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ON A M-ORDER NONLINEAR INTEGRODIFFERENTIAL EQUATION ON TWO VARIABLES

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. In this paper, we study the existence and the compactness of the set of solutions for m-order nonlinear integrodifferential equation on two variables. The main tools are the

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fixed point theorems together with the definition of a suitable Banach space and appropriate conditions for subsets to be relatively compact in this space. Two illustrative examples are given.

1. Introduction

In this paper, we consider the following m-order nonlinear integrodifferential equation in two variables

$$u(x,y) = g(x,y) + \iint_{\Omega} K(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))dsdt, \quad (1.1)$$

where $(x,y) \in \Omega = [0,1] \times [0,1]$ and $g : \Omega \to \mathbb{R}$, $K : \Omega \times \Omega \times \mathbb{R}^{m+1} \to \mathbb{R}$ are given functions. Denote by $D_1^i u = \frac{\partial^i u}{\partial x^i}$, the partial derivative of order $i = 1,...,m$ of a function u defined on Ω, with respect to the first variable.

It is well known that integral and integrodifferential equations have attracted the interest of scientists not only because of their major role in the fields of functional analysis but also because of their important role in numerous applications, for example, mechanics, physics, population dynamics, economics and other fields of science, see Corduneanu [8], Deimling [9].

There are many different methods to solve the integral and integrodifferential equations (see, for example, see [1]-[23] and the references given therein). In [3], his homotopy perturbation method was applied to solve linear and nonlinear systems of integro-differential equations. In [20], based on the applications of the well-known Banach fixed point theorem coupled with Bielecki type norm and a certain integral inequality with explicit estimate, Pachpatte proved uniqueness and other properties of solutions of the following Fredholm type integrodifferential equation

$$x(t) = g(t) + \int_a^b f(t,s,x(s),x'(s),...,x^{(n-1)}(s))ds, \ t \in [a,b],$$

where x, g, f are real valued functions and $n \geq 2$ is an integer. With the same methods, Pachpatte studied the existence, uniqueness and some basic properties of solutions of the Fredholm type integral equation in two variables as follows (see [21]):

$$u(x,y) = f(x,y) + \int_0^a \int_0^b g\left(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)\right)dtds.$$

Abdou et al. also considered the existence of integrable solution of nonlinear integral equation, of type Hammerstein-Volterra of the second kind, by using the technique of measure of weak noncompactness and Schauder fixed point theorem (see [2]).

Recently, in [10]-[12], [15]-[19], using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, we have investigated solvability and asymptotically stable of nonlinear functional integral equations on one variable or two variables, or N variables.

In the base of the above works, we consider (1.1). This paper is organized as follows. In section 2, we present some preliminaries. It consists of the definition of a suitable Banach space and a sufficient condition for relatively compact subsets. In section 3, by applying the Banach theorem and the Schauder theorem, we prove two existence theorems. Furthermore, the compactness of solutions set is also proved. In order to illustrate the results obtained here, two examples are given.

2. Preliminaries

First, we construct an appropriate Banach space for (1.1) as follows. By $X = C(\Omega; \mathbb{R})$, we denote the space of all continuous functions from Ω into $\mathbb{R}$ equipped with the following norm

$$\|u\|_X = \sup_{(x,y)\in\Omega} |u(x, y)|, \ u \in X. \tag{2.1}$$

Put

$$X_m = \{u \in X = C(\Omega; \mathbb{R}) : D_1^i u \in X, \ i = 1, 2, ..., m\}. \tag{2.2}$$

We remark that

$$C^1(\Omega; \mathbb{R})\backslash X_m \neq \phi, \ \ X_m\backslash C^1(\Omega; \mathbb{R}) \neq \phi, \ \ X_m \cap C^1(\Omega; \mathbb{R}) \neq \phi,$$

for all $m = 2, 3, ...$

Indeed,

(i) We have $u = u(x, y) = \left|x - \frac{1}{2}\right| \left(x - \frac{1}{2}\right) \left|y - \frac{1}{3}\right| \left(y - \frac{1}{3}\right) \in C^1(\Omega; \mathbb{R})$, but $u \notin X_m$.

(ii) We also have $v = v(x, y) = x^{m+1} \left|y - \frac{1}{3}\right| \in X_m$, but $v \notin C^1(\Omega; \mathbb{R})$.

(iii) With $w(x, y) = e^{x+y}$, we have $w \in X_m \cap C^1(\Omega; \mathbb{R})$, so

$$X_m \cap C^1(\Omega; \mathbb{R}) \neq \phi.$$

We shall need the following lemma, the proof of which can be found in [10, p.266-267].

Lemma 2.1. ([10]) *X_m is a Banach space with the norm defined by*

$$\|u\|_{X_m} = \|u\|_X + \sum_{i=1}^{m} \left\|D_1^i u\right\|_X, \ u \in X_m. \tag{2.3}$$

Next, we give a sufficient condition for relatively compact subsets of X_m.

