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ITERATIVE ALGORITHM AND CONVERGENCE ANALYSIS FOR SYSTEMS OF VARIATIONAL INEQUALITIES

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. The aim of this paper is to study several classes of problems of systems of variational inequalities and variational inclusions in Hilbert spaces and Banach spaces, respectively. We obtain several existence theorems for systems of variational inequalities and variational inclusions and give the corresponding iterative algorithms together with the convergence analysis. The above results are based on the method of three-step projection and the resolvent operator technique. Our works are valuable supplements to [4, 6, 7, 12, 14-16] and relevant literatures.

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1. Introduction

In recent decades, studies on problems of variational inequality have been extended to studies on problems of systems of variational inequalities by many scholars and handful of meaningful results have been obtained(see, [5],[11]). By using the two-step iterative algorithms, Nie et al. in [8] have obtained the existence of solutions to systems of variational inequalities involving strongly monotone mappings. In [15] the authors have considered an algorithm for a generalized system for relaxed coercive nonlinear inequalities involving three different operators in Hilbert spaces by the convergence of projection methods. By using three-step projection methods the authors in [7], [16] have considered the approximation solvability and convergence of a generalized system for relaxed cocoercive variational inequalities with error estimate in Hilbert spaces. In [12] the authors have introduced the two-step projection methods and have proved that the convergence of the algorithm for the case that the mapping T satisfies relaxed-(γ, r)-cocoercity. These convergence results that obtained generalize and improve results in [9]. Based on the resolvent operator technique, a system of variational-like inclusion problems in the framework of Hilbert spaces is discussed in [4] and the strong convergence theorems for the solution of the systems of variational-like inclusion are obtained. Applying the property of the sunny nonexpansive retraction mapping Q_k, the authors in [14] have introduced the implicit iteration methods. In [6] the authors have introduced a new class of generalized accretive operators named (H, η)-accretive in Banach spaces including the concept of resolvent operators associated with m-accretive operators as special cases, and have proved the existence and uniqueness of solutions for a new system of variational inclusions in terms of the new resolvent operator technique. Based on the studies mentioned above, we investigate in the present paper several classes of problems of systems of variational inequalities and systems of variational inclusions in Hilbert spaces and Banach spaces, respectively. By using the projection operator methods and the resolvent operator methods, etc., we obtain the existence of solutions and give the related iterated algorithm as well as the analysis of the convergence of the solutions. Results obtained in this paper can be viewed as refinements and improvements of the previously known results in [4, 6, 7, 12, 14-16].

This paper is mainly concerned with the investigation on the following four classes of problems of systems of variational inequalities.

Let H be a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

Problem A. Let K be a nonempty closed convex subset of a Hilbert space H and $T_i : K \times K \times K \to H (i = 1, 2, 3)$ be relaxed coercive mappings.

Generalized relaxed coercive problems of systems of variational inequalities, denoted by **Problem A**, is to find x^*, y^*, $z^* \in K$ satisfying

$$\langle \rho T_1(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K, \rho > 0,$$

$$\langle \eta T_2(z^*, x^*, y^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in K, \eta > 0,$$

$$\langle \delta T_3(x^*, y^*, z^*) + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in K, \delta > 0.$$

Problem B. Let $T_i : H \times H \to H (i = 1, 2, 3)$ be mappings and $M_i : H \to 2^H (i = 1, 2, 3)$ be set-valued mappings. Problems of systems of nonlinear variational inclusions, denoted by **Problem B**, is to find $x^*, y^*, z^* \in H, \rho_1, \rho_2, \rho_3 > 0$ satisfying

$$y^* \in x^* + \rho_1 (T_1(y^*, x^*) + M_1(x^*)),$$

$$z^* \in y^* + \rho_2 (T_2(z^*, y^*) + M_2(y^*)),$$

$$x^* \in z^* + \rho_3 (T_3(x^*, z^*) + M_3(z^*)).$$

Problem C. Let K be a nonempty closed convex subset of a real $2-$uniform smooth Banach space B, T_1, T_2, T_3, f, g, h be the given nonlinear mappings. Problems of systems of generalized variational inequalities, denoted by **Problem C**, is to find $x^*, y^*, z^* \in K$ satisfying

(a) $\langle \rho_1 T_1(y^*, z^*, x^*) + x^* - f(y^*), J(f(x) - x^*) \rangle \geq 0,$

(b) $\langle \rho_2 T_2(z^*, x^*, y^*) + y^* - g(z^*), J(g(x) - y^*) \rangle \geq 0,$

(c) $\langle \rho_3 T_3(x^*, y^*, z^*) + z^* - h(x^*), J(h(x) - z^*) \rangle \geq 0,$

for all $x \in B$, $f(x) \in K, \rho_1 > 0$, $g(x) \in K, \rho_2 > 0$, $h(x) \in K, \rho_3 > 0$.

Problem D. Let X_1, X_2 be two q-uniformly smooth Banach spaces, $F : X_1 \times X_2 \times X_3 \to X_1$, $G : X_1 \times X_2 \times X_3 \to X_2$, $H : X_1 \times X_2 \times X_3 \to X_3$, $H_1 : X_1 \to X_1$, $H_2 : X_2 \to X_2$, $H_3 : X_3 \to X_3$ and $\eta : X \times X \times X \to X^*$ be mappings, $M : X_1 \to 2^{X_1}$ be (H_1, η)-accretive operator, $N : X_2 \to 2^{X_2}$ be (H_2, η)-accretive operator and $T : X_3 \to 2^{X_3}$ be (H_3, η)-accretive operator. Problems of systems of variational inclusions involving (H, η)-accretive operators, denoted by **Problem D**, is to find $(a, b, c) \in X_1 \times X_2 \times X_3$ satisfying

$$0 \in F(a, b, c) + M(a),$$

$$0 \in G(a, b, c) + N(b),$$

$$0 \in H(a, b, c) + T(c).$$

2. Prerequisites and Preliminaries

For the reader's convenience we present in this section the definitions, the lemmas and the algorithms that will be used in the sequel. We present first the basic definitions of the paper in the following.

Definition 2.1. ([2]) Let H be a Hilbert space and $T : H \times H \to H$ be a two-variable single-valued mapping. T is said to be

(1) relaxed $(a, r)-$cocoercive, if there exist constants $a, r > 0$ such that

$$\langle T(x,u) - T(y,v), x - y\rangle \geq (-a)\|T(x,u) - T(y,v)\|^2 + r\|x - y\|,$$

for all $x, y, u, v \in H$.

(2) μ-Lipschitz continuous in the first variable, if there exists a constant $\mu > 0$ such that for all $x, y \in H$,

$$\|T(x,u) - T(y,v)\| \leq \mu\|x - y\|, \quad \forall u, v \in H.$$

Definition 2.2. ([15]) Let K be a closed convex subset of a Hilbert space H and $T : K \times K \times K \to H$ be a single-valued mapping.

(1) T is said to be relaxed (γ, r)-coercive if there exist constants $\gamma, r > 0$ such that, for all $x, x' \in K$,

$$\langle T(x,y,z) - T(x',y',z'), x - x'\rangle$$
$$\geq (-\gamma)\|T(x,y,z) - T(x',y',z')\|^2 + r\|x - x'\|^2, \quad \forall y, y', z, z' \in K.$$

(2) T is said to be μ-Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that, for all $x, x' \in K$,

$$\|T(x,y,z) - T(x',y',z')\| \leq \mu\|x - x'\|, \quad \forall y, y', z, z' \in K.$$

We can define similarly the μ-Lipschitz continuity of T in the second variable and the third variable, respectively.

Definition 2.3. ([4]) Let H be a real Hilbert space. A single-valued operator $\eta : H \times H \to H$ is said to be

(1) monotone, if

$$\langle \eta(u,v), u - v\rangle \geq 0, \quad \forall u, v \in H;$$

(2) strictly monotone, if η is monotone and $\langle \eta(u,v), u-v\rangle = 0$ if and only if $u = v$;

(3) δ-strongly monotone, if there exists a constant $\delta > 0$ such that

$$\langle \eta(u,v), u - v\rangle \geq \delta\|u - v\|^2, \quad \forall u, v \in H;$$

(4) τ-Lipschitz, if there exists a constant $\tau > 0$ such that

$$\|\eta(u,v)\| \leq \tau\|u-v\|, \quad \forall u,v \in H,$$

where τ is called the Lipschitz constant.

Definition 2.4. ([4]) Let H be a real Hilbert space and $\eta : H \times H \to H$ be a single-valued operator. A set-valued operator $M : H \to 2^H$ is said to be

(1) η-monotone, if

$$\langle \eta(u,v), x-y \rangle \geq 0, \quad \forall u,v \in H, x \in Mu, y \in Mv;$$

(2) strictly η-monotone, if M is η-monotone and $\langle \eta(u,v), x-y \rangle = 0$ if and only if $u = v$;

(3) (η,δ)-strongly monotone, if there exists a constant $\delta > 0$ such that

$$\langle \eta(u,v), x-y \rangle \geq \delta\|u-v\|^2, \quad \forall u,v \in H, x \in Mu, y \in Mv;$$

(4) maximal η-monotone, if M is $\eta-$monotone and $(I+\rho M)(H) = H, \rho > 0$.

Definition 2.5. ([6],[14]) Let X be a real Banach space with dual space X^*, $\langle \cdot,\cdot \rangle$ the dual pair between X and X^*. Let K be a nonempty closed convex subset of X.

(1) The modulus of smoothness of X is the function $\rho_X : [0,+\infty) \to [0,+\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

X is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0;$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad q > 1.$$

(2) The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant.

(3) A mapping $Q : X \to K$ is said to be sunny, provided that

$$Q(Qx + t(x-Qx)) = Qx, \quad \forall x \in X, \quad t \geq 0.$$

Q is said to be retractive if for all $x \in K$, $Qx = x$.

Remark 2.1. In the sequel, unless otherwise specified, we will always assume that X is a real Banach space such that J_q is single-valued. Note that J_q is single-valued if X is uniformly smooth and that q-uniformly smooth always implies uniformly smooth when $q > 1$. The sunny nonexpansive retraction mapping $Q : X \to K$ is uniformly definite provided that X is smooth.

Definition 2.6. ([16]) A mapping $T : X \times X \times X \to X$ is called (γ, r)-relaxed cocoercive, if there exist constants $\gamma > 0, r > 0$ such that for all $x_1, x_2, y, z \in X$,

$$\langle T(x_1, y, z) – T(x_2, y, z), J(x_1 – x_2)\rangle \geq -\gamma\|T(x_1, y, z) – T(x_2, y, z)\|^2 + r\|x_1 - x_2\|^2.$$

Definition 2.7. ([6]) Let $T, H : X \to X$ be two single-valued operators. The operator T is said to be

(1) accretive if $\langle Tx - Ty, J_q(x - y)\rangle \geq 0, \quad \forall x, y \in X$;

(2) strictly accretive if T is accretive and $\langle Tx - Ty, J_q(x - y)\rangle = 0$ if and only if $x = y$;

(3) r-strongly accretive if there exists some constant $r > 0$ such that

$$\langle Tx - Ty, J_q(x - y)\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X;$$

(4) γ-strongly accretive with respect to H if there exists some constant $\gamma > 0$ such that

$$\langle Tx - Ty, J_q(H(x) - H(y))\rangle \geq \gamma\|x - y\|^q, \quad \forall x, y \in X;$$

(5) Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \quad \forall x, y \in X.$$

Definition 2.8. ([6]) A single-valued operator $\eta : X \times X \to X^*$ is said to be Lipschitz continuous if there exists a constant $\tau > 0$, such that

$$\|\eta(u, v)\| \leq \tau\|u - v\|, \quad \forall u, v \in X.$$

Definition 2.9. ([6]) Let $M : X \to 2^X$ be a multi-valued operator, $H : X \to X$ and $\eta : X \times X \to X^*$ be two single-valued operators, and I be the identity operator on X. M is said to be

(1) accretive if $\langle x - y, J_q(u - v)\rangle \geq 0, \quad \forall u, v \in X, x \in Mu, y \in Mv$;

(2) η-accretive if $\langle x - y, \eta(u - v)\rangle \geq 0, \quad \forall u, v \in X, x \in Mu, y \in Mv$;

(3) strictly η-accretive if M is η-accretive and $\langle x - y, \eta(u - v)\rangle = 0$ if and only if $u = v$;

(4) strongly η-accretive if there exists some constant $r > 0$ such that

$$\langle x - y, \eta(u - v)\rangle \geq r\|u - v\|^2, \quad \forall u, v \in X, x \in Mu, y \in Mv;$$

(5) m-accretive if M is accretive and $(I + \lambda M)(X) = X$, for all $\lambda > 0$;

(6) (m, η)-accretive if M is η-accretive and $(I + \lambda M)(X) = X$, for all $\lambda > 0$;
(7) H-accretive if M is accretive and $(H + \lambda M)(X) = X$, for all $\lambda > 0$;
(8) (H, η)-accretive if M is η-accretive and $(H + \lambda M)(X) = X$, for all $\lambda > 0$.

Definition 2.10. ([6]) Let X be a real Banach space and $\eta : X \times X \to X^*$ be a single-valued operator. Let $H : X \to X$ be a strictly η-accretive operator and $M : X \to 2^X$ be a multi-valued (H, η)-accretive operator. The resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ associated with H, η and M is defined by

$$R^{H,\eta}_{M,\lambda}(u) = (H + \lambda M)^{-1}(u), \quad \forall u \in X.$$

Remark 2.2. Note that $(H + \lambda M)^{-1}$ is single-valued, where λ is a positive constant(see [6]).

The following lemmas are used in this paper.

Lemma 2.1. ([2]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers, satisfying*

$$a_{n+1} \leq (1 - \Omega_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is a nonnegative integer, $\Omega_n \in (0, 1)$ with $\sum_{n=0}^{\infty} \Omega_n = \infty, b_n = o(\Omega_n), \sum_{n=0}^{\infty} c_n < \infty$, then $a_n \to 0$ as $n \to \infty$.

