Nonlinear Functional Analysis and Applications
Vol. 22, No. 5 (2017), pp.935-946
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa
Copyright © 2017 Kyungnam University Press

INCLUSION PROPERTIES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH THE EXTENDED CHO-KWON-SRIVASTAVA OPERATOR BY USING HYPERGEOMETRIC FUNCTION

Khalid Abdulameer Challab[1], Maslina Darus[2] and Firas Ghanim[3]

[1]School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia Bangi 43600 Selangor D. Ehsan, Malaysia
e-mail: `khalid_math1363@yahoo.com`

[2]School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia Bangi 43600 Selangor D. Ehsan, Malaysia
e-mail: `maslina@ukm.edu.my`

[3]Department of Mathematics, College of Sciences
University of Sharjah, Sharjah, United Arab Emirates
e-mail: `fgahmed@sharjah.ac.ae`

Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. The purpose of the present paper is to introduce several new classes of meromorphic functions defined the generalized Cho-Kwon-Srivastava operator and investigate various inclusion properties of these classes. Some interesting applications involving these and other classes of integral operators are also considered.

1. Introduction

Cho et al. [2] introduced new multiplier transformation $\tau^n_{\lambda,\mu}$ by using Hadamard product in 2004. In literature many authors have concentrated Cho-Kwon-Srivastava operator such as [9], [18], [19] and other authors studied

[0]Received June 8, 2017. Revised December 4, 2017.
[0]2010 Mathematics Subject Classification: 30D30, 30E05.
[0]Keywords: Meromorphic functions, integral operator, Hurtiz-Lerch zeta function.
[0]Corresponding author: Maslina Darus.

Cho-Kwon-Srivastava operator as [10], [23] but their study related to analytic function.

Normally, we are considering the class of meromorphic function f of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open disk $U^* = \{z : z \in C \ \ and \ \ 0 < |z| < 1\} = U \setminus \{0\}$ and denoted by Σ.

The Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

was derived by the two functions $f_j(z) \in \Sigma \quad (j = 1, 2)$, which equals

$$f_j(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2). \tag{1.2}$$

Several interesting characteristics and properties of Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined by (see [21])

$$\begin{aligned} &\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \\ &a \in C \setminus Z_0^- = \{0, -1, -2, \cdots\}, \\ &\begin{cases} s \in C & \text{when } \ |z| < 1; \\ Re \ s > 1 & \text{when } \ |z| = 1, \end{cases} \end{aligned} \tag{1.3}$$

had been found in investigations held by several authors (see [8], [14], [22]).

Through the use of [[20], p.1496, Remark 7]:

$$\lim_{b \to 0} \left\{ H_{0,2}^{2,0} \left[(a+n) b^{\frac{1}{\lambda}} \mid \overline{(s,1), \left(0, \frac{1}{\lambda}\right)} \right] \right\} = \lambda \Gamma(s) \quad (\lambda > 0),$$

we were able to introduce a new family of generalized Hurwitz-Lerch zeta functions $\Phi(z, s, a, b, \lambda)$ which equates to

$$\Phi(z, s, a, b, \lambda) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \frac{\Lambda(a+n, b, s, \lambda)}{\lambda \Gamma(s)}. \tag{1.4}$$

El-Ashwah[8] termed the function of $H_a^s(z)$ $(a \in C \setminus Z_0^-; s \in C)$ by

$$H_a^s(z) = \frac{a^s}{z}\Phi(z, s, a) \ \ (z \in U^*), \tag{1.5}$$

and $H_{\lambda,b}^{s,a}(z)$ to be represented by

$$H_{\lambda,b}^{s,a}(z) = \frac{a^s}{z}\Phi(z, s, a, b, \lambda) \ \ (z \in U^*), \tag{1.6}$$

with

$$\Lambda(a, b, s, \lambda) := H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}} \mid \overline{(s, 1), \left(0, \frac{1}{\lambda}\right)}\right].$$