Lemma 2.2. *Let $\mathcal{F} \subset X_m$. Then $\mathcal{F}$ is relatively compact in X_m if and only if the following conditions are satisfied*

(i) $\exists M > 0 : \|u\|_{X_m} \le M,\ \forall u \in \mathcal{F}$;

(ii) $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall (x,y),\ (\bar{x},\bar{y}) \in \Omega$,

$$|x-\bar{x}| + |y-\bar{y}| < \delta \quad \Longrightarrow \quad \sup_{u\in\mathcal{F}} [u(x,y) - u(\bar{x},\bar{y})]_* < \varepsilon, \tag{2.4}$$

where we denote $[u(x,y)]_* = |u(x,y)| + \sum_{i=1}^{m} \left|D_1^i u(x,y)\right|$.

Proof. Let $\mathcal{F}$ be relatively compact in X_m. Then $\mathcal{F}$ is bounded, so (2.4) (i) is true. Now, we show that (2.4) (ii) is also true.

For every $\varepsilon > 0$, considering a collection of open balls in X_m, with center at $u \in \mathcal{F}$ and radius $\frac{\varepsilon}{3}$, as follows

$$B(u, \frac{\varepsilon}{3}) = \{\bar{u} \in X_m : \|u - \bar{u}\|_{X_m} < \frac{\varepsilon}{3}\},\ u \in \mathcal{F}.$$

It is clear that $\overline{\mathcal{F}} \subset \bigcup_{u\in\mathcal{F}} B(u, \frac{\varepsilon}{3})$. Because $\overline{\mathcal{F}}$ compact in X_m, the open cover $\bigcup_{u\in\mathcal{F}} B(u, \frac{\varepsilon}{3})$ of $\overline{\mathcal{F}}$ contains a finite subcover, so there are $u_1, ..., u_q \in \mathcal{F}$ such that

$$\overline{\mathcal{F}} \subset \bigcup\nolimits_{j=1}^{q} B(u_j, \frac{\varepsilon}{3}).$$

By the functions $u_j,\ D_1^i u_j,\ i = 1, ..., m,\ j = 1, ..., q$ are uniformly continuous on Ω, there exists $\delta > 0$ such that, for all $(x,y),\ (\bar{x},\bar{y}) \in \Omega,\ |x-\bar{x}| + |y-\bar{y}| < \delta$, we have

$$[u_j(x,y) - u_j(\bar{x},\bar{y})]_* < \frac{\varepsilon}{3},\ \forall j = 1, ..., q.$$

For all $u \in \mathcal{F}$, $u \in B(u_j, \frac{\varepsilon}{3})$ for some $j = 1, ..., q$. Thus, for all $(x,y),\ (\bar{x},\bar{y}) \in \Omega$, if $|x-\bar{x}| + |y-\bar{y}| < \delta$ then we obtain

$$\begin{aligned} [u(x,y) - u(\bar{x},\bar{y})]_* &\le [u(x,y) - u_j(x,y)]_* + [u_j(x,y) - u_j(\bar{x},\bar{y})]_* \\ &\quad + [u_j(\bar{x},\bar{y}) - u(\bar{x},\bar{y})]_* \\ &\le 2\|u - u_j\|_{X_m} + [u_j(x,y) - u_j(\bar{x},\bar{y})]_* \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

It implies that (2.4) (ii) is true.

Conversely, suppose that the conditions (i), (ii) hold. To prove that $\mathcal{F}$ is relatively compact in X_m, let $\{u_p\}$ be a sequence in $\mathcal{F}$, we show that $\{u_p\}$ contains a convergent subsequence. By (2.4), $\mathcal{F}_1 = \{u_p : p \in \mathbb{N}\}$ and $\mathcal{F}_2^i = \{D_1^i u_p : p \in \mathbb{N}\}$ are uniformly bounded and equicontinuous in X. Applying the Ascoli-Arzela theorem to $\mathcal{F}_1$, it is relatively compact in X, so there exists a subsequence $\{u_{p_k}\}$ of $\{u_p\}$ and $u \in X$ such that

$$\|u_{p_k} - u\|_X \to 0, \text{ as } k \to \infty.$$

Note that $\{D_1^i u_{p_k} : k \in \mathbb{N}\} \subset \mathcal{F}_2^i$ is also uniformly bounded and equicontinuous in X, so it is also relatively compact in X. We obtain the existence of a subsequence of $\{D_1^i u_{p_k}\}$, denoted by the same symbol, and $w^{(i)} \in X$, such that

$$\left\| D_1^i u_{p_k} - w^{(i)} \right\|_X \to 0, \text{ as } k \to \infty.$$

Since

$$u_{p_k}(x,y) - u_{p_k}(0,y) = \int_0^x D_1 u_{p_k}(s,y)ds, \ \forall (x,y) \in \Omega,$$

furthermore $\|u_{p_k} - u\|_X \to 0$ and $\left\|D_1 u_{p_k} - w^{(1)}\right\|_X \to 0$, we obtain

$$u(x,y) - u(0,y) = \int_0^x w^{(1)}(s,y)ds, \ \forall (x,y) \in \Omega.$$

It gives $D_1 u = w^{(1)} \in X$. Let $D_1^i u = w^{(i)}$, $i = 1, 2, ..., r < m$. We have

$$D_1^r u_p(x,y) - D_1^r u_p(0,y) = \int_0^x D_1^{r+1} u_p(s,y)ds, \ \forall (x,y) \in \Omega. \tag{2.5}$$

Since $\|D_1^r u_p - D_1^r u\|_X \to 0$ and $\left\|D_1^{r+1} u_p - w^{(r+1)}\right\|_X \to 0$, (2.5) leads to

$$D_1^r u(x,y) - D_1^r u(0,y) = \int_0^x w^{(r+1)}(s,y)ds, \ \forall (x,y) \in \Omega. \tag{2.6}$$

Then $D_1^{r+1} u = w^{(r+1)} \in X$. By induction, we deduce that $D_1^i u = w^{(i)}$, $i = 1, 2, ..., m$. Therefore $u \in X_m$ and $u_{p_k} \to u$ in X_m. This completes the proof. □

3. The existence theorems

We make the following assumptions.