Lemma 2.2. ([10]) *Let $\eta : H \times H \to H$ be a strictly monotone operator and $M : H \to 2^H$ a maximal η-monotone multi-valued operator. Then for any $\rho > 0$, the operator $(I + \rho M)^{-1}$ is single-valued.*

Lemma 2.3. ([10]) *Let $\eta : H \times H \to H$ be a δ-strongly and strictly monotone and τ-Lipschitz continuous multi-valued operator, $M : H \to 2^H$ a maximal η-monotone multi-valued operator, denote by $J^M_\rho = (I + \rho M)^{-1}$ the resolvent operator of M. Then J^M_ρ is $(\frac{\tau}{\delta})$-Lipschitz continuous, that is,*

$$\|J^M_\rho(u) - J^M_\rho(v)\| \leq \frac{\tau}{\delta}\|u - v\|, \quad \forall u, v \in H, (\rho > 0).$$

Lemma 2.4. ([1],[3]) *Let B be a smooth Banach space and K a nonempty closed convex subset of B. Assume that $Q_k : B \to K$ is kernel preserving and J is a normal symmtric mapping on B. Then the following conclusions are equivalent:*

(a) *Q_k is sunny nonexpansive;*
(b) *$\|Q_kx - Q_ky\|^2 \leq \langle x - y, J(Q_kx - Q_ky)\rangle$, for any $x, y \in B$;*
(c) *$\langle x - Q_kx, J(Q_kx - y)\rangle \geq 0$, for any $y \in K$.*

Lemma 2.5. ([13]) *Let B be a real q-uniformly smooth Banach space with $q > 1$. Then there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q x\rangle + c_q\|y\|^q, \quad \forall x, y \in B.$$

Particularly, if B is a real 2-uniformly smooth Banach space, then there exists an optimal smooth constant $c_2 > 0$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J_2 x\rangle + c_2\|y\|^2, \quad \forall x, y \in B.$$

Lemma 2.6. ([6]) *Let $\{c_n\}$ and $\{k_n\}$ be two sequences of nonnegative real numbers that satisfy the following conditions:*

(1) $0 \leq k_n < 1, n = 0, 1, 2, \cdots$ *and* $\limsup_{n\to\infty} k_n < 1$;
(2) $c_{n+1} \leq k_n c_n, n = 0, 1, 2, \cdots$.

Then $\{c_n\}$ converges to 0 as $n \to \infty$.

Lemma 2.7. ([6]) *Let X be a real Banach space and $\eta : X \times X \to X^*$ be a Lipschitz continuous operator with constant $\sigma > 0$. Let $H : X \to X$ be a strongly η-accretive operator with constant $\gamma > 0$, $M : X \to 2^X$ be a multi-valued (H, η)-accretive operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ is Lipschitz continuous with constant $\frac{\sigma}{\gamma}$, that is,*

$$\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \leq \frac{\sigma}{\gamma}\|u - v\|, \quad \forall u, v \in X.$$

We can easily show the following.

Lemma 2.8. *$(x^*, y^*, z^*) \in H \times H \times H$ is a solution to Problem B if and only if*

$$x^* = J^{M_1}_{\rho_1}(y^* - \rho_1 T_1(y^*, x^*)),$$

$$y^* = J^{M_2}_{\rho_2}(z^* - \rho_2 T_2(z^*, y^*)),$$

$$z^* = J^{M_3}_{\rho_3}(x^* - \rho_3 T_3(x^*, z^*)).$$

Lemma 2.9. *Let B be a real smooth Banach space and $T_i : B \times B \times B \to B (i = 1, 2, 3)$, ρ_1, ρ_2, ρ_3 be any positive constants. Then $x^*, y^*, z^* \in K$ is a solution to the following operator equations in $B \times B \times B$,*

$$x^* = Q_k[f(y^*) - \rho_1 T_1(y^*, z^*, x^*)], \quad \rho_1 > 0,$$

$$y^* = Q_k[g(z^*) - \rho_2 T_2(z^*, x^*, y^*)], \quad \rho_2 > 0,$$

$$z^* = Q_k[h(x^*) - \rho_3 T_3(x^*, y^*, z^*)], \quad \rho_3 > 0.$$

Proof. The variational inequality (a) can be written as

$$\langle [f(y^*) - \rho_1 T_1(y^*, z^*, x^*)] - x^*, J(x^* - f(x))\rangle \geq 0, \quad \forall x \in B, f(x) \in K, \rho_1 > 0.$$

By Lemma 2.4-(c) of the properties of Q_k, the above formula is equivalent to

$$x^* = Q_k[f(y^*) - \rho_1 T_1(y^*, z^*, x^*)], \quad \rho_1 > 0.$$

Similarly, the variational inequality (b) is equivalent to

$$y^* = Q_k[g(z^*) - \rho_2 T_2(z^*, x^*, y^*)], \quad \rho_2 > 0,$$

and the variational inequality (c) is equivalent to

$$z^* = Q_k[h(x^*) - \rho_3 T_3(x^*, y^*, z^*)], \quad \rho_3 > 0.$$

This completes the proof. □

Lemma 2.10. *Let $\eta : X \times X \times X \to X^*$ be a single-valued mapping and $H_i : X_i \to X_i (i = 1, 2, 3)$ be strictly $\eta-$accretive operators. Assume that $M : X_1 \to 2^{X_1}$ is $(H_1, \eta)-$accretive, $N : X_2 \to 2^{X_2}$ is $(H_2, \eta)-$accretive and $T : X_3 \to 2^{X_3}$ is $(H_3, \eta)-$accretive. Then for any given $(a, b, c) \in X_1 \times X_2 \times X_3$, (a, b, c) is a solution to Problem D if and only if*

$$a = R^{H_1\eta}_{M\rho_1}[H_1(a) - \rho_1 F(a, b, c)],$$

$$b = R^{H_2\eta}_{N\rho_2}[H_2(b) - \rho_2 G(a, b, c)],$$

$$c = R^{H_3\eta}_{T\rho_3}[H_3(c) - \rho_3 H(a, b, c)].$$

The proof of Lemma 2.10 can be proceed by direct logic reasoning from Definition 2.10 and we omit the details.

We present finally in this section the algorithms that will be used in the proofs of our main results.

Algorithm 2.1. For any $x_0, y_0, z_0 \in K$, calculating the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$x_{n+1} = (1 - \alpha_n - r_n)f(x_n) + \alpha_n P_k[y_n - \rho T_1(y_n, z_n, x_n)] + r_n u_n,$$

$$y_n = (1 - \beta_n - \delta_n)f(x_n) + \beta_n P_k[z_n - \eta T_2(z_n, x_n, y_n)] + \delta_n v_n,$$

$$z_n = (1 - a_n - \lambda_n)f(x_n) + a_n P_k[x_n - \delta T_3(x_n, y_n, z_n)] + \lambda_n w_n,$$

where $P_k : H \to K$ is the projection mapping, $f : K \to K$ is a nonexpansive mapping, $0 \leq \alpha_n, \beta_n, a_n, r_n, \delta_n, \lambda_n \leq 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are the bounded sequences in K which is introduced for the purpose of escalating the accuracy of the calculations.

Algorithm 2.2. For any fixed $x_0, y_0, z_0 \in H$, we present the following iterative sequences

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)f(x_n) + \alpha_n J_{\rho_1}^{M_1}(y_n - \rho_1 T_1(y_n, x_n)), \\ y_n &= (1-\beta_n)f(x_n) + \beta_n J_{\rho_2}^{M_2}(z_n - \rho_2 T_2(z_n, y_n)), \\ z_n &= (1-\delta_n)f(x_n) + \delta_n J_{\rho_3}^{M_3}(x_n - \rho_3 T_3(x_n, z_n)), \end{aligned}$$

where $J_{\rho_1}^{M_1}, J_{\rho_2}^{M_2}, J_{\rho_3}^{M_3}$ are the resolvent operators of M_1, M_2, M_3, respectively, $f : H \to H$ is a nonexpansive mapping, ρ_1, ρ_2, ρ_3 are positive constants and $0 \le \alpha_n, \beta_n \le 1$.

Algorithm 2.3. For any $x_0, y_0, z_0 \in K$, calculating the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Q_k[f(y_n) - \rho_1 T_1(y_n, z_n, x_n)], \quad \rho_1 > 0, \\ y_{n+1} &= Q_k[g(z_{n+1}) - \rho_2 T_2(z_{n+1}, x_{n+1}, y_{n+1})], \quad \rho_2 > 0, \\ z_{n+1} &= Q_k[h(x_{n+1}) - \rho_3 T_3(x_{n+1}, y_{n+1}, z_{n+1})], \quad \rho_3 > 0, \end{aligned}$$

where Q_k is a sunny nonexpansive retraction mapping, ρ_1, ρ_2, ρ_3 are positive constants and $0 \le \alpha_n \le 1$.

Algorithm 2.4. Let $\eta, H_1, H_2, H_3, M, N, T, F, G$ and H be the same as Problem D. For any $(a_0, b_0, c_0) \in X_1 \times X_2 \times X_3$, the iterative sequence $\{(a_n, b_n, c_n)\}$ is given by the following iterative formula:

$$\begin{aligned} a_{n+1} &= \beta_n a_n + (1-\beta_n) R_{M\rho_1}^{H_1 \eta}[H_1(a_n) - \rho_1 F(a_n, b_n, c_n)], \\ b_{n+1} &= \beta_n b_n + (1-\beta_n) R_{N\rho_2}^{H_2 \eta}[H_2(b_n) - \rho_2 G(a_n, b_n, c_n)], \\ c_{n+1} &= \beta_n c_n + (1-\beta_n) R_{T\rho_3}^{H_3 \eta}[H_3(c_n) - \rho_3 H(a_n, b_n, c_n)], \end{aligned}$$

where $0 \le \beta_n < 1$ and $\limsup_{n\to\infty} \beta_n < 1$.

3. Syetems of variational inequalities in Hilbert spaces

Theorem 3.1. *Let H be a real Hilbert space, K a nonempty closed convex subset of H, $T_i : K \times K \times K \to H$ relaxed $(\gamma_i, r_i)-$coercive and μ_i-Lipschitz continuous in the first variable$(i = 1, 2, 3)$, $f : K \to K$ a nonexpansive mapping. Assume that $(x^*, y^*, z^*) \in K$ is the solution to Problem A, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences defined by Algorithm 2.1, where $\{u_n\}$, $\{v_n\}$ and $\{c_n\}$ are bounded sequences in K which is introduced for the purpose of escalating the accuracy of the calculations. Assume further that the sequences $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{r_n\}, \{\delta_n\}, \{\lambda_n\}$ which lie in $[0, 1]$ satisfying*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \rho, \eta, \delta < \min\left\{\frac{2(r_1-\gamma_1\mu_1^2)}{\mu_1^2}, \frac{2(r_2-\gamma_2\mu_2^2)}{\mu_2^2}, \frac{2(r_3-\gamma_3\mu_3^2)}{\mu_3^2}\right\}$;
(iii) $r_1 > \gamma_1\mu_1^2,\ \ r_2 > \gamma_2\mu_2^2,\ \ r_3 > \gamma_3\mu_3^2$;
(iv) $\lim_{n\to\infty}\|u_n\| = 0,\ \ \lim_{n\to\infty}\|v_n\| = 0,\ \ \lim_{n\to\infty}\|w_n\| = 0.$

Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to x^*, y^* and z^*, respectively.

Proof. Since $x^*, y^*, z^* \in K$ are the solutions to Problem A, we have

$$x^* = P_K[y^* - \rho T_1(y^*, z^*, x^*)], \qquad \rho > 0,$$

$$y^* = P_K[z^* - \eta T_2(z^*, x^*, y^*)], \qquad \eta > 0,$$

$$z^* = P_K[x^* - \delta T_3(x^*, y^*, z^*)], \qquad \delta > 0.$$

By Algorithm 2.1 we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|(1-\alpha_n - r_n)f(x_n) \\
&\quad + \alpha_n P_K[y_n - \rho T_1(y_n, z_n, x_n)] \\
&\quad + r_n u_n - (1-\alpha_n - r_n)f(x^*) - \alpha_n P_K[y^* - \rho T_1(y^*, z^*, x^*)] - r_n x^*\| \\
&\leq (1-\alpha_n - r_n)\|f(x_n) - f(x^*)\| \\
&\quad + \alpha_n\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| + r_n\|u_n - x^*\| \\
&\leq (1-\alpha_n - r_n)\|x_n - x^*\| \\
&\quad + \alpha_n\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| + r_n\|u_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| \\
&\quad + \alpha_n\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| + r_n\|u_n - x^*\|. \qquad (3.1)
\end{aligned}$$

Observe that T_1 is relaxed $(\gamma_1, r_1)-$coercive from the above and μ_1-Lipschitz continuous in the first variable, we then have

$$\begin{aligned}
&\|y_n - y^* - \rho[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|^2 \\
&= \|y_n - y^*\|^2 + \rho^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\
&\quad - 2\rho\langle T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*), y_n - y^*\rangle \\
&\leq \|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 + 2\rho\gamma_1\|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\
&\quad - 2\rho r_1\|y_n - y^*\|^2 \\
&\leq \|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 + 2\rho\gamma_1\mu_1^2\|y_n - y^*\|^2 \\
&\quad - 2\rho r_1\|y_n - y^*\|^2 \\
&= (1 + \rho^2\mu_1^2 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1)\|y_n - y^*\|^2. \qquad (3.2)
\end{aligned}$$