We also denote by

$$L_{\lambda,b}^{s,a}f(z) : \Sigma \to \Sigma.$$

The equation

$$L_{\lambda,b}^{s,a}f(z) = H_{\lambda,b}^{s,a}(z) * f(z) \ \ (a \in C \setminus Z_0^-; s \in C; z \in U^*)$$

can be signified and defined as a new linear operator:

$$L_{\lambda,b}^{s,a}f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}\left(\frac{a}{n+a+1}\right)^s \frac{\Lambda(a+n, b, s, \lambda)}{\lambda\Gamma(s)} a_n z^n, \tag{1.7}$$

with $\min\{\Re(a), \Re(s)\} > 0$; $\lambda > 0$ if $R(b) > 0$ and $s \in C$ and $a \in C \setminus Z_0^-$ if $b = 0$.

This was clear in El-Ashwah's study [8] which examined that by taking the limit as $b \to 0$ one would have obtained $L_a^s f(z)$.

Also we note that:

(i) $L_\beta^\alpha f(z) = P_\beta^\alpha f(z)$ $(\alpha, \beta > 0)$ (see Lashin[12]);
(ii) $L_1^\alpha f(z) = P^\alpha f(z)$ $(\alpha > 0)$ (see Aqlan et al. [1], with $p = 1$);
(iii) $L_\nu^1 f(z) = F_\nu f(z)$ $(\nu > 0)$ (see[16], p.11 and 389).

The function

$$\Psi(d, c; z) = \frac{1}{z} + \sum_{n=0}^{\infty}\frac{(d)_{n+1}}{(c)_{n+1}} z^n \ \ (d \in C^* = C \setminus \{0\}; c \in C \setminus Z_0^-; z \in U^*), \tag{1.8}$$

can be derived when $(\gamma)_n$ also defined as the Pochhammer symbol and the Gamma function Γ, was represented by

$$(\gamma)_n := \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} \gamma(\gamma+1)\ldots(\gamma+n-1), & (n \in \mathbb{N}), \\ 1, & (n = 0). \end{cases}$$

It was noted that

$$\Psi(d,c;z)=\frac{1}{z}F_1(d,1,c;z),$$

$$F_1(d,b,c;z)=\sum_{n=0}^{\infty}\frac{(d)_n(b)_n}{(c)_n(1)_n}z^n \ \ (d,b,c\in C \ \ and \ \ c\notin Z_0^-;z\in U),$$

was also defined as the (Gaussian) hypergeometric function.

Set

$$L_{\lambda,b}^{s,a}*F_{\lambda,b}^{s,a}(z)=\frac{1}{z(1-z)},$$

we have

$$F_{\lambda,b}^{s,a}(z)=\frac{1}{z}+\sum_{n=0}^{\infty}\left(\frac{n+a+1}{a}\right)^s\frac{\lambda\Gamma(s)}{\Lambda(a+n,b,s,\lambda)}z^n. \tag{1.9}$$

Through the Hadamard product (or convolution), we have

$$F_{\lambda,b}^{s,a}(z)*F_{\lambda,b}^{s,a}(d,c;z)=\Psi(d,c;z) \ \ (z\in U^*). \tag{1.10}$$

A new operator $F_{\lambda,b}^{s,a}(d,c;z)$, was derived from the original operator $F_{\lambda,b}^{s,a}(z)$.

The linear operator $F_{\lambda,b}^{s,a}(d,c;z):\Sigma\to\Sigma$, had been verified from the classification of the operator $F_{\lambda,b}^{s,a}(d,c;z)f(z)$, which equaled:

$$F_{\lambda,b}^{s,a}(d,c;z)f(z)=F_{\lambda,b}^{s,a}(d,c;z)*f(z) \ \ (s\in C;d\in C^*;c,a\in C\setminus Z_0^-) \tag{1.11}$$

and

$$F_{\lambda,b}^{s,a}(d,c;z)f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}\frac{(d)_{n+1}}{(c)_{n+1}}\left(\frac{a}{n+a+1}\right)^s\frac{\Lambda(a+n,b,s,\lambda)}{\lambda\Gamma(s)}a_nz^n. \tag{1.12}$$

