S(z)) = G(N).$$

Then $G|_M : M \multimap E$ is a n-KKM map. It is easily checked that $(E, M; \Gamma|_{\langle M\rangle})$ also satisfies the n-KKM principle (1) [19, Lemma 2]; and hence there exists a

$$\widehat{y} \in \bigcap_{z\in N} G(z) = \bigcap_{z\in N} (E \setminus S(z)).$$

Hence $\widehat{y} \notin S(z)$ for all $z \in N$. Since N is arbitrary, this violates condition (ii).
(2) ⇒ (1): Let $G : D \multimap E$ be a closed [resp., open] valued n-KKM map. Define a map $S : D \multimap E$ by $S(z) := E \setminus G(z)$ for $z \in D$. Then S is open [resp., closed] valued. Hence (i) holds. Suppose that there exists an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} G(z) = \emptyset$ on the contrary to the conclusion of (1). Then

$$E = \bigcup_{z\in M} (E \setminus G(z)) = \bigcup_{z\in M} S(z),$$

that is, (ii) holds. Therefore, by (2), there exists an $N \in \langle M\rangle$ such that $|N| \leq n + 1$ and

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

Hence there exists an $x_0 \in \Gamma_N$ such that $x_0 \in \bigcap_{z\in N} S(z) = \bigcap_{z\in N}(E \setminus G(z))$ or $x_0 \notin G(N)$. Therefore, $\Gamma_N \not\subset G(N)$ and G is not a n-KKM map. A contradiction. □

The above proof is almost same to the one for Theorem 2.5 with an additional requirement imposed on the size of the set N.

Note also that the closed-valued case of (1) characterizes the partial n-KKM space. Moreover, we have the following:

Corollary 4.6. *An absolute convex space* $(X;\Lambda)$ *is a partial n-KKM space if and only if it is an* L_n^**-space.*

Now the following diagram for abstract convex spaces clearly holds:

$$\begin{array}{ccc} L^*\text{-space or KKM space} & \Longrightarrow & \text{Partial KKM spaces} \\ \Downarrow & & \Downarrow \\ L_n^*\text{-space or } n\text{-KKM space} & \Longrightarrow & \text{Partial } n\text{-KKM spaces} \end{array}$$

5. Problems for further study

In this section, we introduce three articles containing materials closely related to the contents of this present paper. Each of them can be used to obtain some new generalized results if possible. After giving short abstracts of those three articles, we raise possible problems.

1. (S. Park [22, 25]) *Abstract:* " We clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces."

A problem is to obtain equivalent formulations of the n-KKM principle, other than the Fan matching property as shown in Theorem 4.5. Similarly, another problem is to find some equivalent formulation of the partial n-KKM principle and their applications. In fact, it might be possible to generalize the contents of [22] or [23] to the (partial) n-KKM spaces.

2. (W. Kulpa et al. [11]) *Abstract:* "Within the framework of spaces admitting special L^*-operators (such as continuous or L_n^*-operators) we prove fixed point theorems (of Brouwer or Schauder type) and discuss some related issues (e.g. the existence symmetric equilibria)."

A problem is to obtain generalizations of applicable parts of [11] to partial n-KKM spaces.

3. (M.-H. Shih [27]) *Abstract:* "The hidden convex structure of the Knaster-Kuratowski-Mazurkiewicz theorem allows us to reveal the hidden convex structure of Klee's geometric transversal structure. This enables us to describe two forms of Klee's theorem in arbitrary topological vector spaces which are not required to be Hausdorff or locally convex, and to determine the Helly number of two classes of families of convex sets in topological vector spaces."

A problem is to generalize any parts of [27] to our abstract convex structure; more precisely, generalize topological vector spaces in [27] or in many other works of Shih to our abstract convex spaces.

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