Letting

$$L = \max\{\sup_{n\geq 0}\|u_n - x^*\|, \sup_{n\geq 0}\|v_n - x^*\|, \sup_{n\geq 0}\|w_n - x^*\|, \sup_{n\geq 0}\|x^* - z^*\|, \sup_{n\geq 0}\|x^* - y^*\|\},$$

combining (3.1) and (3.2) we then deduce

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| + r_nL, \tag{3.3}$$

where $\theta_1 = \sqrt{1+\rho^2\mu_1^2 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1} < 1$. Again by Algorithm 2.1, we have

$$\begin{aligned}
&\|y_n - y^*\| \\
&= \|(1-\beta_n-\delta_n)f(x_n) + \beta_n P_K[z_n - \eta T_2(z_n, x_n, y_n)] \\
&\quad + \delta_n v_n - (1-\beta_n-\delta_n)f(y^*) - \beta_n P_K[z^* - \eta T_2(z^*, x^*, y^*)] - \delta_n y^*\| \\
&\leq (1-\beta_n-\delta_n)\|f(x_n) - f(y^*)\| \\
&\quad + \beta_n\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| + \delta_n\|v_n - y^*\| \\
&\leq (1-\beta_n-\delta_n)\|x_n - y^*\| \\
&\quad + \beta_n\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| + \delta_n\|v_n - y^*\| \\
&\leq (1-\beta_n)\|x_n - y^*\| \\
&\quad + \beta_n\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\| + \delta_n\|v_n - y^*\|.
\end{aligned} \tag{3.4}$$

Since T_2 is relaxed $(\gamma_2, r_2)-$coercive from above and μ_2-Lipschitz continuous in the first variable, we then have

$$\begin{aligned}
&\|z_n - z^* - \eta[T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)]\|^2 \\
&= \|z_n - z^*\|^2 + \eta^2\|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\
&\quad - 2\eta\langle T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*), z_n - z^*\rangle \\
&\leq \|z_n - z^*\|^2 + \eta^2\mu_2^2\|z_n - z^*\|^2 \\
&\quad + 2\eta\gamma_2\|T_2(z_n, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 - 2\eta r_2\|z_n - z^*\|^2 \\
&\leq \|z_n - z^*\|^2 + \eta^2\mu_2^2\|z_n - z^*\|^2 \\
&\quad + 2\eta\gamma_2\mu_2^2\|z_n - z^*\|^2 - 2\eta r_2\|z_n - z^*\|^2 \\
&= (1+\eta^2\mu_2^2 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2)\|z_n - z^*\|^2.
\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) we have

$$\|y_n - y^*\| \leq (1-\beta_n)\|x_n - y^*\| + \beta_n\theta_2\|z_n - z^*\| + \delta_nL, \tag{3.6}$$

where $\theta_2 = \sqrt{1+\eta^2\mu_2^2+2\eta\gamma_2\mu_2^2-2\eta r_2} < 1$. By Algorithm 2.1 once more we still have

$$
\begin{aligned}
&\|z_n - z^*\| \\
&= \|(1-a_n-\lambda_n)f(x_n) + a_nP_K[x_n - \delta T_3(x_n,y_n,z_n)] \\
&\quad + \lambda_n w_n - (1-a_n-\lambda_n)f(z^*) - a_nP_K[x^* - \delta T_3(x^*,y^*,z^*)] - \lambda_n z^*\| \\
&\le (1-a_n-\lambda_n)\|f(x_n)-f(z^*)\| \\
&\quad + a_n\|x_n - z^* - \delta[T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*)]\| + \lambda_n\|w_n - z^*\| \\
&\le (1-a_n-\lambda_n)\|x_n - z^*\| \\
&\quad + a_n\|x_n - z^* - \delta[T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*)]\| + \lambda_n\|w_n - z^*\| \\
&\le (1-a_n)\|x_n - z^*\| \\
&\quad + a_n\|x_n - z^* - \delta[T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*)]\| + \lambda_n\|w_n - z^*\|. \qquad (3.7)
\end{aligned}
$$

Since T_3 is relaxed $(\gamma_3, r_3)-$coercive from above and μ_3-Lipschitz continuous in the first variable, we then have

$$
\begin{aligned}
&\|x_n - x^* - \delta[T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*)]\|^2 \\
&= \|x_n - x^*\|^2 + \delta^2\|T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*)\|^2 \\
&\quad - 2\delta\langle T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*), x_n - x^*\rangle \\
&\le \|x_n - x^*\|^2 + \delta^2\mu_3^2\|x_n - x^*\|^2 \\
&\quad + 2\delta\gamma_3\|T_3(x_n,y_n,z_n) - T_3(x^*,y^*,z^*)\|^2 - 2\delta r_3\|x_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 + \delta^2\mu_3^2\|x_n - x^*\|^2 \\
&\quad + 2\delta\gamma_3\mu_3^2\|x_n - x^*\|^2 - 2\delta r_3\|x_n - x^*\|^2 \\
&= (1+\delta^2\mu_3^2+2\delta\gamma_3\mu_3^2-2\delta r_3)\|x_n - x^*\|^2. \qquad (3.8)
\end{aligned}
$$

Combining (3.7) and (3.8) we have

$$\|z_n - z^*\| \le (1-a_n)\|x_n - z^*\| + a_n\theta_3\|x_n - x^*\| + \lambda_n L, \qquad (3.9)$$

where $\theta_3 = \sqrt{1+\delta^2\mu_3^2+2\delta\gamma_3\mu_3^2-2\delta r_3} < 1$. By (ii), we have

$$
\begin{aligned}
&\|z_n - z^*\| \\
&\le (1-a_n)\|x_n - z^*\| + a_n\theta_3\|x_n - x^*\| + \lambda_n L \\
&\le (1-a_n)\|x_n - z^*\| + a_n\|x_n - x^*\| + \lambda_n L \\
&\le (1-a_n)\|x_n - x^*\| + (1-a_n)\|x^* - z^*\| + a_n\|x_n - x^*\| + \lambda_n L \\
&\le \|x_n - x^*\| + (1-a_n+\lambda_n)L. \qquad (3.10)
\end{aligned}
$$

Combining (3.10) and (3.6) we then have

$$\begin{aligned}
&\|y_n - y^*\| \\
&\leq (1-\beta_n)\|x_n - y^*\| + \beta_n\theta_2\|z_n - z^*\| + \delta_n L \\
&\leq (1-\beta_n)\|x_n - y^*\| + \beta_n\|z_n - z^*\| + \delta_n L \\
&\leq (1-\beta_n)\|x_n - y^*\| + \beta_n\{\|x_n - x^*\| + (1-a_n+\lambda_n)L\} + \delta_n L \\
&\leq (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\|x^* - y^*\| \\
&\quad + \beta_n\|x_n - x^*\| + \beta_n(1-a_n+\lambda_n)L + \delta_n L \\
&\leq \|x_n - x^*\| + (1-\beta_n a_n + \beta_n\lambda_n + \delta_n)L. \qquad (3.11)
\end{aligned}$$

Combining (3.10) and (3.11) we then have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| + r_n L \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\{\|x_n - x^*\| + (1-\beta_n a_n + \beta_n\lambda_n + \delta_n)L\} + r_n L \\
&\leq (1-\alpha_n + \alpha_n\theta_1)\|x_n - x^*\| + [\alpha_n\theta_1(1-\beta_n a_n + \beta_n\lambda_n + \delta_n) + r_n]L \\
&\leq [1-\alpha_n(1-\theta_1)]\|x_n - x^*\| + [\alpha_n\theta_1(1-\beta_n a_n + \beta_n\lambda_n + \delta_n) + r_n]L, \quad (3.12)
\end{aligned}$$

Let $\Omega_n = \alpha_n(1-\theta_1)$; $a_n = \|x_n - x^*\|$; $b_n = 0$; $c_n = [\alpha_n\theta_1(1-\beta_n a_n + \beta_n\lambda_n + \delta_n) + r_n]L < \infty$, combining (3.12) and Lemma 2.1 we then deduce $\|x_n - x^*\| \to 0$ as $n \to \infty$, that is $x_n \to x^*(n \to \infty)$.

With the same arguments as above we could deduce $y_n \to y^*(n \to \infty)$ and $z_n \to z^*(n \to \infty)$. The proof is complete. □

Theorem 3.2. *Let H be a real Hilbert space and $\eta : H \times H \to H$ be a $\delta-$strongly and strictly monotone and $\tau-$Lipschitz continuous multi-valued operator. Assume that $M_1, M_2, M_3 : H \to 2^H$ are maximal $\eta-$monotone multi-valued operators, $f : H \to H$ is a nonexpansive mapping, $T_1, T_2, T_3 : H \times H \to 2^H$ are $u, v, w-$Lipschitz continuous in the first variable and relaxed $(a, r), (b, s), (c, t)-$cocoercive, respectively. If the following conditions are satisfied:*

(i) $\sum_{n=0}^{+\infty} \alpha_n = +\infty, \alpha_n \in [0, 1]$;
(ii) $\sum_{n=0}^{+\infty}(1-\beta_n\delta_n) < +\infty, \beta_n, \delta_n \in [0, 1]$;
(iii) $\sum_{n=0}^{+\infty}(1-\delta_n) < +\infty, \delta_n \in [0, 1]$;
(iv) $\min\left\{\frac{[(u+r-au^2)(u-r+au^2)]^{1/2}}{u}, \frac{[(v+s-bv^2)(v-s+bv^2)]^{1/2}}{v}, \frac{[(w+t-cw^2)(w-t+cw^2)]^{1/2}}{w}\right\} < \frac{\delta}{\tau}$;
(v) $r > au^2, \ s > bv^2, \ t > cw^2$,

then there exist $\rho_1 \in (0, \frac{2(r-au^2)}{u^2})$, $\rho_2 \in (0, \frac{2(s-bv^2)}{v^2})$, $\rho_3 \in (0, \frac{2(t-cw^2)}{w^2})$ *satisfying, for any solution* (x^*, y^*, z^*) *to Problem B, the sequence* $\{(x_n, y_n, z_n)\}$ *generated by Algorithm 2.2 converges strongly to* (x^*, y^*, z^*).

Proof. Let $\theta_1 = (1+\rho_1^2 u^2+2\rho_1 a u^2-2\rho_1 r)^{1/2}$, $\theta_2 = (1+\rho_2^2 v^2+2\rho_2 b v^2-2\rho_2 s)^{1/2}$, and $\theta_3 = (1 + \rho_3^2 w^2 + 2\rho_3 c w^2 - 2\rho_3 t)^{1/2}$. From conditions (iv) and (v) we deduce that there exist $\rho_1 \in (0, \frac{2(s-bv^2)}{v^2})$ satisfying $\frac{\tau\cdot\theta_1}{\delta} < 1$, $\rho_2 \in (0, \frac{2(s-bv^2)}{v^2})$ satisfying $\frac{\tau\cdot\theta_2}{\delta} < 1$ and $\rho_3 \in (0, \frac{2(s-bv^2)}{v^2})$ satisfying $\frac{\tau\cdot\theta_3}{\delta} < 1$.

Assume that $(x^*, y^*, z^*) \in H \times H \times H$ is any solution to Problem B, then we know from Lemma 2.9 that

$$\begin{aligned} x^* &= J_{\rho_1}^{M_1}(y^* - \rho_1 T_1(y^*, x^*)),\\ y^* &= J_{\rho_2}^{M_2}(z^* - \rho_2 T_2(z^*, y^*)),\\ z^* &= J_{\rho_3}^{M_3}(x^* - \rho_3 T_3(x^*, z^*)). \end{aligned} \tag{3.13}$$

By Algorithm 2.2 we have

$$\begin{aligned} &\|x_{n+1} - x^*\|\\ &= \|(1-\alpha_n)f(x_n) + \alpha_n J_{\rho_1}^{M_1}(y_n - \rho_1 T_1(y_n, x_n)) - (1-\alpha_n)f(x^*)\\ &\quad - \alpha_n J_{\rho_1}^{M_1}(y^* - \rho_1 T_1(y^*, x^*))\|\\ &\le (1-\alpha_n)\|f(x_n) - f(x^*)\|\\ &\quad + \alpha_n\|J_{\rho_1}^{M_1}(y_n - \rho_1 T_1(y_n, x_n)) - J_{\rho_1}^{M_1}(y^* - \rho_1 T_1(y^*, x^*))\|\\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|J_{\rho_1}^{M_1}(y_n - \rho_1 T_1(y_n, x_n)) - J_{\rho_1}^{M_1}(y^* - \rho_1 T_1(y^*, x^*))\|\\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\|y_n - y^* - \rho_1(T_1(y_n, x_n) - T_1(y^*, x^*))\|. \end{aligned} \tag{3.14}$$

Since $T_1 : H \times H \to H$ is $\mu-$Lipschitz continuous in the first variable and relaxed $(a, r)-$coercive, we have

$$\begin{aligned} &\|y_n - y^* - \rho_1(T_1(y_n, x_n) - T_1(y^*, x^*))\|^2\\ &= \|y_n - y^*\|^2 + \rho_1^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2\\ &\quad - 2\rho_1\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^*\rangle\\ &\le \|y_n - y^*\|^2 + \rho_1^2 u^2\|y_n - y^*\|^2\\ &\quad + 2\rho_1 a\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 - 2\rho_1 r\|y_n - y^*\|^2\\ &\le \|y_n - y^*\|^2 + \rho_1^2 u^2\|y_n - y^*\|^2 + 2\rho_1 a u^2\|y_n - y^*\|^2 - 2\rho_1 r\|y_n - y^*\|^2\\ &= (1 + \rho_1^2 u^2 + 2\rho_1 a u^2 - 2\rho_1 r)\|y_n - y^*\|^2. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we have

$$\|x_{n+1} - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1\|y_n - y^*\|, \tag{3.16}$$

where $\theta_1 = (1 + \rho_1^2 u^2 + 2\rho_1 a u^2 - 2\rho_1 r)^{1/2} < 1$. By Algorithm 2.2, we have

$$\begin{aligned}
&\|y_n - y^*\| \\
&= \|(1-\beta_n) f(x_n) + \beta_n J_{\rho_2}^{M_2}(z_n - \rho_2 T_2(z_n, y_n)) \\
&\quad - (1-\beta_n) f(y^*) - \beta_n J_{\rho_2}^{M_2}(z^* - \rho_2 T_2(z^*, y^*))\| \\
&\le (1-\beta_n)\|f(x_n) - f(y^*)\| + \beta_n \frac{\tau}{\delta}\|z_n - z^* - \rho_2(T_2(z_n, y_n) - T_2(z^*, y^*))\| \\
&\le (1-\beta_n)\|x_n - y^*\| + \beta_n \frac{\tau}{\delta}\|z_n - z^* - \rho_2(T_2(z_n, y_n) - T_2(z^*, y^*))\|. \quad (3.17)
\end{aligned}$$