It is easily verified from the definition of the operator $F_{\lambda,b}^{s,a}(d,c;z)f(z)$, that

$$z(F_{\lambda,b}^{s+1,a}(d,c;z)f(z))'=aF_{\lambda,b}^{s,a}(d,c;z)f(z)-(a+1)F_{\lambda,b}^{s+1,a}(d,c;z)f(z) \tag{1.13}$$

and

$$z(F_{\lambda,b}^{s,a}(d,c;z)f(z))' = dF_{\lambda,b}^{s,a}(d+1,c;z)f(z)-(d+1)F_{\lambda,b}^{s,a}(d,c;z)f(z) \ \ (d\in C\setminus\{-1\}). \tag{1.14}$$

Clearly, upon setting $d=\mu$ and $c=1$ in (1.12) and taking the limit as $b\to 0$, we obtain the operator $I_{a,\mu}^s f(z)$ $(a,\mu\in R^+,\ s\in N_0=N\cup\{0\})$ (see Cho et al. [4]).

Allowing f and g to be analytic in U. it can be concluded that f is subordinate to g and would be written as $f \prec g$ or $f(z) \prec g(z)$. The Schwarz function ω in U presented the formula $f(z) = g(\omega(z))$.

To further substantiate the existence of $0 \leq \eta$ and $\beta < 1$, three key terms and their subclasses had to be addressed. The mathematical terms and their meromorphic functions which were subclasses of Σ included $\mathcal{MS}(\eta)$ starlike of order η, $\mathcal{MK}(\eta)$ convex of order η and $\mathcal{ML}(\eta, \beta)$ close-to-convex of order β and type η in U ([11],[13],[16]).

In this context, $\mathcal{N}$ indicated the class of all functions ϕ which were analytic and univalent in U, whereby $\phi(U)$ was convex with $\phi(0) = 1$ and $Re\{\phi(z)\} > 0 (z \in U)$.

Through the use of the principle of subordination between analytic functions, the subclasses $\mathcal{MS}(\eta, \phi)$, $\mathcal{MK}(\eta, \phi)$ and $\mathcal{ML}(\eta, \beta; \phi, \psi)$ of the class Σ were introduced respectively for $0 \leq \eta, \beta < 1$ and $\phi, \psi \in \mathcal{N}$, which had been defined by

$$\mathcal{MS}(\eta, \phi) := \left\{ f \in \Sigma : \frac{1}{1-\eta} \left(-\frac{zf'(z)}{f(z)} - \eta \right) \prec \phi(z) \ in \ U \right\},$$

$$\mathcal{MK}(\eta, \phi) := \left\{ f \in \Sigma : \frac{1}{1-\eta} \left(-\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \eta \right) \prec \phi(z) \ in \ U \right\}$$

and

$$\mathcal{ML}(\eta, \beta; \phi, \psi) := \left\{ f \in \Sigma : \exists g \in \mathcal{MS}(\eta, \phi) \ s.t. \ \frac{1}{1-\beta} \left(-\frac{zf'(z)}{g(z)} - \beta \right) \prec \psi(z) \ in \ U \right\}.$$

It was noted that the classes mentioned above were similar to those commonly applied on the space of analytic and univalent functions in U ([5],[15]).
Subclasses of Σ entails special choices for the functions of ϕ and ψ as:

$$\mathcal{MS}\left(\eta; \frac{1+z}{1-z}\right) = \mathcal{MS}(\eta),$$

$$\mathcal{MK}\left(\eta; \frac{1+z}{1-z}\right) = \mathcal{MK}(\eta)$$

and

$$\mathcal{ML}(\eta, \beta; \frac{1+z}{1-z}, \frac{1+z}{1-z}) = \mathcal{ML}(\eta, \beta).$$