(A_1) $g \in X_m$,

(A_2) $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$, such that

$$\frac{\partial K}{\partial x}, \frac{\partial^2 K}{\partial x^2}, ..., \frac{\partial^m K}{\partial x^m} \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R}),$$

and there exist nonnegative functions $k_0, k_1, ..., k_m : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\beta = \sum_{i=0}^{m} \sup_{(x,y)\in\Omega} \iint_\Omega k_i(x,y,s,t)dsdt < 1$,

(ii) $\forall(x,y,s,t)\in\Omega\times\Omega,\ \forall(u_0,u_1,...,u_m),(\bar{u}_0,\bar{u}_1,...,\bar{u}_m)\in\mathbb{R}^{m+1}$,

$$|K(x,y,s,t;u_0,u_1,...,u_m)-K(x,y,s,t;\bar{u}_0,\bar{u}_1,...,\bar{u}_m)| \le k_0(x,y,s,t)\sum_{j=0}^{m}|u_j-\bar{u}_j|,$$

(iii) $\forall(x,y,s,t)\in\Omega\times\Omega,\ \forall(u_0,u_1,...,u_m),(\bar{u}_0,\bar{u}_1,...,\bar{u}_m)\in\mathbb{R}^{m+1}$,

$$\left|\frac{\partial^i K}{\partial x^i}(x,y,s,t;u_0,u_1,...,u_m)-\frac{\partial^i K}{\partial x^i}(x,y,s,t;\bar{u}_0,\bar{u}_1,...,\bar{u}_m)\right| \le k_i(x,y,s,t)\sum_{j=0}^{m}|u_j-\bar{u}_j|,$$

Theorem 3.1. *Let the functions g, K in (1.1) satisfy the assumptions (A_1), (A_2). Then the equation (1.1) has a unique solution in X_m.*

Proof. For every $u\in X_m$, we put

$$(Au)(x,y)=g(x,y) + \iint_\Omega K(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))dsdt,\ (x,y)\in\Omega. \tag{3.1}$$

It is obvious that $Au\in X_m$. We shall show that $A:X_m\to X_m$ is a contraction map, by proving

$$\|Au-A\bar{u}\|_{X_m}\le\beta\|u-\bar{u}\|_{X_m},\ \forall u,\ \bar{u}\in X_m. \tag{3.2}$$

For any u, $\bar{u}\in X_m$, and $(x,y)\in\Omega$, from (A_2)-(ii), (3.1) leads to

$$\begin{aligned}
&|(Au)(x,y)-(A\bar{u})(x,y)|\\
&\le\iint_\Omega |K(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))\\
&\qquad -K(x,y,s,t;\bar{u}(s,t),D_1\bar{u}(s,t),...,D_1^m\bar{u}(s,t))|\,dsdt\\
&\le\iint_\Omega k_0(x,y,s,t)\sum_{j=0}^{m}\left|D_1^j u(s,t)-D_1^j\bar{u}(s,t)\right|dsdt\\
&\le\left(\sup_{(x,y)\in\Omega}\iint_\Omega k_0(x,y,s,t)dsdt\right)\|u-\bar{u}\|_{X_m}.
\end{aligned}$$

Hence we have

$$\|Au-A\bar{u}\|_X\le\left(\sup_{(x,y)\in\Omega}\iint_\Omega k_0(x,y,s,t)dsdt\right)\|u-\bar{u}\|_{X_m}. \tag{3.3}$$

Similarly, by

$$D_1^i(Au)(x,y) = D_1^i g(x,y) + \iint_\Omega \frac{\partial^i K}{\partial x^i}(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))dsdt,$$

and (A_2)-(ii) we obtain

$$\begin{aligned}
&\left|D_1^i(Au)(x,y) - D_1^i(A\bar{u})(x,y)\right| \\
&\leq \iint_\Omega \left| \frac{\partial^i K}{\partial x^i}(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t)) \right. \\
&\quad \left. -\frac{\partial^i K}{\partial x^i}(x,y,s,t;\bar{u}(s,t),D_1\bar{u}(s,t),...,D_1^m \bar{u}(s,t)) \right| dsdt \\
&\leq \iint_\Omega k_i(x,y,s,t) \sum_{j=0}^m \left| D_1^j u(s,t) - D_1^j \bar{u}(s,t) \right| dsdt \\
&\leq \left(\sup_{(x,y)\in\Omega} \iint_\Omega k_i(x,y,s,t)dsdt \right) \|u-\bar{u}\|_{X_m}.
\end{aligned}$$

Hence we have

$$\left\|D_1^i(Au) - D_1^i(A\bar{u})\right\|_X \leq \left(\sup_{(x,y)\in\Omega} \iint_\Omega k_i(x,y,s,t)dsdt \right) \|u-\bar{u}\|_{X_m}. \quad (3.4)$$

From (3.3) and (3.4), we have (3.2). Applying the Banach fixed point theorem, Theorem 3.1 is proved. □

Next, we also obtain the existence of solutions of (1.1) in X_m via the Schauder fixed point theorem, by making the following assumptions.