Since $T_2 : H \times H \to H$ is $v-$Lipschitz continuous in the first variable and relaxed $(b, s)-$coercive, we have

$$\begin{aligned}
&\|z_n - z^* - \rho_2(T_2(z_n, y_n) - T_2(z^*, y^*))\|^2 \\
&= \|z_n - z^*\|^2 + \rho_2^2 \|T_2(z_n, y_n) - T_2(z^*, y^*)\|^2 \\
&\quad - 2\rho_2 \langle T_2(z_n, y_n) - T_2(z^*, y^*), z_n - z^* \rangle \\
&\le \|z_n - z^*\|^2 + \rho_2^2 v^2 \|z_n - z^*\|^2 + 2\rho_2 b \|T_2(z_n, y_n) - T_2(z^*, y^*)\|^2 \\
&\quad - 2\rho_2 s \|z_n - z^*\|^2 \\
&\le \|z_n - z^*\|^2 + \rho_2^2 v^2 \|z_n - z^*\|^2 + 2\rho_2 b v^2 \|z_n - z^*\|^2 - 2\rho_2 s \|z_n - z^*\|^2 \\
&= (1 + \rho_2^2 v^2 + 2\rho_2 b v^2 - 2\rho_2 s)\|z_n - z^*\|^2. \quad (3.18)
\end{aligned}$$

Combining (3.17) with (3.18) we have

$$\|y_n - y^*\| \le (1-\beta_n)\|x_n - y^*\| + \beta_n \frac{\tau}{\delta}\theta_2 \|z_n - z^*\|, \quad (3.19)$$

where $\theta_2 = (1 + \rho_2^2 v^2 + 2\rho_2 b v^2 - 2\rho_2 s)^{1/2} < 1$. By Algorithm 2.2, we have

$$\begin{aligned}
&\|z_n - z^*\| \\
&= \|(1-\delta_n) f(x_n) + \delta_n J_{\rho_3}^{M_3}(x_n - \rho_3 T_3(x_n, z_n)) - (1-\delta_n) f(z^*) \\
&\quad - \delta_n J_{\rho_3}^{M_3}(x^* - \rho_3 T_3(x^*, z^*))\| \\
&\le (1-\delta_n)\|f(x_n) - f(z^*)\| \\
&\quad + \delta_n \|J_{\rho_3}^{M_3}(x_n - \rho_3 T_3(x_n, z_n)) - J_{\rho_3}^{M_3}(x^* - \rho_3 T_3(x^*, z^*))\| \\
&\le (1-\delta_n)\|x_n - z^*\| + \delta_n \|J_{\rho_3}^{M_3}(x_n - \rho_3 T_3(x_n, z_n)) - J_{\rho_3}^{M_3}(x^* - \rho_3 T_3(x^*, z^*))\| \\
&\le (1-\delta_n)\|x_n - z^*\| + \delta_n \frac{\tau}{\delta}\|x_n - x^* - \rho_3(T_3(x_n, z_n) - T_3(x^*, z^*))\|. \quad (3.20)
\end{aligned}$$

Since $T_3 : H \times H \to H$ is $w-$Lipschitz continuous in the first variable and relaxed $(c,t)-$coercive, we have

$$\begin{aligned}
&\|x_n - x^* - \rho_3(T_3(x_n, z_n) - T_3(x^*, z^*))\|^2 \\
&= \|x_n - x^*\|^2 + \rho_3^2\|T_3(x_n, z_n) - T_3(x^*, z^*)\|^2 \\
&\quad - 2\rho_3\langle T_3(x_n, z_n) - T_3(x^*, z^*), x_n - x^*\rangle \\
&\le \|x_n - x^*\|^2 + \rho_3^2 w^2\|x_n - x^*\|^2 + 2\rho_3 c\|T_3(x_n, z_n) - T_3(x^*, z^*)\|^2 \\
&\quad - 2\rho_3 t\|x_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 + \rho_3^2 w^2\|x_n - x^*\|^2 + 2\rho_3 cw^2\|x_n - x^*\|^2 - 2\rho_3 t\|x_n - x^*\|^2 \\
&= (1 + \rho_3^2 w^2 + 2\rho_3 cw^2 - 2\rho_3 t)\|x_n - x^*\|^2. \qquad (3.21)
\end{aligned}$$

Combining (3.21) with (3.20) we have

$$\|z_n - z^*\| \le (1 - \delta_n)\|x_n - z^*\| + \delta_n \frac{\tau}{\delta}\theta_3\|x_n - x^*\|, \qquad (3.22)$$

where $\theta_3 = (1 + \rho_3^2 w^2 + 2\rho_3 cw^2 - 2\rho_3 t)^{1/2} < 1$.

Let $L = \max\{\|x^* - z^*\|, \|x^* - y^*\|\}$. Then from (3.22) and the condition $\frac{\tau}{\delta}\theta_3 < 1$, we have

$$\begin{aligned}
\|z_n - z^*\| &\le (1 - \delta_n)\|x_n - z^*\| + \delta_n \frac{\tau}{\delta}\theta_3\|x_n - x^*\|, \\
&\le (1 - \delta_n)\|x_n - x^*\| + (1 - \delta_n)\|x^* - z^*\| + \delta_n \frac{\tau}{\delta}\theta_3\|x_n - x^*\|, \\
&\le \|x_n - x^*\| + (1 - \delta_n)\|x^* - z^*\| \\
&\le \|x_n - x^*\| + (1 - \delta_n)L. \qquad (3.23)
\end{aligned}$$

Combining (3.19) with (3.23) and the condition $\frac{\tau}{\delta}\theta_2 < 1$, we have

$$\begin{aligned}
&\|y_n - y^*\| \\
&\le (1 - \beta_n)\|x_n - y^*\| + \beta_n \frac{\tau}{\delta}\theta_2\|z_n - z^*\| \\
&\le (1 - \beta_n)\|x_n - y^*\| + \beta_n\|z_n - z^*\| \\
&\le (1 - \beta_n)\|x_n - y^*\| + \beta_n\{\|x_n - x^*\| + (1 - \delta_n)L\} \\
&\le (1 - \beta_n)\|x_n - x^*\| + (1 - \beta_n)\|x^* - y^*\| + \beta_n\{\|x_n - x^*\| + (1 - \delta_n)L\} \\
&\le (1 - \beta_n)\|x_n - x^*\| + (1 - \beta_n)\|x^* - y^*\| + \beta_n\|x_n - x^*\| + \beta_n(1 - \delta_n)L \\
&\le \|x_n - x^*\| + (1 - \beta_n)L + \beta_n(1 - \delta_n)L \\
&\le \|x_n - x^*\| + (1 - \beta_n\delta_n)L. \qquad (3.24)
\end{aligned}$$

Combining (3.16) with (3.24) and the condition $\frac{\tau}{\delta}\theta_1 < 1$ we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1\|y_n - y^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1\{\|x_n - x^*\| + (1-\beta_n\delta_n)L\} \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1(1-\beta_n\delta_n)L \\ &\leq (1-\alpha_n+\alpha_n\frac{\tau}{\delta}\theta_1)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1(1-\beta_n\delta_n)L \\ &\leq (1-(1-\frac{\tau}{\delta}\theta_1)\alpha_n)\|x_n - x^*\| + \alpha_n\frac{\tau}{\delta}\theta_1(1-\beta_n\delta_n)L.\end{aligned}$$

Let $a_n = \|x_n - x^*\|$, $\Omega_n = (1-\frac{\tau}{\delta}\theta_1)\alpha_n$, $b_n = 0$, $c_n = \alpha_n\frac{\tau}{\delta}\theta_1(1-\beta_n\delta_n)L$. From Condition (i) we deduce

$$\sum_{n=0}^{\infty}\Omega_n = \sum_{n=0}^{\infty}(1-\frac{\tau}{\delta}\theta_1)\alpha_n = (1-\frac{\tau}{\delta}\theta_1)\sum_{n=0}^{\infty}\alpha_n = \infty,$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}c_n &= \sum_{n=0}^{\infty}\alpha_n\frac{\tau}{\delta}\theta_1(1-\beta_n\delta_n)L \\ &= \frac{\tau}{\delta}\theta_1 L\sum_{n=0}^{\infty}\alpha_n(1-\beta_n\delta_n) \\ &< \frac{\tau}{\delta}\theta_1 L\sum_{n=0}^{\infty}(1-\beta_n\delta_n) \\ &< +\infty.\end{aligned}$$

By Lemma 2.1, we have $\|x_n - x^*\| = a_n \to 0(n\to\infty)$. (3.24) together with $\sum_{n=0}^{\infty}(1-\beta_n\delta_n) < +\infty$, $\beta_n, \delta_n \in [0,1]$ imply $\|y_n - y^*\| \to 0(n\to\infty)$. (3.23) together with $\sum_{n=0}^{\infty}(1-\delta_n) < +\infty$, $\delta_n \in [0,1]$ imply $\|z_n - z^*\| \to 0(n\to\infty)$. This completes the proof. □

4. Variational inequalities in Banach spaces

Theorem 4.1. *Let B be a real $2-$uniform smooth Banach space, (x^*, y^*, z^*) be a solution to Problem C. Assume that $T_1 : B\times B\times B \to B$ is $(\gamma_1, r_1)-$relaxed cocoercive and Lipschitz continuous with constant μ_{11} in the first variable, with constant μ_{12} in the second variable and with constant μ_{13} in the third variable, $T_2 : B\times B\times B\to B$ is $(\gamma_2, r_2)-$relaxed cocoercive and Lipschitz continuous with constant μ_{21} in the first variable, with constant μ_{22} in the second variable and with constant μ_{23} in the third variable and $T_3 : B\times B\times B\to B$ is*

$(\gamma_3, r_3)-$relaxed cocoercive and Lipschitz continuous with constant μ_{31} in the first variable, with constant μ_{32} in the second variable and with constant μ_{33} in the third variable. Assume further that f is $(\gamma_4, r_4)-$relaxed cocoercive and μ_4-Lipschitz continuous, g is $(\gamma_5, r_5)-$relaxed cocoercive and μ_5-Lipschitz continuous, h is $(\gamma_6, r_6)-$relaxed cocoercive and μ_6-Lipschitz continuous. If

(a) $1-\rho_1\mu_{13}-\frac{\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}}-(\frac{\theta_1\theta_2}{1-\rho_2\mu_{23}}+\rho_1\mu_{12})[\frac{\frac{\theta_3}{1-\rho_3\mu_{33}}+\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1-\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}]\in[0,1]$;

(b) $\frac{\frac{\theta_3}{1-\rho_3\mu_{33}}+\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1-\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\in[0,1]$;

(c) $\frac{\theta_2}{1-\rho_2\mu_{23}}\frac{\frac{\theta_3}{1-\rho_3\mu_{33}}+\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1-\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}+\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\in[0,1]$;

(d) $0<k_1+(1+2\rho_1\gamma_1\mu_{11}^2-2\rho_1 r_1+c_2\rho_1^2\mu_{11}^2)^{1/2}<1$;

(e) $0<k_2+(1+2\rho_2\gamma_2\mu_{21}^2-2\rho_2 r_2+c_2\rho_2^2\mu_{21}^2)^{1/2}<1$;

(f) $0<k_3+(1+2\rho_3\gamma_3\mu_{31}^2-2\rho_3 r_3+c_2\rho_3^2\mu_{31}^2)^{1/2}<1$,

where

$$k_1=[1-2(r_4-\gamma_4\mu_4^2)+c_2\mu_4^2]^{1/2},$$

$$k_2=[1-2(r_5-\gamma_5\mu_5^2)+c_2\mu_5^2]^{1/2},$$

$$k_3=[1-2(r_6-\gamma_6\mu_6^2)+c_2\mu_6^2]^{1/2}$$

and $\alpha_n\in[0,1]$, $\sum_{n=0}^{\infty}\alpha_n=\infty$, then for any initial value $x_0, y_0, z_0\in K$, the sequence $\{(x_n, y_n, z_n)\}$ generated by Algorithm 2.3 converges strongly to (x^*, y^*, z^*).