For example, through the use of operator $F^{s,a}_{\lambda,b}(d,c;z)$, classes of meromorphic functions for $\phi,\psi \in \mathcal{N}$,$\lambda > 0$ and $0 \leq \eta,\ \beta < 1$ we have:

$$\mathcal{MS}_{d,a}(s,\lambda,\eta,\phi) := \left\{ f \in \Sigma : F^{s,a}_{\lambda,b}(d,c;z)f \in \mathcal{MS}(\eta,\phi) \right\},$$

$$\mathcal{MK}_{d,a}(s,\lambda,\eta,\phi) := \left\{ f \in \Sigma : F^{s,a}_{\lambda,b}(d,c;z)f \in \mathcal{MK}(\eta,\phi) \right\},$$

and

$$\mathcal{ML}_{d,a}(s,\lambda,\eta,\beta;\phi,\psi) := \left\{ f \in \Sigma : F^{s,a}_{\lambda,b}(d,c;z)f \in \mathcal{ML}(\eta,\beta;\phi,\psi) \right\}$$

were presented. We also note that

$$f(z) \in \mathcal{MK}_{d,a}(s,\lambda,\eta,\phi) \Leftrightarrow -zf'(z) \in \mathcal{MS}_{d,a}(s,\lambda,\eta,\phi). \tag{1.15}$$

In particular, we set

$$\mathcal{MS}_{d,a}\left(s,\lambda,\eta,\frac{1+Az}{1+Bz}\right) =: \mathcal{MS}_{d,a}\left(s,\lambda,\eta,A,B\right) \quad (-1 < B < A \leq 1)$$

and

$$\mathcal{MK}_{d,a}\left(s,\lambda,\eta,\frac{1+Az}{1+Bz}\right) =: \mathcal{MK}_{d,a}\left(s,\lambda,\eta,A,B\right) \quad (-1 < B < A \leq 1).$$

In this paper, we investigated several inclusion properties of the classes $\mathcal{MS}_{d,a}(s,\lambda,\ \eta,\phi)$, $\mathcal{MK}_{d,a}(s,\lambda,\eta,\phi)$ and $\mathcal{ML}_{d,a}(s,\lambda,\eta,\beta;\phi,\psi)$ associated with the operator $F^{s,a}_{\lambda,b}(d,c;z)$. Some applications involving the integral operators had been considered by the previous works such that [3] and [8].

2. Inclusion properties involving the operator $F^{s,a}_{\lambda,b}(d,c;z)$

The results that followed were required in the investigation.

Lemma 2.1. ([6]) *Let ϕ be convex univalent in U with $\phi(0) = 1$ and $Re\{\mathcal{K}\phi(z)+\nu\} > 0$ $(\mathcal{K},\nu \in C)$. If p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\mathcal{K}p(z)+\nu} \prec \phi(z) \quad (z \in U)$$

implies

$$p(z) \prec \phi(z) \quad (z \in U).$$

Lemma 2.2. ([17]) *Let ϕ be convex univalent in U and ω be analytic in U with $Re\{\omega(z)\} \geq 0$. If p is analytic in U and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \quad (z \in U)$$

implies

$$p(z) \prec \phi(z) \quad (z \in U).$$

Firstly, with the assistance provided by Lemma 2.1, several key theorems were highlighted and explained.

Theorem 2.3. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in U} Re\{\phi(z)\} < \min\left\{\frac{Re(d)+1-\eta}{1-\eta}, \frac{Re(a)+1-\eta}{1-\eta}\right\},$$

for $Re(d), Re(a) > 0; \quad 0 \leq \eta < 1$. Then we have

$$\mathcal{MS}_{d+1,a}(s,\lambda,\eta,\phi) \subset \mathcal{MS}_{d,a}(s,\lambda,\eta,\phi) \subset \mathcal{MS}_{d,a}(s+1,\lambda,\eta,\phi).$$