(A_1) $g \in X_m$,

$(\bar{A}_2)$ $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1};\mathbb{R})$, such that

$$\frac{\partial K}{\partial x}, \frac{\partial^2 K}{\partial x^2}, ..., \frac{\partial^m K}{\partial x^m} \in C(\Omega \times \Omega \times \mathbb{R}^{m+1};\mathbb{R}),$$

and there exist nonnegative functions $\bar{k}_0, \bar{k}_1, ..., \bar{k}_m : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\bar{\beta} = \sum\limits_{i=0}^m \sup\limits_{(x,y)\in\Omega} \iint_\Omega \bar{k}_i(x,y,s,t)dsdt < 1$,

(ii) $|K(x,y,s,t;u_0,u_1,...,u_m)| \leq \bar{k}_0(x,y,s,t)\left(1 + \sum\limits_{j=0}^m |u_j|\right)$,

$\forall (x,y,s,t) \in \Omega \times \Omega,\ \forall (u_0,u_1,...,u_m) \in \mathbb{R}^{m+1}$,

(iii) $\left|\frac{\partial^i K}{\partial x^i}(x,y,s,t;u_0,u_1,...,u_m)\right| \leq \bar{k}_i(x,y,s,t)\left(1+\sum_{j=0}^{m}|u_j|\right),$

$\forall(x,y,s,t)\in\Omega\times\Omega,\ \forall(u_0,u_1,...,u_m)\in\mathbb{R}^{m+1},\ i=1,2,...,m.$

Theorem 3.2. *Let the functions g, K in* (1.1) *satisfy the assumptions* (A_1), $(\bar{A}_2)$. *Then the equation* (1.1) *has a solution in X_m. Furthermore, the set of solutions is compact.*

Proof. With the operator A as in (3.1), it is clear that $A: X_m \to X_m$. For $M>0$, we define a closed ball in X_m as follows:

$$B_M = \{u \in X_m : \|u\|_{X_m} \leq M\}.$$

We shall show that there exists $M>0$ such that $A: B_M \to B_M$. For every $u \in B_M$ and $(x,y)\in\Omega$, we have

$$\begin{aligned}|(Au)(x,y)| &\leq |g(x,y)| + \iint_\Omega |K(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))|\,dsdt\\ &\leq \|g\|_X + \iint_\Omega \bar{k}_0(x,y,s,t)\left(1+\sum_{i=0}^{m}\left|D_1^i u(s,t)\right|\right)dsdt\\ &\leq \|g\|_X + \iint_\Omega \bar{k}_0(x,y,s,t)\left(1+\|u\|_{X_m}\right)dy\\ &\leq \|g\|_X + (1+M)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_0(x,y,s,t)dsdt\right).\end{aligned}$$

It implies that

$$\|Au\|_X \leq \|g\|_X + (1+M)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_0(x,y,s,t)dsdt\right). \tag{3.5}$$

Similarly,

$$\begin{aligned}\left|D_1^i(Au)(x,y)\right| &\leq \left|D_1^i g(x,y)\right|\\ &\quad+\iint_\Omega \left|\frac{\partial^i K}{\partial x^i}(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))\right| dsdt\\ &\leq \left\|D_1^i g\right\|_X + (1+M)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_i(x,y,s,t)dsdt\right),\end{aligned}$$

so we have

$$\left\|D_1^i(Au)\right\|_X \leq \left\|D_1^i g\right\|_X + (1+M)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_i(x,y,s,t)dsdt\right). \tag{3.6}$$

Therefore, we have

$$\|Au\|_{X_m} \leq \|g\|_{X_m} + (1+M) \sum_{i=0}^{m} \sup_{(x,y)\in\Omega} \iint_{\Omega} \bar{k}_i(x,y,s,t)dsdt \qquad (3.7)$$
$$\leq \|g\|_{X_m} + (1+M)\bar{\beta}.$$

Choosing $M \geq \|g\|_{X_m} + (1+M)\bar{\beta}$, i.e. $M \geq \frac{\|g\|_{X_m}+\bar{\beta}}{1-\bar{\beta}}$, then $A : B_M \to B_M$.

Now we show that two conditions as below are satisfied.

(i) $A : B_M \to B_M$ is continuous.

(ii) $\mathcal{F} = A(B_M)$ is relatively compact in X_m.