Proof. By Algorithm 2.3, Lemma 2.10 and the sunny nonexpansive kernel-preserving propertoies of Q_k, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|\\
&=\|(1-\alpha_n)x_n+\alpha_n Q_k[f(y_n)-\rho_1T_1(y_n,z_n,x_n)]\\
&\quad-(1-\alpha_n)x^*-\alpha_n Q_k[f(y^*)-\rho_1T_1(y^*,z^*,x^*)]\|\\
&\le(1-\alpha_n)\|x_n-x^*\|\\
&\quad+\alpha_n\|[f(y_n)-\rho_1T_1(y_n,z_n,x_n)]-[f(y^*)-\rho_1T_1(y^*,z^*,x^*)]\|\\
&=(1-\alpha_n)\|x_n-x^*\|+\alpha_n\|-(y_n-y^*)+[f(y_n)-f(y^*)]+(y_n-y^*)\\
&\quad-\rho_1[T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)]\|\\
&\le(1-\alpha_n)\|x_n-x^*\|+\alpha_n\|y_n-y^*-[f(y_n)-f(y^*)]\|\\
&\quad+\alpha_n\|(y_n-y^*)-\rho_1[T_1(y_n,z_n,x_n)-T_1(y^*,z^*,x^*)]\|. \qquad (4.1)
\end{aligned}$$

Since T_1 is relaxed $(\gamma_1, r_1)-$cocoercive and $\mu_{11}-$Lipschitz continuous in the first variable, we deduce from Lemma 2.5 that

$$\begin{aligned}
&\|y_n - y^* - \rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \\
&= \|y_n - y^* - \rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n) \\
&\quad + T_1(y^*, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \\
&\leq \|y_n - y^* - \rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)]\| \\
&\quad + \rho_1\|[T_1(y^*, z_n, x_n) - T_1(y^*, z^*, x^*)]\| \\
&\leq \|y_n - y^* - \rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)]\| \\
&\quad + \rho_1\|[T_1(y^*, z_n, x_n) - T_1(y^*, z^*, x_n) + T_1(y^*, z^*, x_n) - T_1(y^*, z^*, x^*)]\| \\
&\leq \|y_n - y^* - \rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)]\| \\
&\quad + \rho_1\|[T_1(y^*, z_n, x_n) - T_1(y^*, z^*, x_n)]\| + \rho_1\|T_1(y^*, z^*, x_n) - T_1(y^*, z^*, x^*)]\|.
\end{aligned} \tag{4.2}$$

Observe that

$$\begin{aligned}
&\|y_n - y^* - \rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)]\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\langle -\rho_1[T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)], J(y_n - y^*)\rangle \\
&\quad + c_2(-\rho_1)^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)\|^2 \\
&= \|y_n - y^*\|^2 - 2\rho_1\langle [T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)], J(y_n - y^*)\rangle \\
&\quad + c_2\rho_1^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)\|^2 \\
&\leq \|y_n - y^*\|^2 - 2\rho_1[-\gamma_1\|T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)\|^2 + r_1\|y_n - y^*\|^2] \\
&\quad + c_2\rho_1^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\rho_1\gamma_1\|T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)\|^2 \\
&\quad - 2\rho_1 r_1\|y_n - y^*\|^2 + c_2\rho_1^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z_n, x_n)\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\rho_1\gamma_1\mu_{11}^2\|y_n - y^*\|^2 - 2\rho_1 r_1\|y_n - y^*\|^2 + c_2\rho_1^2\mu_{11}^2\|y_n - y^*\|^2 \\
&\leq (1 + 2\rho_1\gamma_1\mu_{11}^2 - 2\rho_1 r_1 + c_2\rho_1^2\mu_{11}^2)\|y_n - y^*\|^2.
\end{aligned} \tag{4.3}$$

Since T_1 is $\mu_{12}-$Lipschitz continuous in the second variable, we have

$$\|T_1(y^*, z_n, x_n) - T_1(y^*, z^*, x_n)\| \leq \mu_{12}\|z_n - z^*\|. \tag{4.4}$$

Since T_1 is $\mu_{13}-$Lipschitz continuous in the third variable, we have

$$\|T_1(y^*, z^*, x_n) - T_1(y^*, z^*, x^*)\| \leq \mu_{13}\|x_n - x^*\|. \tag{4.5}$$

Since f is relaxed $(\gamma_4, r_4)-$cocoercive and μ_4-Lipschitz continuous, we deduce that

$$\begin{aligned}
&\|y_n - y^* - [f(y_n) - f(y^*)]\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\langle -[f(y_n) - f(y^*)], J(y_n - y^*)\rangle + c_2\|f(y_n) - f(y^*)\|^2 \\
&= \|y_n - y^*\|^2 - 2\langle [f(y_n) - f(y^*)], J(y_n - y^*)\rangle + c_2\|f(y_n) - f(y^*)\|^2 \\
&\leq \|y_n - y^*\|^2 - 2[-\gamma_4\|f(y_n) - f(y^*)\|^2 + r_4\|y_n - y^*)\|^2] + c_2\|f(y_n) - f(y^*)\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\gamma_4\|f(y_n) - f(y^*)\|^2 - 2r_4\|y_n - y^*)\|^2 + c_2\|f(y_n) - f(y^*)\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\gamma_4\mu_4^2\|y_n - y^*\|^2 - 2r_4\|y_n - y^*)\|^2 + c_2\mu_4^2\|y_n - y^*\|^2 \\
&= (1 + 2\gamma_4\mu_4^2 - 2r_4 + c_2\mu_4^2)\|y_n - y^*\|^2 \\
&= k_1^2\|y_n - y^*\|^2. \qquad (4.6)
\end{aligned}$$

Let $\theta_1 = k_1 + (1 + 2\rho_1\gamma_1\mu_{11}^2 - 2\rho_1 r_1 + c_2\rho_1^2\mu_{11}^2)^{1/2}$. By condition (d) we have $0 < \theta_1 < 1$, we can therefore deduce from (4.1) – (4.6) that

$$\begin{aligned}
\|x_{n+1} - x^*\| \leq (1 - \alpha_n + \alpha_n\rho_1\mu_{13})\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| \\
+ \alpha_n\rho_1\mu_{12})\|z_n - z^*\|. \qquad (4.7)
\end{aligned}$$

But

$$\begin{aligned}
&\|y_{n+1} - y^*\| \\
&= \|Q_k[g(z_{n+1}) - \rho_2 T_2(z_{n+1}, x_{n+1}, y_{n+1})] - Q_k[g(z^*) - \rho_2 T_2(z^*, x^*, y^*)]\| \\
&\leq \|[g(z_{n+1}) - \rho_2 T_2(z_{n+1}, x_{n+1}, y_{n+1})] - [g(z^*) - \rho_2 T_2(z^*, x^*, y^*)]\| \\
&= \|g(z_{n+1}) - g(z^*) - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y^*)]\| \\
&= \| - (z_{n+1} - z^*) + g(z_{n+1}) - g(z^*) + (z_{n+1} - z^*) \\
&\quad - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y^*)]\|. \qquad (4.8)
\end{aligned}$$

Since T_2 is relaxed $(\gamma_2, r_2)-$cocoercive and $\mu_{21}-$Lipschitz continuous in the first variable, we deduce that

$$\begin{aligned}
&\|z_{n+1} - z^* - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y^*)]\| \\
&= \|z_{n+1} - z^* - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1}) \\
&\quad + T_2(z^*, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y^*)]\| \\
&\le \|z_{n+1} - z^* - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})]\| \\
&\quad + \rho_2\|T_2(z^*, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y^*)\| \\
&\le \|z_{n+1} - z^* - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})]\| \\
&\quad + \rho_2\|T_2(z^*, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y_{n+1}) + T_2(z^*, x^*, y_{n+1}) - T_2(z^*, x^*, y^*)\| \\
&\le \|z_{n+1} - z^* - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})]\| \\
&\quad + \rho_2\|T_2(z^*, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y_{n+1})\| \\
&\quad + \rho_2\|T_2(z^*, x^*, y_{n+1}) - T_2(z^*, x^*, y^*)\|. \qquad (4.9)
\end{aligned}$$

But

$$\begin{aligned}
&\|z_{n+1} - z^* - \rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})]\|^2 \\
&\le \|z_{n+1} - z^*\|^2 \\
&\quad + 2\langle -\rho_2[T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})], J(\|z_{n+1} - z^*\|)\rangle \\
&\quad + c_2\rho_2^2\|T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})\|^2 \\
&= \|z_{n+1} - z^*\|^2 \\
&\quad - 2\rho_2\langle [T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})], J(\|z_{n+1} - z^*\|)\rangle \\
&\quad + c_2\rho_2^2\|T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})\|^2 \\
&\le \|z_{n+1} - z^*\|^2 \\
&\quad - 2\rho_2[-\gamma_2\|T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})\| + r_2\|z_{n+1} - z^*\|^2] \\
&\quad + c_2\rho_2^2\|T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})\|^2 \qquad (4.10)
\end{aligned}$$

$$\begin{aligned}
&= \|z_{n+1} - z^*\|^2 \\
&\quad + 2\rho_2\gamma_2\|T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})\| - 2\rho_2 r_2\|z_{n+1} - z^*\|^2 \\
&\quad + c_2\rho_2^2\|T_2(z_{n+1}, x_{n+1}, y_{n+1}) - T_2(z^*, x_{n+1}, y_{n+1})\|^2 \\
&\le \|z_{n+1} - z^*\|^2 + 2\rho_2\gamma_2\mu_{21}^2\|z_{n+1} - z^*\|^2 - 2\rho_2 r_2\|z_{n+1} - z^*\|^2 \\
&\quad + c_2\rho_2^2\mu_{21}^2\|z_{n+1} - z^*\|^2 \\
&= (1 + 2\rho_2\gamma_2\mu_{21}^2 - 2\rho_2 r_2 + c_2\rho_2^2\mu_{21}^2)\|z_{n+1} - z^*\|^2.
\end{aligned}$$

Since T_2 is $\mu_{22}-$Lipschitz continuous in the second variable, we have

$$\|T_2(z^*, x_{n+1}, y_{n+1}) - T_2(z^*, x^*, y_{n+1})\| \leq \mu_{22}\|x_{n+1} - x^*\|. \tag{4.11}$$

Since T_2 is $\mu_{23}-$Lipschitz continuous in the third variable, we have

$$\|T_2(z^*, x^*, y_{n+1}) - T_2(z^*, x^*, y^*)\| \leq \mu_{23}\|y_{n+1} - y^*\|. \tag{4.12}$$

Since g is relaxed $(\gamma_5, r_5)-$cocoercive and μ_5-Lipschitz continuous, we deduce that

$$\begin{aligned}
&\|z_{n+1} - z^* - [g(z_{n+1}) - g(z^*)]\|^2 \\
&\leq \|z_{n+1} - z^*\|^2 + 2\langle -[g(z_{n+1}) - g(z^*)], J(z_{n+1} - z^*)\rangle + c_2\|g(z_{n+1}) - g(z^*)\|^2 \\
&= \|z_{n+1} - z^*\|^2 - 2\langle [g(z_{n+1}) - g(z^*)], J(z_{n+1} - z^*)\rangle + c_2\|g(z_{n+1}) - g(z^*)\|^2 \\
&\leq \|z_{n+1} - z^*\|^2 - 2[-\gamma_5\|g(z_{n+1}) - g(z^*)\|^2 + r_5\|z_{n+1} - z^*\|^2] \\
&\quad + c_2\|g(z_{n+1}) - g(z^*)\|^2 \\
&= \|z_{n+1} - z^*\|^2 + 2\gamma_5\|g(z_{n+1}) - g(z^*)\|^2 - 2r_5\|z_{n+1} - z^*\|^2 \\
&\quad + c_2\|g(z_{n+1}) - g(z^*)\|^2 \\
&\leq \|z_{n+1} - z^*\|^2 + 2\gamma_5\mu_5^2\|z_{n+1} - z^*\|^2 - 2r_5\|z_{n+1} - z^*\|^2 + c_2\mu_5^2\|z_{n+1} - z^*\|^2 \\
&= (1 + 2\gamma_5\mu_5^2 - 2r_5 + c_2\mu_5^2)\|z_{n+1} - z^*\|^2 \\
&= k_2^2\|z_{n+1} - z^*\|^2.
\end{aligned} \tag{4.13}$$

Let $\theta_2 = k_2 + (1 + 2\rho_2\gamma_2\mu_{21}^2 - 2\rho_2 r_2 + c_2\rho_2^2\mu_{21}^2)^{1/2}$. By condition (e) we have $0 < \theta_2 < 1$, we can therefore deduce from (4.8) – (4.13) that

$$\begin{aligned}
\|y_{n+1} - y^*\| \leq \theta_2\|z_{n+1} - z^*\| + \rho_2\mu_{22})\|x_{n+1} - x^*\| \\
+ \rho_2\mu_{23}\|y_{n+1} - y^*\|.
\end{aligned} \tag{4.14}$$

This implies that

$$\|y_{n+1} - y^*\| \leq \frac{\theta_2}{1 - \rho_2\mu_{23}}\|z_{n+1} - z^*\| + \frac{\rho_2\mu_{22}}{1 - \rho_2\mu_{23}}\|x_{n+1} - x^*\|$$

and

$$\|y_n - y^*\| \leq \frac{\theta_2}{1 - \rho_2\mu_{23}}\|z_n - z^*\| + \frac{\rho_2\mu_{22}}{1 - \rho_2\mu_{23}}\|x_n - x^*\|.$$

Combining (4.14) and (4.7) we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le (1-\alpha_n+\alpha_n\rho_1\mu_{13})\|x_n-x^*\| \\
&\quad + \alpha_n\theta_1\left\{\frac{\theta_2}{1-\rho_2\mu_{23}}\|z_n-z^*\| + \frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\|x_n-x^*\|\right\} \\
&\quad + \alpha_n\rho_1\mu_{12}\|z_n-z^*\| \\
&= (1-\alpha_n+\alpha_n\rho_1\mu_{13})\|x_n-x^*\| + \frac{\alpha_n\theta_1\theta_2}{1-\rho_2\mu_{23}}\|z_n-z^*\| \\
&\quad + \frac{\alpha_n\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\|x_n-x^*\| + \alpha_n\rho_1\mu_{12}\|z_n-z^*\| \\
&= \left(1-\alpha_n+\alpha_n\rho_1\mu_{13}+\frac{\alpha_n\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\right)\|x_n-x^*\| \\
&\quad + \left(\frac{\alpha_n\theta_1\theta_2}{1-\rho_2\mu_{23}}+\alpha_n\rho_1\mu_{12}\right)\|z_n-z^*\|. \qquad (4.15)
\end{aligned}$$

$$\begin{aligned}
&\|z_{n+1}-z^*\| \\
&= \|Q_k[h(x_{n+1})-\rho_3T_3(x_{n+1},y_{n+1},z_{n+1})] - Q_k[h(x^*)-\rho_3T_3(x^*,y^*,z^*)]\| \\
&\le \|[h(x_{n+1})-\rho_3T_3(x_{n+1},y_{n+1},z_{n+1})] - [h(x^*)-\rho_3T_3(x^*,y^*,z^*)]\| \\
&= \|h(x_{n+1})-h(x^*)-\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1})-T_3(x^*,y^*,z^*)]\| \\
&= \|-(x_{n+1}-x^*)+h(x_{n+1})-h(x^*)+(x_{n+1}-x^*) \\
&\quad -\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1})-T_3(x^*,y^*,z^*)]\| \\
&\le \|(x_{n+1}-x^*)-[h(x_{n+1})-h(x^*)]\| + \|(x_{n+1}-x^*) \\
&\quad -\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1})-T_3(x^*,y^*,z^*)]\|. \qquad (4.16)
\end{aligned}$$