Proof. Theorem 2.3 was derived by adding $f \in \mathcal{MS}_{d+1,a}(s,\lambda,\eta,\phi)$ with setting

$$p(z) = \frac{1}{1-\eta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d,c;z)f(z))'}{F^{s,a}_{\lambda,b}(d,c;z)f(z)} - \eta\right), \tag{2.1}$$

where p is analytic in U with $p(0) = 1$. Applying (1.14) and (2.1), we obtain

$$\frac{1}{1-\eta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d+1,c;z)f(z))'}{F^{s,a}_{\lambda,b}(d+1,c;z)f(z)} - \eta\right)$$
$$= p(z) + \frac{zp'(z)}{-(1-\eta)p(z)+d+1-\eta} \qquad (z \in U). \tag{2.2}$$

Since $\max_{z\in U} Re\{\phi(z)\} < \frac{Re(d)+1-\eta}{1-\eta}$ $(Re(d) > 0; 0 \leq \eta < 1; z \in U)$, we note that

$$Re\{-(1-\eta)\phi(z) + a + 1 - \eta\} > 0 \qquad (z \in U).$$

The second part of Theorem 2.3 which incorporated Lemma 2.1 to (2.2) shown as $p \prec \phi$, so that $f \in \mathcal{MS}_{d,a}(s,\lambda,\eta,\phi)$, was completed when it was proven through the application of similar arguments detailed in point (1.13). □

Theorem 2.4. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in U} Re\{\phi(z)\} < \min\left\{\frac{Re(d)+1-\eta}{1-\eta}, \frac{Re(a)+1-\eta}{1-\eta}\right\},$$

for $Re(d), Re(a) > 0; \quad 0 \leq \eta < 1$. *Then we have*

$$\mathcal{MK}_{d+1,a}(s, \lambda, \eta, \phi) \subset \mathcal{MK}_{d,a}(s, \lambda, \eta, \phi) \subset \mathcal{MK}_{d,a}(s+1, \lambda, \eta, \phi).$$

Proof. The utilization seen in (1.15) and Theorem 2.3 proved the outcome found in Theorem 2.4, that is

$$\begin{aligned} f(z) \in \mathcal{MK}_{d+1,a}(s, \lambda, \eta, \phi) &\Leftrightarrow -zf'(z) \in \mathcal{MS}_{d+1,a}(s, \lambda, \eta, \phi) \\ &\Rightarrow -zf'(z) \in \mathcal{MS}_{d,a}(s, \lambda, \eta, \phi) \\ &\Leftrightarrow f(z) \in \mathcal{MK}_{d,a}(s, \lambda, \eta, \phi) \end{aligned}$$

and

$$\begin{aligned} f(z) \in \mathcal{MK}_{d,a}(s, \lambda, \eta, \phi) &\Leftrightarrow -zf'(z) \in \mathcal{MS}_{d,a}(s, \lambda, \eta, \phi) \\ &\Rightarrow -zf'(z) \in \mathcal{MS}_{d,a}(s+1, \lambda, \eta, \phi) \\ &\Leftrightarrow f(z) \in \mathcal{MK}_{d,a}(s+1, \lambda, \eta, \phi), \end{aligned}$$

which evidently proves Theorem 2.4. □

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \leq 1; z \in U)$$

whereby both Theorems equates to

Corollary 2.5. *Let*

$$\frac{(A+1)}{(B+1)} < \min\left\{\frac{Re(d)+1-\eta}{1-\eta}, \frac{Re(a)+1-\eta}{1-\eta}\right\},$$
$$Re(d), Re(a) > 0; \quad 0 \leq \eta < 1; \quad -1 < B < A \leq 1.$$

Then

$$\mathcal{MS}_{d+1,a}(s, \lambda, \eta; A, B) \subset \mathcal{MS}_{d,a}(s, \lambda, \eta; A, B) \subset \mathcal{MS}_{d,a}(s+1, \lambda, \eta; A, B)$$

and

$$\mathcal{MK}_{d+1,a}(s, \lambda, \eta; A, B) \subset \mathcal{MK}_{d,a}(s, \lambda, \eta; A, B) \subset \mathcal{MK}_{d,a}(s+1, \lambda, \eta; A, B).$$

The following inclusion relation for the class $\mathcal{ML}_{d,a}(s, \lambda, \eta, \beta; \phi, \psi)$ was attained through from Lemma 2.2.