To prove (i), let $\{u_p\} \subset B_M$, $\|u_p - u\|_{X_m} \to 0$, as $m \to \infty$, we need to show that

$$\|Au_p - Au\|_X \to 0 \text{ and } \sum_{i=1}^{m} \left\|D_1^i(Au_p) - D_1^i(Au)\right\|_X \to 0, \text{ as } p \to \infty. \qquad (3.8)$$

Note that

$$|(Au_p)(x,y) - (Au)(x,y)| \qquad (3.9)$$
$$\leq \iint_{\Omega} |K(x,y,s,t;u_p(s,t), D_1u_p(s,t), ..., D_1^m u_p(s,t))$$
$$-K(x,y,s,t;u(s,t), D_1u(s,t), ..., D_1^m u(s,t))|\, dsdt.$$

Since K is uniformly continuous on $\Omega \times \Omega \times [-M,M]^{m+1}$, for $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(u_0, u_1, ..., u_m)$, $(\bar{u}_0, \bar{u}_1, ..., \bar{u}_m) \in [-M,M]^{m+1}$ and $(x,y,s,t) \in \Omega \times \Omega$,

$$\sum_{i=0}^{m} |u_i - \bar{u}_i| < \delta$$
$$\Longrightarrow |K(x,y,s,t;u_0,u_1,...,u_m) - K(x,y,s,t;\bar{u}_0,\bar{u}_1,...,\bar{u}_m)| < \varepsilon.$$

Since $\|u_p - u\|_X \to 0$ and $\sum_{i=1}^{m} \left\|D_1^i u_p - D_1^i u\right\|_X \to 0$, there is $p_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ with $p \geq p_0$,

$$\|u_p - u\|_X + \sum_{i=1}^{m} \left\|D_1^i u_p - D_1^i u\right\|_X < \delta.$$

It implies that for all $p \in \mathbb{N}$ with $p \geq p_0$ and $(x,y,s,t) \in \Omega \times \Omega$,

$$|K(x,y,s,t;u_p(s,t), D_1u_p(s,t), ..., D_1^m u_p(s,t))$$
$$-K(x,y,s,t;u(s,t), D_1u(s,t), ..., D_1^m u(s,t))| < \varepsilon.$$

Hence we have

$$|(Au_p)(x,y) - (Au)(x,y)| < \varepsilon, \ \forall (x,y) \in \Omega, \ \forall p \geq p_0.$$

It means that

$$\|Au_p - Au\|_X < \varepsilon, \ \forall p \geq p_0, \tag{3.10}$$

i.e., $\|Au_p - Au\|_X \to 0$, as $p \to \infty$. By the same argument, we obtain that

$$\left\|D_1^i(Au_p) - D_1^i(Au)\right\|_X \to 0,$$

as $p \to \infty$, for each $i = 1, ..., m$.

To prove (ii), we use Lemma 2.2. Condition (2.4) (i) holds because of $\mathcal{F} = A(B_M) \subset B_M$. It remains to show (2.4) (ii). We note that

$$\begin{aligned}
&(Au)(x,y) - (Au)(\bar{x},\bar{y}) \\
&= g(x,y) - g(\bar{x},\bar{y}) \\
&\quad + \iint_\Omega [K(x,y,s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t)) \\
&\quad - K(\bar{x},\bar{y},s,t;u(s,t),D_1u(s,t),...,D_1^m u(s,t))]\, dsdt,
\end{aligned} \tag{3.11}$$

for all (x,y), $(\bar{x},\bar{y}) \in \Omega$, $u \in B_M$. Let $\varepsilon > 0$. By the fact that K is uniformly continuous on $\Omega \times \Omega \times [-M,M]^{m+1}$, there exists $\delta_1 > 0$ such that for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned}
&|x-\bar{x}| + |y-\bar{y}| < \delta_1 \\
&\Longrightarrow |K(x,y,s,t;u_0,u_1,...,u_m) - K(\bar{x},\bar{y},s,t;u_0,u_1,...,u_m)| < \frac{\varepsilon}{4},
\end{aligned}$$

for all $(s,t;u_0,u_1,...,u_m) \in \Omega \times [-M,M]^{m+1}$. Then, for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned}
|x-\bar{x}| + |y-\bar{y}| < \delta_1 \Longrightarrow &|K(x,y,s,t;u(y),D_1u(y),...,D_1^m u(y)) \\
&-K(\bar{x},\bar{y},s,t;u(y),D_1u(y),...,D_1^m u(y))| < \frac{\varepsilon}{4},
\end{aligned}$$

for all $(s,t,u) \in \Omega \times B_M$. Hence, for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned}
&|x-\bar{x}| + |y-\bar{y}| < \delta_1 \\
&\Longrightarrow \iint_\Omega |K(x,y,s,t;u(y),D_1u(y),...,D_1^m u(y)) \\
&\qquad -K(\bar{x},\bar{y},s,t;u(y),D_1u(y),...,D_1^m u(y))|\, dsdt < \frac{\varepsilon}{4}, \ \forall u \in B_M.
\end{aligned}$$

Since g is also uniformly continuous on Ω, there is $\delta_2 > 0$ such that

$$\forall (x,y), \ (\bar{x},\bar{y}) \in \Omega, \ \ |x-\bar{x}| + |y-\bar{y}| < \delta_2 \Longrightarrow |g(x,y) - g(\bar{x},\bar{y})| < \frac{\varepsilon}{4}.$$