Since T_3 is relaxed (γ_3, r_3)-cocoercive and $\mu_{31}-$Lipschitz continuous in the first variable, we have

$$\begin{aligned}
&\|(x_{n+1}-x^*)-\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1})-T_3(x^*,y^*,z^*)]\| \\
&= \|(x_{n+1}-x^*)-\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1})-T_3(x^*,y_{n+1},z_{n+1}) \\
&\quad + T_3(x^*,y_{n+1},z_{n+1})-T_3(x^*,y^*,z^*)]\| \\
&\le \|(x_{n+1}-x^*)-\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1})-T_3(x^*,y_{n+1},z_{n+1})]\| \\
&\quad + \rho_3\|T_3(x^*,y_{n+1},z_{n+1})-T_3(x^*,y^*,z^*)\|
\end{aligned}$$

$$
\begin{aligned}
&= \|(x_{n+1}-x^*) - \rho_3[T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})]\| \\
&\quad + \rho_3\|T_3(x^*,y_{n+1},z_{n+1}) - T_3(x^*,y^*,z_{n+1}) + T_3(x^*,y^*,z_{n+1}) - T_3(x^*,y^*,z^*)\| \\
&\le \|(x_{n+1}-x^*) - \rho_3[T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})]\| \\
&\quad + \rho_3\|T_3(x^*,y_{n+1},z_{n+1}) - T_3(x^*,y^*,z_{n+1})\| \\
&\quad + \rho_3\|T_3(x^*,y^*,z_{n+1}) - T_3(x^*,y^*,z^*)\|. \qquad (4.17)
\end{aligned}
$$

But

$$
\begin{aligned}
&\|(x_{n+1}-x^*) - \rho_3[T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})]\|^2 \\
&\le \|x_{n+1}-x^*\|^2 \\
&\quad + 2\langle -\rho_3[T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})], J(\|x_{n+1}-x^*\|)\rangle \\
&\quad + c_2\rho_3^2\|T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})\|^2 \\
&= \|x_{n+1}-x^*\|^2 \\
&\quad - 2\rho_3\langle [T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})], J(\|x_{n+1}-x^*\|)\rangle \\
&\quad + c_2\rho_3^2\|T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})\|^2 \\
&= \|x_{n+1}-x^*\|^2 \\
&\quad - 2\rho_3[-\gamma_3\|T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})\| + r_3\|x_{n+1}-x^*\|^2] \\
&\quad + c_2\rho_3^2\|T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})\|^2 \\
&= \|x_{n+1}-x^*\|^2 + 2\rho_3\gamma_3\|T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})\| \\
&\quad - 2\rho_3 r_3\|x_{n+1}-x^*\|^2 \\
&\quad + c_2\rho_3^2\|T_3(x_{n+1},y_{n+1},z_{n+1}) - T_3(x^*,y_{n+1},z_{n+1})\|^2 \\
&\le \|x_{n+1}-x^*\|^2 + 2\rho_3\gamma_3\mu_{31}^2\|x_{n+1}-x^*\|^2 - 2\rho_3 r_3\|x_{n+1}-x^*\|^2 \\
&\quad + c_2\rho_3^2\mu_{31}^2\|x_{n+1}-x^*\|^2 \\
&= (1 + 2\rho_3\gamma_3\mu_{31}^2 - 2\rho_3 r_3 + c_2\rho_3^2\mu_{31}^2)\|x_{n+1}-x^*\|^2. \qquad (4.18)
\end{aligned}
$$

Since T_3 is $\mu_{32}-$Lipschitz continuous in the second variable, we have

$$
\|T_3(x^*,y_{n+1},z_{n+1}) - T_3(x^*,y^*,z_{n+1})\| \le \mu_{32}\|y_{n+1}-y^*\|. \qquad (4.19)
$$

Since T_3 is $\mu_{33}-$Lipschitz continuous in the third variable, we have

$$
\|T_3(x^*,y^*,z_{n+1}) - T_3(x^*,y^*,z^*)\| \le \mu_{33}\|z_{n+1}-z^*\|. \qquad (4.20)
$$

Since h is relaxed $(\gamma_6, r_6)-$cocoercive and μ_6-Lipschitz continuous, we have that

$$\begin{aligned}
&\|x_{n+1} - x^* - [h(x_{n+1}) - h(x^*)]\|^2 \\
&\le \|x_{n+1} - x^*\|^2 + 2\langle -[h(x_{n+1}) - h(x^*)], J(x_{n+1} - x^*)\rangle \\
&\quad + c_2\|h(x_{n+1}) - h(x^*)\|^2 \\
&= \|x_{n+1} - x^*\|^2 - 2\langle [h(x_{n+1}) - h(x^*)], J(x_{n+1} - x^*)\rangle + c_2\|h(x_{n+1}) - h(x^*)\|^2 \\
&\le \|x_{n+1} - x^*\|^2 - 2[-\gamma_6\|h(x_{n+1}) - h(x^*)\|^2 + r_6\|x_{n+1} - x^*\|^2] \\
&\quad + c_2\|h(x_{n+1}) - h(x^*)\|^2 \\
&= \|x_{n+1} - x^*\|^2 + 2\gamma_6\|h(x_{n+1}) - h(x^*)\|^2 - 2r_6\|x_{n+1} - x^*\|^2 \\
&\quad + c_2\|h(x_{n+1}) - h(x^*)\|^2 \\
&\le \|x_{n+1} - x^*\|^2 + 2\gamma_6\mu_6^2\|x_{n+1} - x^*\|^2 - 2r_6\|x_{n+1} - x^*\|^2 \\
&\quad + c_2\mu_6^2\|x_{n+1} - x^*\|^2 \\
&= (1 + 2\gamma_6\mu_6^2 - 2r_6 + c_2\mu_6^2)\|x_{n+1} - x^*\|^2 \\
&= k_3^2\|x_{n+1} - x^*\|^2.
\end{aligned} \tag{4.21}$$

Let $\theta_3 = k_3 + (1 + 2\rho_3\gamma_3\mu_{31}^2 - 2\rho_3 r_3 + c_2\rho_3^2\mu_{31}^2)^{1/2}$. Then, from condition (f) we have $0 < \theta_3 < 1$. Therefore, we can obtain from (4.14) – (4.21) that

$$\|z_{n+1} - z^*\| \le \theta_3\|x_{n+1} - x^*\| + \rho_3\mu_{32}\|y_{n+1} - y^*\| + \rho_3\mu_{33}\|z_{n+1} - z^*\|.$$

It implies that

$$\|z_{n+1} - z^*\| \le \frac{\theta_3}{1 - \rho_3\mu_{33}}\|x_{n+1} - x^*\| + \frac{\rho_3\mu_{32}}{1 - \rho_3\mu_{33}}\|y_{n+1} - y^*\|, \tag{4.22}$$

and then

$$\begin{aligned}
\|z_n - z^*\| &\le \frac{\theta_3}{1 - \rho_3\mu_{33}}\|x_n - x^*\| + \frac{\rho_3\mu_{32}}{1 - \rho_3\mu_{33}}\|y_n - y^*\| \\
&\le \frac{\theta_3}{1 - \rho_3\mu_{33}}\|x_n - x^*\| \\
&\quad + \frac{\rho_3\mu_{32}}{1 - \rho_3\mu_{33}}\left[\frac{\theta_2}{1 - \rho_2\mu_{23}}\|z_n - z^*\| + \frac{\rho_2\mu_{22}}{1 - \rho_2\mu_{23}}\|x_n - x^*\|\right] \\
&= \frac{\theta_3}{1 - \rho_3\mu_{33}}\|x_n - x^*\| + \frac{\rho_3\mu_{32}}{1 - \rho_3\mu_{33}}\frac{\theta_2}{1 - \rho_2\mu_{23}}\|z_n - z^*\| \\
&\quad + \frac{\rho_3\mu_{32}}{1 - \rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1 - \rho_2\mu_{23}}\|x_n - x^*\|
\end{aligned}$$

$$= \left[\frac{\theta_3}{1-\rho_3\mu_{33}} + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\right]\|x_n - x^*\| + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}\|z_n - z^*\|.$$

That is,

$$\|z_n - z^*\| \leq \frac{\frac{\theta_3}{1-\rho_3\mu_{33}} + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1 - \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\|x_n - x^*\|. \tag{4.23}$$

From (4.15), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq (1-\alpha_n+\alpha_n\rho_1\mu_{13})\|x_n - x^*\| \\
&\quad + \frac{\alpha_n\theta_1\theta_2}{1-\rho_2\mu_{23}}\|z_n - z^*\| + \frac{\alpha_n\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\|x_n - x^*\| + \alpha_n\rho_1\mu_{12}\|z_n - z^*\| \\
&= \left(1-\alpha_n+\alpha_n\rho_1\mu_{13} + \frac{\alpha_n\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\right)\|x_n - x^*\| \\
&\quad + \left(\frac{\alpha_n\theta_1\theta_2}{1-\rho_2\mu_{23}} + \alpha_n\rho_1\mu_{12}\right)\|z_n - z^*\| \\
&\leq (1-\alpha_n+\alpha_n\rho_1\mu_{13} + \frac{\alpha_n\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}})\|x_n - x^*\| \\
&\quad + \left(\frac{\alpha_n\theta_1\theta_2}{1-\rho_2\mu_{23}} + \alpha_n\rho_1\mu_{12}\right)\left[\frac{\frac{\theta_3}{1-\rho_3\mu_{33}} + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1 - \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\right]\|x_n - x^*\| \\
&= \Bigg\{\left(1-\alpha_n+\alpha_n\rho_1\mu_{13} + \frac{\alpha_n\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\right) \\
&\quad + \left(\frac{\alpha_n\theta_1\theta_2}{1-\rho_2\mu_{23}} + \alpha_n\rho_1\mu_{12}\right)\left[\frac{\frac{\theta_3}{1-\rho_3\mu_{33}} + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1 - \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\right]\Bigg\}\|x_n - x^*\| \\
&= \Bigg\{1-\alpha_n\Bigg\{1-\rho_1\mu_{13} - \frac{\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}} \\
&\quad - \left(\frac{\theta_1\theta_2}{1-\rho_2\mu_{23}} + \rho_1\mu_{12}\right)\left[\frac{\frac{\theta_3}{1-\rho_3\mu_{33}} + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1 - \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\right]\Bigg\}\Bigg\}\|x_n - x^*\|.
\end{aligned}$$

By condition (a) we know

$$1-\rho_1\mu_{13} - \frac{\theta_1\rho_2\mu_{22}}{1-\rho_2\mu_{23}} - \left(\frac{\theta_1\theta_2}{1-\rho_2\mu_{23}} + \rho_1\mu_{12}\right)\left[\frac{\frac{\theta_3}{1-\rho_3\mu_{33}} + \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1 - \frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\right] \in [0,1].$$

Since $\sum_{n=1}^{\infty}\alpha_n=\infty$, from Lemma 2.6 that
$$\lim_{n\to\infty}\|x_n-x^*\|=0.$$
It follows from (4.23) that
$$\lim_{n\to\infty}\|z_n-z^*\|=0.$$
Combining (4.23) with (4.14) we have
$$\begin{aligned}
&\|y_n-y^*\|\\
&\le \frac{\theta_2}{1-\rho_2\mu_{23}}\|z_n-z^*\|+\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\|x_n-x^*\|\\
&\le \frac{\theta_2}{1-\rho_2\mu_{23}}\frac{\frac{\theta_3}{1-\rho_3\mu_{33}}+\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1-\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}\|x_n-x^*\|+\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\|x_n-x^*\|\\
&\le \left[\frac{\theta_2}{1-\rho_2\mu_{23}}\frac{\frac{\theta_3}{1-\rho_3\mu_{33}}+\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}}{1-\frac{\rho_3\mu_{32}}{1-\rho_3\mu_{33}}\frac{\theta_2}{1-\rho_2\mu_{23}}}+\frac{\rho_2\mu_{22}}{1-\rho_2\mu_{23}}\right]\|x_n-x^*\|. \qquad (4.24)
\end{aligned}$$
It implies that
$$\lim_{n\to\infty}\|y_n-y^*\|=0,$$
which completes the proof. □

Theorem 4.2. *Let $\eta: X\times X\times X\to X^*$ be $\sigma-$Lipschitz continuous. Assume that $H_i: X_i\to X_i$ is γ_i-strongly $\eta-$accretive and τ_i-Lipschitz continuous, $i=1,2,3$; and that $M: X_1\to 2^{X_1}$ is $(H_1,\eta)-$accretive, $N: X_2\to 2^{X_2}$ is $(H_2,\eta)-$accretive and $T: X_3\to 2^{X_3}$ is $(H_3,\eta)-$accretive. The nonlinear operator $F: X_1\times X_2\times X_3\to X_1$ is r_1-H_1 strongly accretive s_1-Lipschitz continuous in the first variable, θ_1- Lipschitz continuous in the second variable and θ_2- Lipschitz continuous in the third variable. The nonlinear operator $G: X_1\times X_2\times X_3\to X_2$ is r_2-H_2 strongly accretive s_2- Lipschitz continuous in the second variable, ε_1- Lipschitz continuous in the first variable and ε_2- Lipschitz continuous in the third variable. The nonlinear operator $H: X_1\times X_2\times X_3\to X_3$ is r_3-H_3 strongly accretive s_3- Lipschitz continuous in the third variable, α_1- Lipschitz continuous in the first variable and α_2- Lipschitz continuous in the second variable. If there exist constants ρ_1,ρ_2,ρ_3 satisfying*
$$\gamma_2\gamma_3\sigma(\tau_1^q-q\rho_1r_1+\rho_1^qc_qs_1^q)^{1/q}+\gamma_1\gamma_3\sigma\rho_2\varepsilon_1+\gamma_1\gamma_2\sigma\rho_3\alpha_1<\gamma_1\gamma_2\gamma_3,$$
$$\gamma_1\gamma_3\sigma(\tau_2^q-q\rho_2r_2+\rho_2^qc_qs_2^q)^{1/q}+\gamma_2\gamma_3\sigma\rho_1\theta_1+\gamma_1\gamma_2\sigma\rho_3\alpha_2<\gamma_1\gamma_2\gamma_3,$$
$$\gamma_1\gamma_2\sigma(\tau_3^q-q\rho_3r_3+\rho_3^qc_qs_3^q)^{1/q}+\gamma_2\gamma_3\sigma\rho_1\theta_2+\gamma_1\gamma_3\sigma\rho_2\varepsilon_2<\gamma_1\gamma_2\gamma_3,$$
where $c_q>0$ is the same as that in Lemma 2.5, then Problem D has a unique solution.