Theorem 2.6. *Let* $\phi, \psi \in \mathcal{N}$ *with*

$$\max_{z \in U} Re\{\phi(z)\} < \min\left\{\frac{Re(d)+1-\eta}{1-\eta}, \frac{Re(a)+1-\eta}{1-\eta}\right\},$$

for $Re(d), Re(a) > 0;\ \ 0 \le \eta < 1$. *Then*

$$\mathcal{ML}_{d+1,a}(s,\lambda,\eta,\beta;\phi,\psi) \subset \mathcal{ML}_{d,a}(s,\lambda,\eta,\beta;\phi,\psi) \subset \mathcal{ML}_{d,a}(s+1,\lambda,\eta,\beta;\phi,\psi).$$

Proof. The inclusion of Theorem 2.6 had been verified by going through the addition of $f \in \mathcal{ML}_{d+1,a}(s,\lambda,\eta,\beta;\phi,\psi)$ which formed the definition of $\mathcal{ML}_{d+1,a}$ $(s,\lambda,\eta,\beta;\phi,\psi)$, that indicate the function of $g \in \mathcal{MS}_{d+1,a}(s,\lambda,\eta,\phi)$ as

$$\frac{1}{1-\beta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d+1,c;z)f(z))'}{F^{s,a}_{\lambda,b}(d+1,c;z)g(z)}-\beta\right) \prec \psi(z)\quad (z \in U).$$

Now let

$$p(z) = \frac{1}{1-\beta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d,c;z)f(z))'}{F^{s,a}_{\lambda,b}(d,c;z)g(z)}-\beta\right), \tag{2.3}$$

where p is analytic in U with $p(0) = 1$. Using (1.14), we obtain

$$\frac{1}{1-\beta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d+1,c;z)f(z))'}{F^{s,a}_{\lambda,b}(d+1,c;z)g(z)}-\beta\right)$$
$$= \frac{1}{1-\beta}\left(\frac{\frac{z(F^{s,a}_{\lambda,b}(d,c;z)(-zf'(z)))'}{F^{s,a}_{\lambda,b}(d,c;z)g(z)}+(d+1)\frac{F^{s,a}_{\lambda,b}(d,c;z)(-zf'(z))}{F^{s,a}_{\lambda,b}(d,c;z)g(z)}}{\frac{z(F^{s,a}_{\lambda,b}(d,c;z)g(z))'}{F^{s,a}_{\lambda,b}(d,c;z)g(z)}+d+1}-\beta\right). \tag{2.4}$$

Theorem 2.3 indicates $g \in \mathcal{MS}_{d+1,a}(s,\lambda,\eta,\phi) \subset \mathcal{MS}_{d,a}(s,\lambda,\eta,\phi)$,

$$q(z) = \frac{1}{1-\eta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d,c;z)g(z))'}{F^{s,a}_{\lambda,b}(d,c;z)g(z)}-\eta\right), \tag{2.5}$$

was set, where $q \prec \phi$ in U had been assumed as $\phi \in \mathcal{N}$. By virtue of (2.3), (2.4) and (2.5), that

$$\frac{1}{1-\beta}\left(-\frac{z(F^{s,a}_{\lambda,b}(d+1,c;z)f(z))'}{F^{s,a}_{\lambda,b}(d+1,c;z)g(z)}-\beta\right)$$
$$= p(z) + \frac{zp'(z)}{-(1-\eta)q(z)+d+1-\eta} \prec \psi(z) \quad (z \in U) \tag{2.6}$$

had been discerned. Since $Re(d) > 0$ and $q \prec \phi$ in U with $\max_{z\in U} Re\{\phi(z)\} < \frac{Re(d)+1-\eta}{1-\eta}$, we have

$$Re\{-(1-\eta)q(z)+d+1-\eta\}\quad (z \in U).$$

Furthermore, taking

$$\omega(z) = \frac{1}{-(1-\eta)q(z)+d+1-\eta}$$

in (2.6), and applying Lemma 2.2, revealed that $p \prec \psi$ in U, and $f \in \mathcal{ML}_{d,a}(s,\lambda,\eta,\beta;\phi,\psi)$ attributed to the second inclusion through similar points detailed above with (1.13). Therefore, the proof of Theorem 2.6 is completed. □