Choose $\bar{\delta}_1 = \min\{\delta_1, \delta_2\}$, it gives for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$|x-\bar{x}| + |y-\bar{y}| < \bar{\delta}_1 \Longrightarrow |(Au)(x,y) - (Au)(\bar{x},\bar{y})| < \frac{\varepsilon}{2}, \ \forall u \in B_M. \tag{3.12}$$

It is similar to $\frac{\partial^i K}{\partial x^i}$, $D_1^i g$, so there is $\bar{\delta}_2 > 0$ such that for all (x, y), $(\bar{x}, \bar{y}) \in \Omega$,

$$\begin{aligned} &|x-\bar{x}| + |y-\bar{y}| < \bar{\delta}_2 \\ &\Longrightarrow \left|D_1^i(Au)(x,y) - D_1^i(Au)(\bar{x},\bar{y})\right| < \frac{\varepsilon}{2m}, \ \forall u \in B_M. \end{aligned} \tag{3.13}$$

It follows that, by choosing $\delta = \min\{\bar{\delta}_1, \bar{\delta}_2\}$, we have, for all (x, y), $(\bar{x}, \bar{y}) \in \Omega$,

$$\begin{aligned} |x-\bar{x}| + |y-\bar{y}| < \delta \Longrightarrow &[(Au)(x,y) - (Au)(\bar{x},\bar{y})]_* \\ &= |(Au)(x,y) - (Au)(\bar{x},\bar{y})| \\ &\quad + \sum_{i=1}^m \left|D_1^i(Au)(x,y) - D_1^i(Au)(\bar{x},\bar{y})\right| \\ &< \frac{\varepsilon}{2} + m\frac{\varepsilon}{2m} = \varepsilon, \ \forall u \in B_M. \end{aligned} \tag{3.14}$$

Using Lemma 2.2, $\mathcal{F} = A(B_M)$ is relatively compact in X_m. And applying the Schauder fixed point theorem, the existence of a solution is proved.

Next, we show that the set of solutions, $S = \{u \in B_M : u = Au\}$, is compact in X_m. By the compactness of the operator $A : B_M \to B_M$ and $S = A(S)$, we only prove that S is closed. Let $\{u_p\} \subset S$, $\|u_p - u\|_{X_m} \to 0$. The continuity of A leads to

$$\begin{aligned} \|u - Au\|_{X_m} &\le \|u - u_p\|_{X_m} + \|u_p - Au\|_{X_m} \\ &= \|u - u_p\|_{X_m} + \|Au_p - Au\|_{X_m} \to 0, \end{aligned}$$

so $u = Au \in S$. Theorem 3.2 is proved. □

To the end, we illustrate the results obtained here by two examples.

Example 3.3. We consider (1.1), with the functions g, K as follows

$$\begin{cases} K(x,y,s,t;u_0,u_1,...,u_m) \\ = k(x,y)\left[(st)^{\alpha_0}\sin\left(\frac{\pi u_0}{2w_0(s,t)}\right) + \sum\limits_{i=1}^m (st)^{\alpha_i}\cos\left(\frac{2\pi u_i}{D_1^i w_0(s,t)}\right)\right], \\ g(x,y) = w_0(x,y) - \sum\limits_{j=0}^m \frac{1}{(1+\alpha_j)^2} k(x,y), \end{cases} \tag{3.15}$$

where

$$w_0(x,y) = e^x + x^{\gamma_1}|y-\alpha|^{\gamma_2}, \ k(x,y) = x^{\tilde{\gamma}_1}|y-\tilde{\alpha}|^{\tilde{\gamma}_2}, \tag{3.16}$$

and α, γ_1, γ_2, $\tilde{\alpha}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, α_0, $\alpha_1, ..., \alpha_m$ are positive constants satisfying

$$\begin{cases} 0 < \alpha < 1,\ 0 < \gamma_2 \le 1,\ \gamma_1 > m, \\ 0 < \tilde{\alpha} < 1,\ 0 < \tilde{\gamma}_2 \le 1,\ \tilde{\gamma}_1 > m, \\ \alpha_0,\ \alpha_1, ..., \alpha_m > 0, \\ 2\pi \sum_{j=0}^{m} \frac{1}{(1+\alpha_j)^2} \left(1 + \sum_{i=1}^{m} \tilde{\gamma}_1 (\tilde{\gamma}_1 - 1) ... (\tilde{\gamma}_1 - i + 1)\right) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} < 1. \end{cases} \tag{3.17}$$

We have

$$\begin{aligned} w_0(x,y) &= e^x + x^{\gamma_1} |y-\alpha|^{\gamma_2}, \\ D_1^i w_0(x,y) &= e^x + \gamma_1 (\gamma_1 - 1) ... (\gamma_1 - i + 1) x^{\gamma_1 - i} |y-\alpha|^{\gamma_2}, \end{aligned}$$

so w_0, $D_1^i w_0 \in X$ and $w_0(x,y) \ge 1$, $D_1^i w_0(x,y) \ge 1$. Hence $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$. We now prove that (A_1), (A_2) hold. It is obviously that (A_1) holds, by w_0, $k \in X_m$.