Proof. For any fixed $\rho_1 > 0, \rho_2 > 0$ and $\rho_3 > 0$, we define

$$
\begin{aligned}
T_{\rho_1} &: X_1 \times X_2 \times X_3 \to X_1, \\
S_{\rho_2} &: X_1 \times X_2 \times X_3 \to X_2, \\
P_{\rho_3} &: X_1 \times X_2 \times X_3 \to X_3
\end{aligned}
$$

by

$$
\begin{aligned}
T_{\rho_1}(u,v,w) &= R^{H_1\eta}_{M\rho_1}[H_1(u) - \rho_1 F(u,v,w)], \\
S_{\rho_2}(u,v,w) &= R^{H_2\eta}_{N\rho_2}[H_2(v) - \rho_2 G(u,v,w)], \\
P_{\rho_3}(u,v,w) &= R^{H_3\eta}_{T\rho_3}[H_3(w) - \rho_3 H(u,v,w)], \qquad (4.25)
\end{aligned}
$$

for all $(u,v,w) \in X_1 \times X_2 \times X_3$. Then from Lemma 2.7, we have the following three inequalities, respectively:

$$
\begin{aligned}
&\|T_{\rho_1}(u_1,v_1,w_1) - T_{\rho_1}(u_2,v_2,w_2)\| \\
&= \|R^{H_1\eta}_{M\rho_1}[H_1(u_1) - \rho_1 F(u_1,v_1,w_1)] - R^{H_1\eta}_{M\rho_1}[H_1(u_2) - \rho_1 F(u_2,v_2,w_2)]\| \\
&\le \frac{\sigma}{\gamma_1}\|[H_1(u_1) - \rho_1 F(u_1,v_1,w_1)] - [H_1(u_2) - \rho_1 F(u_2,v_2,w_2)]\| \\
&= \frac{\sigma}{\gamma_1}\|[H_1(u_1) - H_1(u_2)] - \rho_1[F(u_1,v_1,w_1) - F(u_2,v_2,w_2)]\| \\
&= \frac{\sigma}{\gamma_1}\|[H_1(u_1) - H_1(u_2)] - \rho_1[F(u_1,v_1,w_1) - F(u_2,v_1,w_1) \\
&\quad + F(u_2,v_1,w_1) - F(u_2,v_2,w_1) + F(u_2,v_2,w_1) - F(u_2,v_2,w_2)]\| \\
&\le \frac{\sigma}{\gamma_1}\|[H_1(u_1) - H_1(u_2)] - \rho_1[F(u_1,v_1,w_1) - F(u_2,v_1,w_1)]\| \\
&\quad + \frac{\sigma\rho_1}{\gamma_1}\|F(u_2,v_1,w_1) - F(u_2,v_2,w_1)\| \\
&\quad + \frac{\sigma\rho_1}{\gamma_1}\|F(u_2,v_2,w_1) - F(u_2,v_2,w_2)]\|, \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
&\|S_{\rho_2}(u_1,v_1,w_1) - S_{\rho_2}(u_2,v_2,w_2)\| \\
&= \|R^{H_2\eta}_{N\rho_2}[H_2(v_1) - \rho_2 G(u_1,v_1,w_1)] - R^{H_2\eta}_{N\rho_2}[H_2(v_2) - \rho_2 G(u_2,v_2,w_2)]\| \\
&\le \frac{\sigma}{\gamma_2}\|[H_2(v_1) - \rho_2 G(u_1,v_1,w_1)] - [H_2(v_2) - \rho_2 G(u_2,v_2,w_2)]\| \\
&= \frac{\sigma}{\gamma_2}\|[H_2(v_1) - H_2(v_2)] - \rho_2[G(u_1,v_1,w_1) - G(u_2,v_2,w_2)]\|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma}{\gamma_2}\|[H_2(v_1) - H_2(v_2)] - \rho_2[G(u_1, v_1, w_1) - G(u_1, v_2, w_1) \\
&\quad + G(u_1, v_2, w_1) - G(u_2, v_2, w_1) + G(u_2, v_2, w_1) - G(u_2, v_2, w_2)]\| \\
&\leq \frac{\sigma}{\gamma_2}\|[H_2(v_1) - H_2(v_2)] - \rho_2[G(u_1, v_1, w_1) - G(u_1, v_2, w_1)]\| \\
&\quad + \frac{\sigma\rho_2}{\gamma_2}\|G(u_1, v_2, w_1) - G(u_2, v_2, w_1)\| \\
&\quad + \frac{\sigma\rho_2}{\gamma_2}\|G(u_2, v_2, w_1) - G(u_2, v_2, w_2)]\|
\end{aligned} \tag{4.27}
$$

and

$$
\begin{aligned}
&\|P_{\rho_3}(u_1, v_1, w_1) - P_{\rho_3}(u_2, v_2, w_2)\| \\
&= \|R_{T\rho_3}^{H_3\eta}[H_3(w_1) - \rho_3 H(u_1, v_1, w_1)] - R_{T\rho_3}^{H_3\eta}[H_3(w_2) - \rho_3 H(u_2, v_2, w_2)]\| \\
&\leq \frac{\sigma}{\gamma_3}\|[H_3(w_1) - \rho_3 H(u_1, v_1, w_1)] - [H_3(w_2) - \rho_3 H(u_2, v_2, w_2)]\| \\
&= \frac{\sigma}{\gamma_3}\|[H_3(w_1) - H_3(w_2)] - \rho_3[H(u_1, v_1, w_1) - H(u_2, v_2, w_2)]\| \\
&= \frac{\sigma}{\gamma_3}\|[H_3(w_1) - H_3(w_2)] - \rho_3[H(u_1, v_1, w_1) - H(u_1, v_1, w_2) \\
&\quad + H(u_1, v_1, w_2) - H(u_2, v_1, w_2) + H(u_2, v_1, w_2) - H(u_2, v_2, w_2)]\| \\
&\leq \frac{\sigma}{\gamma_3}\|[H_3(w_1) - H_3(w_2)] - \rho_3[H(u_1, v_1, w_1) - H(u_1, v_1, w_2)]\| \\
&\quad + \frac{\sigma\rho_3}{\gamma_3}\|H(u_1, v_1, w_2) - H(u_2, v_1, w_2)\| \\
&\quad + \frac{\sigma\rho_3}{\gamma_3}\|H(u_2, v_1, w_1) - H(u_2, v_2, w_2)]\|.
\end{aligned} \tag{4.28}
$$

Since X_1, X_2, X_3 are q-uniformly smooth Banach spaces, we have

$$
\begin{aligned}
&\|H_1(u_1) - H_1(u_2) - \rho_1[F(u_1, v_1, w_1) - F(u_2, v_1, w_1)]\|^q \\
&\leq \|H_1(u_1) - H_1(u_2)\|^q \\
&\quad + q\langle -\rho_1[F(u_1, v_1, w_1) - F(u_2, v_1, w_1)], J_q[H_1(u_1) - H_1(u_2)]\rangle \\
&\quad + c_q\rho_1^q\|F(u_1, v_1, w_1) - F(u_2, v_1, w_1)\|^q \\
&= \|H_1(u_1) - H_1(u_2)\|^q \\
&\quad - q\rho_1\langle F(u_1, v_1, w_1) - F(u_2, v_1, w_1), J_q[H_1(u_1) - H_1(u_2)]\rangle \\
&\quad + c_q\rho_1^q\|F(u_1, v_1, w_1) - F(u_2, v_1, w_1)\|^q \\
&\leq \tau_1^q\|u_1 - u_2\|^q - q\rho_1 r_1\|u_1 - u_2\|^q + c_q\rho_1^q s_1^q\|u_1 - u_2\|^q \\
&= (\tau_1^q - q\rho_1 r_1 + c_q\rho_1^q s_1^q)\|u_1 - u_2\|^q,
\end{aligned} \tag{4.29}
$$

$$\begin{aligned}
&\|H_2(v_1) - H_2(v_2) - \rho_2[G(u_1, v_1, w_1) - G(u_1, v_2, w_1)]\|^q \\
&\le \|H_2(v_1) - H_2(v_2)\|^q \\
&\quad + q\langle -\rho_2[G(u_1, v_1, w_1) - G(u_1, v_2, w_1)], J_q[H_2(v_1) - H_2(v_2)]\rangle \\
&\quad + c_q\rho_2^q\|G(u_1, v_1, w_1) - G(u_1, v_2, w_1)\|^q \\
&= \|H_2(v_1) - H_2(v_2)\|^q \\
&\quad - q\rho_2\langle G(u_1, v_1, w_1) - G(u_1, v_2, w_1), J_q[H_2(v_1) - H_2(v_2)]\rangle \\
&\quad + c_q\rho_2^q\|G(u_1, v_1, w_1) - G(u_1, v_2, w_1)\|^q \\
&\le \tau_2^q\|v_1 - v_2\|^q - q\rho_2 r_2\|v_1 - v_2\|^q + c_q\rho_2^q s_2^q\|v_1 - v_2\|^q \\
&= (\tau_2^q - q\rho_2 r_2 + c_q\rho_2^q s_2^q)\|v_1 - v_2\|^q \qquad (4.30)
\end{aligned}$$

and

$$\begin{aligned}
&\|H_3(w_1) - H_3(w_2) - \rho_3[H(u_1, v_1, w_1) - H(u_1, v_1, w_2)]\|^q \\
&\le \|H_3(w_1) - H_3(w_2)\|^q \\
&\quad + q\langle -\rho_3[H(u_1, v_1, w_1) - H(u_1, v_1, w_2)], J_q[H_3(w_1) - H_3(w_2)]\rangle \\
&\quad + c_q\rho_3^q\|H(u_1, v_1, w_1) - H(u_1, v_1, w_2)\|^q \\
&= \|H_3(w_1) - H_3(w_2)\|^q \\
&\quad - q\rho_3\langle H(u_1, v_1, w_1) - H(u_1, v_1, w_2), J_q[H_3(w_1) - H_3(w_2)]\rangle \\
&\quad + c_q\rho_3^q\|H(u_1, v_1, w_1) - H(u_1, v_1, w_2)\|^q \\
&\le \tau_3^q\|w_1 - w_2\|^q - q\rho_3 r_3\|w_1 - w_2\|^q + c_q\rho_3^q s_3^q\|w_1 - w_2\|^q \\
&= (\tau_3^q - q\rho_3 r_3 + c_q\rho_3^q s_3^q)\|w_1 - w_2\|^q. \qquad (4.31)
\end{aligned}$$

We also have

$$\|F(u_2, v_1, w_1) - F(u_2, v_2, w_1)\| \le \theta_1\|v_1 - v_2\|, \qquad (4.32)$$

$$\|F(u_2, v_2, w_1) - F(u_2, v_2, w_2)\| \le \theta_2\|w_1 - w_2\|, \qquad (4.33)$$

$$\|G(u_1, v_2, w_1) - G(u_2, v_2, w_1)\| \le \varepsilon_1\|u_1 - u_2\|, \qquad (4.34)$$

$$\|G(u_2, v_2, w_1) - G(u_2, v_2, w_2)\| \le \varepsilon_2\|w_1 - w_2\|, \qquad (4.35)$$

$$\|H(u_1, v_1, w_2) - H(u_2, v_1, w_2)\| \le \alpha_1\|u_1 - u_2\|, \qquad (4.36)$$

$$\|H(u_2, v_1, w_2) - H(u_2, v_2, w_2)\| \le \alpha_2\|v_1 - v_2\|. \qquad (4.37)$$