3. Inclusion properties involving the integral operator F_μ

This section had been considered as the integral operator F_μ (see, e.g., [11]) defined by

$$F_\mu(f) := F_\mu(f)(z) = \frac{\mu}{z^{\mu+1}} \int_0^z t^\mu f(t)dt \quad (f \in \mathcal{N}; \mu > 0). \tag{3.1}$$

The definition of F_μ defined by (3.1) is given as the following:

$$z(F^{s,a}_{\lambda,b}(d,c;z)F_\mu(f)(z))' = \mu F^{s,a}_{\lambda,b}(d,c;z)f(z) - (\mu+1)F^{s,a}_{\lambda,b}(d,c;z)F_\mu(f)(z).$$

Theorem 3.1 discussed below, exhibited proof, similar to that of Theorem 2.3.

Theorem 3.1. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in U} Re\{\phi(z)\} < \frac{\mu+1-\eta}{1-\eta} \quad (\mu > 0;\ \ 0 \le \eta < 1).$$

If $f \in \mathcal{MS}_{d,a}(s,\lambda,\eta,\phi)$, then $F_\mu(f) \in \mathcal{MS}_{d,a}(s,\lambda,\eta,\phi)$.

Next, an inclusion property involving F was derived, which had been obtained by applying (1.15) and Theorem 3.1.

Theorem 3.2. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in U} Re\{\phi(z)\} < \frac{\mu+1-\eta}{1-\eta} \quad (\mu > 0;\ \ 0 \le \eta < 1).$$

If $f \in \mathcal{MK}_{d,a}(s,\lambda,\eta,\phi)$, then $F_\mu(f) \in \mathcal{MK}_{d,a}(s,\lambda,\eta,\phi)$.

From Theorems 3.1 and 3.2, we gathered

Corollary 3.3. *Let $\frac{1+A}{1+B} < \frac{\mu+1-\eta}{1-\eta}$ $(\mu > 0; -1 < B < A \le 1; 0 \le \eta < 1)$. If $f \in \mathcal{MS}_{d,a}(s,\lambda,\eta,A,B)$ (or $\mathcal{MK}_{d,a}(s,\lambda,\eta,A,B)$), then*

$$F_\mu(f) \in \mathcal{MS}_{d,a}(s,\lambda,\eta,A,B)\,(or \mathcal{MK}_{d,a}(s,\lambda,\eta,A,B)).$$

Finally, we obtain Theorem 3.4 as stated below, was collected by using the same techniques as in the proof of Theorem 2.6.

Theorem 3.4. *Let* $\phi, \psi \in \mathcal{N}$ *with*

$$\max_{z\in U} Re\{\phi(z)\} < \frac{\mu+1-\eta}{1-\eta} \quad (\mu > 0;\ \ 0 \le \eta < 1).$$

If $f \in \mathcal{ML}_{d,a}(s, \lambda, \eta, \beta; \phi, \psi)$, *then* $F_\mu(f) \in \mathcal{ML}_{d,a}(s, \lambda, \eta, \beta; \phi, \psi)$.

4. Conclusion

This paper defined a new operator for the class of meromorphic univalent functions via the principle of subordination. Some inclusion properties had been given. Many other results can be showed as [2], [3], [4], [7], [8].

Acknowledgement: The work here is supported by MOHE grant: FRGS/1/2016/STG06/UKM/01/1.

References

[1] E. Aqlan, J.M. Jahangiri, and S.R. Kulkarni, *Certain integral operators applied to meromorphic p-valent functions*, J. of Natural Geometry, **24**(1/2) (2003), 111–120.

[2] N.E. Cho and I.H. Kim, *Inclusion and argument properties for certain classes of meromorphic functions associated with a family of multiplier transformations*, Jour. Math. Anal. Appl., **300**(2) (2004), 505–520.