Assumption (A_2) holds, by the fact that

First, $D_1^i k \in X$,

$$\frac{\partial^i K}{\partial x^i} = D_1^i k(x,y) \left[(st)^{\alpha_0} \sin\left(\frac{\pi u_0}{2 w_0(s,t)}\right) + \sum_{i=1}^{m} (st)^{\alpha_i} \cos\left(\frac{2\pi u_i}{D_1^i w_0(s,t)}\right) \right],$$

so $\frac{\partial^i K}{\partial x^i} \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$;

$$\begin{aligned} &|K(x,y,s,t;u_0,u_1,...,u_m) - K(x,y,s,t;\bar{u}_0,\bar{u}_1,...,\bar{u}_m)| \\ &\le k(x,y) \left((st)^{\alpha_0} \frac{\pi |u_0 - \bar{u}_0|}{2 w_0(s,t)} + \sum_{i=1}^{m} (st)^{\alpha_i} \frac{2\pi |u_i - \bar{u}_i|}{D_1^i w_0(s,t)} \right) \\ &\le 2\pi k(x,y) \sum_{i=0}^{m} (st)^{\alpha_i} \sum_{j=0}^{m} |u_j - \bar{u}_j| \\ &\equiv k_0(x,y,s,t) \sum_{j=0}^{m} |u_j - \bar{u}_j|, \end{aligned} \tag{3.18}$$

in which

$$k_0(x,y,s,t) = 2\pi k(x,y) \sum_{j=0}^{m} (st)^{\alpha_j}; \tag{3.19}$$

$$\left|\frac{\partial^i K}{\partial x^i}(x,y,s,t;u_0,u_1,...,u_m) - \frac{\partial^i K}{\partial x^i}(x,y,s,t;\bar{u}_0,\bar{u}_1,...,\bar{u}_m)\right| \tag{3.20}$$
$$\leq k_i(x,y,s,t)\sum_{j=0}^{m}\left|u_j - \bar{u}_j\right|,$$

where

$$k_i(x,y,s,t) = 2\pi\left|D_1^i k(x,y)\right|\sum_{j=0}^{m}(st)^{\alpha_j}. \tag{3.21}$$

We have

$$\begin{aligned}\iint_\Omega k_i(x,y,s,t)dsdt &= 2\pi\left|D_1^i k(x,y)\right|\sum_{j=0}^{m}\iint_\Omega (st)^{\alpha_j}dsdt \\ &= 2\pi\sum_{j=0}^{m}\frac{1}{(1+\alpha_j)^2}\left|D_1^i k(x,y)\right| \\ &\leq 2\pi\sum_{j=0}^{m}\frac{1}{(1+\alpha_j)^2}\sup_{(x,y)\in\Omega}\left|D_1^i k(x,y)\right|.\end{aligned} \tag{3.22}$$

We also have the following lemma, it is clear, so we omit its proof.

Lemma 3.4. *Let positive constants* α, γ_2, γ_1 *satisfy* $0 < \alpha < 1$, $0 < \gamma_2 \leq 1 < \gamma_1$. *Then*

$$\begin{aligned} 0 &\leq x^{\gamma_1}\left|y-\alpha\right|^{\gamma_2} \leq \max\{\alpha^{\gamma_2},(1-\alpha)^{\gamma_2}\},\ \forall x,y\in[0,1], \\ 0 &\leq x^{\gamma_1-1}\left|y-\alpha\right|^{\gamma_2} \leq \max\{\alpha^{\gamma_2},(1-\alpha)^{\gamma_2}\},\ \forall x,y\in[0,1].\end{aligned}$$

Using Lemma 3.4, we get

$$\begin{aligned} &0 \leq k(x,y) = x^{\tilde{\gamma}_1}\left|y-\tilde{\alpha}\right|^{\tilde{\gamma}_2} \leq \max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}; \\ &0 \leq D_1 k(x,y) = \tilde{\gamma}_1 x^{\tilde{\gamma}_1-1}\left|y-\tilde{\alpha}\right|^{\tilde{\gamma}_2} \leq \tilde{\gamma}_1\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\ &\qquad\vdots \\ &0 \leq D_1^i k(x,y) = \tilde{\gamma}_1(\tilde{\gamma}_1-1)\ldots(\tilde{\gamma}_1-i+1)\,x^{\tilde{\gamma}_1-i}\left|y-\alpha\right|^{\tilde{\gamma}_2} \\ &\qquad \leq \tilde{\gamma}_1(\tilde{\gamma}_1-1)\ldots(\tilde{\gamma}_1-i+1)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\},\ i=1,...,m,\end{aligned} \tag{3.23}$$

so

$$\sum_{i=0}^{m} \sup_{(x,y)\in\Omega} \iint_{\Omega} k_i(x,y,s,t)dsdt \tag{3.24}$$
$$\leq 2\pi \sum_{j=0}^{m} \frac{1}{(1+\alpha_j)^2} \left(1 + \sum_{i=1}^{m} \tilde{\gamma}_1 (\tilde{\gamma}_1 - 1) ... (\tilde{\gamma}_1 - i + 1)\right) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\},$$

consequently

$$\beta = \sum_{i=0}^{m} \sup_{(x,y)\in\Omega} \iint_{\Omega} k_i(x,y,s,t)dsdt < 1. \tag{3.25}$$

Then, Theorem 3.1 is fulfilled. Morever, $w_0 \in X_m$ is also a unique solution of (1.1).