Hence, we can deduce from (4.26) – (4.37) that

$$\begin{aligned}
&\|T_{\rho_1}(u_1, v_1, w_1) - T_{\rho_1}(u_2, v_2, w_2) \\
&\le \frac{\sigma}{\gamma_1}(\tau_1^q - q\rho_1 r_1 + c_q\rho_1^q s_1^q)^{1/q}\|u_1 - u_2\|\| \\
&\quad + \frac{\sigma\rho_1\theta_1}{\gamma_1}\|v_1 - v_2\| + \frac{\sigma\rho_1\theta_2}{\gamma_1}\|w_1 - w_2\|, \qquad (4.38)
\end{aligned}$$

$$\begin{aligned}
&\|S_{\rho_2}(u_1,v_1,w_1)-S_{\rho_2}(u_2,v_2,w_2)\| \\
&\le \frac{\sigma}{\gamma_2}(\tau_2^q-q\rho_2r_2+c_q\rho_2^qs_2^q)^{1/q}\|v_1-v_2\| \\
&\quad+\frac{\sigma\rho_2\varepsilon_1}{\gamma_2}\|u_1-u_2\|+\frac{\sigma\rho_2\varepsilon_2}{\gamma_2}\|w_1-w_2\|
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
&\|P_{\rho_3}(u_1,v_1,w_1)-P_{\rho_3}(u_2,v_2,w_2)\| \\
&\le \frac{\sigma}{\gamma_3}(\tau_3^q-q\rho_3r_3+c_q\rho_3^qs_3^q)^{1/q}\|w_1-w_2\| \\
&\quad+\frac{\sigma\rho_3\alpha_1}{\gamma_3}\|u_1-u_2\|+\frac{\sigma\rho_3\alpha_2}{\gamma_3}\|v_1-v_2\|.
\end{aligned} \tag{4.40}$$

Therefore, we have

$$\begin{aligned}
&\|T_{\rho_1}(u_1,v_1,w_1)-T_{\rho_1}(u_2,v_2,w_2)\| \\
&\quad+\|S_{\rho_2}(u_1,v_1,w_1)-S_{\rho_2}(u_2,v_2,w_2)\|+\|P_{\rho_3}(u_1,v_1,w_1)-P_{\rho_3}(u_2,v_2,w_2)\| \\
&\le [\frac{\sigma}{\gamma_1}(\tau_1^q-q\rho_1r_1+c_q\rho_1^qs_1^q)^{1/q}+\frac{\sigma\rho_2\varepsilon_1}{\gamma_2}+\frac{\sigma\rho_3\alpha_1}{\gamma_3}]\|u_1-u_2\| \\
&\quad+[\frac{\sigma}{\gamma_2}(\tau_2^q-q\rho_2r_2+c_q\rho_2^qs_2^q)^{1/q}+\frac{\sigma\rho_1\theta_1}{\gamma_1}+\frac{\sigma\rho_3\alpha_2}{\gamma_3}]\|v_1-v_2\| \\
&\quad+[\frac{\sigma}{\gamma_3}(\tau_3^q-q\rho_3r_3+c_q\rho_3^qs_3^q)^{1/q}+\frac{\sigma\rho_1\theta_2}{\gamma_1}+\frac{\sigma\rho_2\varepsilon_2}{\gamma_2}]\|w_1-w_2\| \\
&\le k(\|u_1-u_2\|+\|v_1-v_2\|+\|w_1-w_2\|),
\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}
k=\max\Big\{&\frac{\sigma}{\gamma_1}(\tau_1^q-q\rho_1r_1+c_q\rho_1^qs_1^q)^{1/q}+\frac{\sigma\rho_2\varepsilon_1}{\gamma_2}+\frac{\sigma\rho_3\alpha_1}{\gamma_3}, \\
&\frac{\sigma}{\gamma_2}(\tau_2^q-q\rho_2r_2+c_q\rho_2^qs_2^q)^{1/q}+\frac{\sigma\rho_1\theta_1}{\gamma_1}+\frac{\sigma\rho_3\alpha_2}{\gamma_3}, \\
&\frac{\sigma}{\gamma_3}(\tau_3^q-q\rho_3r_3+c_q\rho_3^qs_3^q)^{1/q}+\frac{\sigma\rho_1\theta_2}{\gamma_1}+\frac{\sigma\rho_2\varepsilon_2}{\gamma_2}\Big\}.
\end{aligned}$$

It is easy to show that $(X_1\times X_2\times X_3,\|\cdot\|_1)$ is a Banach space, where $\|\cdot\|_1$ is defined as following, that is, for any $(u,v,w)\in X_1\times X_2\times X_3$,

$$\|(u,v,w)\|_1=\|u\|+\|v\|+\|w\|.$$

For any fixed $\rho_1>0,\rho_2>0$ and $\rho_3>0$, we define

$$Q_{\rho_1,\rho_2,\rho_3}:X_1\times X_2\times X_3\to X_1\times X_2\times X_3$$

as following, that is, for any $(u,v,w)\in X_1\times X_2\times X_3$,

$$Q_{\rho_1,\rho_2,\rho_3}(u,v,w)=(T_{\rho_1}(u,v,w),S_{\rho_2}(u,v,w),P_{\rho_3}(u,v,w)).$$

From condition (d) we know $0 < k < 1$, and then (4.41) implies

$$\begin{aligned}
&\|Q_{\rho_1,\rho_2,\rho_3}(u_1,v_1,w_1) - Q_{\rho_1,\rho_2,\rho_3}(u_2,v_2,w_2)\|_1 \\
&= \|(T_{\rho_1}(u_1,v_1,w_1), S_{\rho_2}(u_1,v_1,w_1), P_{\rho_3}(u_1,v_1,w_1)) \\
&\quad - (T_{\rho_1}(u_2,v_2,w_2), S_{\rho_2}(u_2,v_2,w_2), P_{\rho_3}(u_2,v_2,w_2))\|_1 \\
&= \|(T_{\rho_1}(u_1,v_1,w_1) - T_{\rho_1}(u_2,v_2,w_2), S_{\rho_2}(u_1,v_1,w_1) \\
&\quad - S_{\rho_2}(u_2,v_2,w_2), P_{\rho_3}(u_1,v_1,w_1) - P_{\rho_3}(u_2,v_2,w_2))\|_1 \\
&= \|T_{\rho_1}(u_1,v_1,w_1) - T_{\rho_1}(u_2,v_2,w_2\| \\
&\quad + \|S_{\rho_2}(u_1,v_1,w_1) - S_{\rho_2}(u_2,v_2,w_2)\| + \|P_{\rho_3}(u_1,v_1,w_1) - P_{\rho_3}(u_2,v_2,w_2)\| \\
&\le k(\|u_1 - u_2\| + \|v_1 - v_2\| + \|w_1 - w_2\|) \\
&= k(\|(u_1 - u_2, v_1 - v_2, w_1 - w_2)\|_1 \\
&= k(\|(u_1,v_1,w_1) - (u_2,v_2,w_2)\|_1,
\end{aligned}$$

which shows that $Q_{\rho_1,\rho_2,\rho_3} : X_1 \times X_2 \times X_3 \to X_1 \times X_2 \times X_3$ is an operator. Therefore there exists a unique element $(a,b,c) \in X_1 \times X_2 \times X_3$ such that

$$Q_{\rho_1,\rho_2,\rho_3}(a,b,c) = (a,b,c),$$

which is equivalent to

$$\begin{aligned}
a &= R^{H_1\eta}_{M\rho_1}[H_1(a) - \rho_1 F(a,b,c)], \\
b &= R^{H_2\eta}_{N\rho_2}[H_2(b) - \rho_2 G(a,b,c)], \\
c &= R^{H_3\eta}_{T\rho_3}[H_3(c) - \rho_3 H(a,b,c)].
\end{aligned}$$

By Lemma 2.10 we know that (a,b,c) is the unique solution to Problem D. This completes the proof. □

Theorem 4.3. *Let $\eta, H_1, H_2, H_3, M, N, T, F, G$ and H be the same as those in Theorem 4.2. Assume that all the conditions in Theorem 4.2 are satisfied. Then the sequence $\{(a_n, b_n, c_n)\}$ generated by Algorithm 2.4 converges strongly to the unique solution (a,b,c) to Problem D, and there exists $0 \le d < 1$ satisfying*

$$\|a_n - a\| + \|b_n - b\| + \|c_n - c\| \le d^n(\|a_0 - a\| + \|b_0 - b\| + \|c_0 - c\|), \quad \forall n \ge 0.$$

Proof. Theorem 4.2 gives a unique solution (a,b,c) to Problem D. By Lemma 2.10 we know that

$$a = \beta_n a + (1-\beta_n) R^{H_1\eta}_{M\rho_1}[H_1(a) - \rho_1 F(a,b,c)], \tag{4.42}$$

$$b = \beta_n b + (1-\beta_n) R^{H_2\eta}_{N\rho_2}[H_2(b) - \rho_2 G(a,b,c)], \tag{4.43}$$

$$c = \beta_n c + (1-\beta_n) R^{H_3\eta}_{T\rho_3}[H_3(c) - \rho_3 H(a,b,c)]. \tag{4.44}$$

By Algorithm 2.4 and (4.42) we have

$$\begin{aligned}
&\|a_{n+1}-a\| \\
&= \|\beta_n a_n + (1-\beta_n) R_{M\rho_1}^{H_1\eta}[H_1(a_n) - \rho_1 F(a_n,b_n,c_n)] \\
&\quad - \beta_n a - (1-\beta_n) R_{M\rho_1}^{H_1\eta}[H_1(a) - \rho_1 F(a,b,c)]\| \\
&\le \beta_n \|a_n - a\| + (1-\beta_n)\frac{\sigma}{\gamma_1}(\tau_1^q - q\rho_1 r_1 + c_q \rho_1^q s_1^q)^{1/q}\|a_n - a\| \\
&\quad + (1-\beta_n)\frac{\sigma\rho_1\theta_1}{\gamma_1}\|b_n - b\| + (1-\beta_n)\frac{\sigma\rho_1\theta_2}{\gamma_1}\|c_n - c\|. \qquad (4.45)
\end{aligned}$$

By Algorithm 2.4 and (4.43) we have

$$\begin{aligned}
&\|b_{n+1}-b\| \\
&= \|\beta_n b_n + (1-\beta_n) R_{N\rho_2}^{H_2\eta}[H_2(b_n) - \rho_2 G(a_n,b_n,c_n)] \\
&\quad - \beta_n b - (1-\beta_n) R_{N\rho_2}^{H_2\eta}[H_2(b) - \rho_2 G(a,b,c)]\| \\
&\le \beta_n \|b_n - b\| + (1-\beta_n)\frac{\sigma}{\gamma_2}(\tau_2^q - q\rho_2 r_2 + c_q \rho_2^q s_2^q)^{1/q}\|b_n - b\| \\
&\quad + (1-\beta_n)\frac{\sigma\rho_2\varepsilon_1}{\gamma_2}\|a_n - a\| + (1-\beta_n)\frac{\sigma\rho_2\varepsilon_2}{\gamma_2}\|c_n - c\|. \qquad (4.46)
\end{aligned}$$

And also, from Algorithm 2.4 and (4.44) we have

$$\begin{aligned}
&\|c_{n+1}-c\| \\
&= \|\beta_n c_n + (1-\beta_n) R_{T\rho_3}^{H_3\eta}[H_3(c_n) - \rho_3 H(a_n,b_n,c_n)] \\
&\quad - \beta_n c - (1-\beta_n) R_{T\rho_3}^{H_3\eta}[H_3(c) - \rho_3 H(a,b,c)]\| \\
&\le \beta_n \|c_n - c\| + (1-\beta_n)\frac{\sigma}{\gamma_3}(\tau_3^q - q\rho_3 r_3 + c_q \rho_3^q s_3^q)^{1/q}\|c_n - c\| \\
&\quad + (1-\beta_n)\frac{\sigma\rho_3\alpha_1}{\gamma_3}\|a_n - a\| + (1-\beta_n)\frac{\sigma\rho_3\alpha_2}{\gamma_3}\|b_n - b\|. \qquad (4.47)
\end{aligned}$$

It follows from (4.45) – (4.47) that

$$\begin{aligned}
&\|a_{n+1}-a\| + \|b_{n+1}-b\| + \|c_{n+1}-c\| \\
&\le \beta_n(\|a_n - a\| + \|b_n - b\| + \|c_n - c\|) \\
&\quad + (1-\beta_n)k(\|a_n - a\| + \|b_n - b\| + \|c_n - c\|) \\
&= [k + (1-k)\beta_n](\|a_n - a\| + \|b_n - b\| + \|c_n - c\|), \qquad (4.48)
\end{aligned}$$

where

$$k = \max\left\{ \frac{\sigma}{\gamma_1}(\tau_1^q - q\rho_1 r_1 + c_q \rho_1^q s_1^q)^{1/q} + \frac{\sigma\rho_2\varepsilon_1}{\gamma_2} + \frac{\sigma\rho_3\alpha_1}{\gamma_3}, \frac{\sigma}{\gamma_2}(\tau_2^q - q\rho_2 r_2 + c_q \rho_2^q s_2^q)^{1/q} + \frac{\sigma\rho_1\theta_1}{\gamma_1} + \frac{\sigma\rho_3\alpha_2}{\gamma_3}, \frac{\sigma}{\gamma_3}(\tau_3^q - q\rho_3 r_3 + c_q \rho_3^q s_3^q)^{1/q} + \frac{\sigma\rho_1\theta_2}{\gamma_1} + \frac{\sigma\rho_2\varepsilon_2}{\gamma_2} \right\}.$$

In this case, we know that $0 \leq k < 1$. Let $c_n = \|a_n - a\| + \|b_n - b\| + \|c_n - c\|$ and $k_n = k + (1-k)\beta_n$.,Then (4.48) can be written as

$$c_{n+1} \leq k_n c_n, \quad n = 0, 1, 2 \cdots.$$

By Algorithm 2.4 we have $\limsup_{n\to\infty} k_n < 1$. By using Lemma 2.6 we deduce $0 \leq k_n \leq d < 1$ and

$$\|a_n - a\| + \|b_n - b\| + \|c_n - c\| \leq d^n(\|a_0 - a\| + \|b_0 - b\| + \|c_0 - c\|), \quad n = 0, 1, 2 \cdots.$$

Therefore, $\{(a_n, b_n, c_n)\}$ converges strongly to the unique solution to Problem D. This completes the proof. □

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