[3] N.E. Cho and I.H. Kim, *Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function*, Appl. Math. Comp., **187**(1) (2007), 115–121.

[4] N.E. Cho, O.S. Kwon and H.M. Srivastava, *Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations*, Inte. Tran. Spec. Func., **16**(8) (2005), 647–659.

[5] J.H.Choi, M.Saigo and H.M.Srivastava, *Some inclusion properties of a certain family of integral operators*, Jour. Math. Anal. Appl., **276**(1) (2002), 432–445.

[6] P.Eenigenburg, P.T. Mocanu, S.S. Miller and M.O. Reade, *On a briot-bouquet differential subordination*, In General Inequalities, **3** (1983), 339–348.

[7] R.M.El-Ashwah, *Inclusion relationships for certain subclasses of meromorphic functions defined by using the extended multiplier transformations*, ISRN Math. Anal., **2012** (2012), 1-13.

[8] R.M.El-Ashwah, *Inclusion properties regarding the meromorphic structure of Srivastava-Attiya operator*, Sout. Asia. Bull. Math., **38**(4) (2014), 501-512.

[9] F. Ghanim and M. Darus, *A class of meromorphically analytic functions related to Cho-Kwon-Srivastava operator with applications to generalized hypergemetric functions*, AIP Conference Proc., **1602**(1) (2014), 886–892.

[10] S. Kavitha, S. Sivasubramanian and R. Jayasankar, *Differential subordination and superordination results for Cho-Kwon-Srivastava operator*, Comp. and Math. with Appl., **64**(6) (2012,) 1789–1803.

[11] V. Kumar and S.L. Shukla, *Certain integrals for classes of p-valent meromorphic functions*, Bull. Aust. Math. Soc., **25**(01) (1982), 85–97.

[12] A.Y.Lashin, *On certain subclasses of meromorphic functions associated with certain integral operators*, Comp. and Math.with Appl., **59**(1) (2010), 524–531.

[13] R.J. Liber, M.S. Robertson and others, *Meromorphic close-to-convex functions*, Michigan Math. J., **8**(2) (1961), 167-175.

[14] Q.M. Luo and H.M. Srivastava, *Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials*, Jour. Math. Anal. Appl., **308**(1) (2005), 290–302.

[15] W. Ma and D. Minda, *An internal geometric characterization of strongly starlike functions*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **45** (1991), 89–97.

[16] S.S. Miller and P.T. Mocanu, Diff. subordinations: theory and applications, CRC Press, (2000).

[17] S.S. Miller, P.T. Mocanu et al, *Differential subordinations and univalent functions*, Michigan Math. J., **28**(2) (1981), 157–172.

[18] T. Panigrahi, *A subclass of multivalent meromorphic functions associated withitertions of the Cho-Kwon-Srivastava operator*, Palestine Jour. of Math., **4**(1) (2015). 57-64.

[19] K. Piejko and J. Sokol, *subclass of meromorphic functions associated with the Cho-Kwon-Srivastava operator*, Jour. Math. Anal. Appl., **337**(2) (2008), 1261–1266.

[20] P. Sharma, J.K. Prajapat and R.K. Raina, *Certain subordination results involving a generalized multiplier transformation operator*, Jour. Clas. Anal., **2**(1) (2013), 85–106.

[21] H.M. Srivastava and J. Choi, *Series associated with the zeta and related functions*, Springer Science & Business Media, **530** (2001), 1-92.

[22] H.M. Srivastava, S. Gaboury and F. Ghanim, *Certain subclasses of meromorphically univalent functions defined by a linear operator associated with the λ-generalized Hurwitz–Lerch zeta function*, Inte. Tran. Special Func., **26**(4) (2015), 258–272.

[23] Z.G. Wang, R. Aghalary, M. Darus and R.W. Ibrahim, *Some properties of certain multivalent analytic functions involving the Cho-Kwon-Srivastava operator*, Math. Comp. Mode., **49**(9) (2009), 1969–1984.