Example 3.5. We consider (1.1) with the functions K, g defined by

$$\begin{cases} K(x,y,s,t;u_0,u_1,...,u_m) = k(x,y)K_1(s,t;u_0,u_1,...,u_m), \\ g(x,y) = w_0(x,y) - 2\sum\limits_{j=0}^{m} \frac{1}{(1+\alpha_j)^2} k(x,y), \end{cases} \tag{3.26}$$

where

$$\begin{cases} K_1(s,t;u_0,u_1,...,u_m) \\ = (st)^{\alpha_0} \left(\frac{|u_0|}{w_0(s,t)} + \left| \frac{u_0}{w_0(s,t)} \right|^{1/4} \right) + \sum\limits_{i=1}^{m} (st)^{\alpha_i} \left(\frac{|u_i|}{D_1^i w_0(s,t)} + \left(\frac{u_i}{D_1^i w_0(s,t)} \right)^{1/3} \right), \\ w_0(x,y) = e^x + x^{\gamma_1} |y-\alpha|^{\gamma_2}, \;\; k(x,y) = x^{\tilde{\gamma}_1} |y-\tilde{\alpha}|^{\tilde{\gamma}_2}, \end{cases} \tag{3.27}$$

and α, γ_1, γ_2, $\tilde{\alpha}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, α_0, α_1, ..., α_m are positive constants satisfying

$$\begin{cases} 0 < \alpha < 1, \; 0 < \gamma_2 \leq 1, \; \gamma_1 > m, \\ 0 < \tilde{\alpha} < 1, \; 0 < \tilde{\gamma}_2 \leq 1, \; \tilde{\gamma}_1 > m, \\ \alpha_0, \; \alpha_1, ..., \; \alpha_m > 0, \\ 2\sum\limits_{j=0}^{m} \frac{1}{(1+\alpha_j)^2} \left(1 + \sum\limits_{i=1}^{m} \tilde{\gamma}_1 (\tilde{\gamma}_1 - 1) ... (\tilde{\gamma}_1 - i + 1)\right) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} < 1. \end{cases} \tag{3.28}$$

We can prove that (A_1), $(\bar{A}_2)$ hold, by the following.

First, w_0, $D_1^i w_0 \in X$ and $w_0(x,y) \geq 1$, $D_1^i w_0(x,y) \geq 1$. Then $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$.

By $D_1^i k \in X$, $\frac{\partial^i K}{\partial x^i} = D_1^i k(x,y) K_1(s,t;u_0,u_1,...,u_m)$, so $\frac{\partial^i K}{\partial x^i} \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$. Applying the inequality

$$a \leq 1 + a^q, \;\; \forall a \geq 0, \;\; \forall q \geq 1,$$

we obtain

$$|K_1(s,t;u_0,u_1,...,u_m)| \leq (st)^{\alpha_0}\left(1+\frac{2|u_0|}{w_0(s,t)}\right) + \sum_{i=1}^{m}(st)^{\alpha_i}\left(1+\frac{2|u_i|}{D_1^i w_0(s,t)}\right) \leq 2\sum_{i=0}^{m}(st)^{\alpha_i}\left(1+\sum_{j=0}^{m}|u_j|\right), \tag{3.29}$$

it leads to

$$|K(x,y,s,t;u_0,u_1,...,u_m)| = k(x,y)\,|K_1(s,t;u_0,u_1,...,u_m)| \leq \bar{k}_0(x,y,s,t)\left(1+\sum_{j=0}^{m}|u_j|\right), \tag{3.30}$$

in which

$$\bar{k}_0(x,y,s,t) = 2k(x,y)\sum_{j=0}^{m}(st)^{\alpha_j}. \tag{3.31}$$

Similarly,

$$\left|\frac{\partial^i K}{\partial x_1^i}(x,y,s,t;u_0,u_1,...,u_m)\right| \leq \bar{k}_i(x,y,s,t)\left(1+\sum_{j=0}^{m}|u_j|\right), \tag{3.32}$$

in which

$$\bar{k}_i(x,y,s,t) = 2\left|D_1^i k(x,y)\right|\sum_{j=0}^{m}(st)^{\alpha_j}. \tag{3.33}$$

Next,

$$\begin{aligned}
&\sum_{i=0}^{m}\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_i(x,y,s,t)dsdt \\
&= 2\sum_{i=0}^{m}\sup_{(x,y)\in\Omega}\left|D_1^i k(x,y)\right|\sum_{j=0}^{m}\iint_\Omega (st)^{\alpha_j}dsdt \\
&\leq 2\sum_{j=0}^{m}\frac{1}{(1+\alpha_j)^2}\left(1+\sum_{i=1}^{m}\tilde{\gamma}_1\,(\tilde{\gamma}_1-1)\,...\,(\tilde{\gamma}_1-i+1)\right)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\},
\end{aligned}$$

so

$$\bar{\beta} = \sum_{i=0}^{m}\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_i(x,y,s,t)dsdt < 1. \tag{3.34}$$

Theorem 3.2 is true. Furthermore, $w_0 \in X_m$ is also a solution of (1.1).

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