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A SIMPLE HYBRID BREGMAN PROJECTION ALGORITHMS FOR A FAMILY OF COUNTABLE BREGMAN QUASI-STRICT PSEUDO-CONTRACTIONS

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. In this paper, a simple hybrid Bregman projection iterative algorithm is investigated for finding a common fixed point of a family of countable Bregman quasi-strict pseudo-contractions. Furthermore, strong convergence results are established in a reflexive Banach space.

1. Introduction

Fixed point theory, which serves as a significant branch of nonlinear analysis theory, has been applied in investigating nonlinear phenomena. In fact, many real world problems arising in economics, optimal control, image reconstruction, engineering, and physics can be studied via fixed point techniques.

Construction of iterative algorithm for seeking fixed points of nonlinear mappings as a main task of fixed point theory has been a popular concern.

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Many classical algorithms appeared successively in history; for instance, Picard iterative algorithm, Mann iterative algorithm [9], Ishikawa iterative algorithm [7], and so on. It is well known that, for more general nonexpansive mappings, the Picard iterative algorithm does not converge to fixed points of nonexpansive mappings even when they have fixed points; the Mann (or Ishikawa) iterative algorithm only weakly converges to fixed points of nonexpansive mappings. But strong convergence is often much more desirable than weak convergence in many disciplines, including economics, image recovery, quantum physics, control theory, and problems arise in infinite dimension spaces. For it translates the physically tangible property so that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. In order to get strong convergence result, the projection method which was first introduced by Haugazeau [6] in 1968, is an effective modified method about the Mann (or Ishikawa) iterative algorithm.

In recent years, projection methods have caused widely attention. Many author focused attention on constructing iterative algorithm for seeking fixed points of Bregman nonlinear operators via (Bregman) projection technique, see [16, 11, 17, 14] and the references therein. In 2010, [11] gave the concept of Bregman strongly nonexpansive mapping and proved the strong convergence results of two Bregman hybrid projection algorithms for finding common fixed points of finitely many Bregman strongly nonexpanisve operators in reflexive Banach spaces. In 2014, [17] designed an iterative scheme about a Bregman relatively nonexpansive operator. Recently, Ugwunnadi et al. [14] introduced the concept of Bregman quasi-strict pseudo-contraction and proved the strong convergence by using hybrid Bregman projection iterative algorithm for Bregman quasi-strict pseudo-contractions.

The purpose of this paper is to give a simple hybrid Bregman projection iterative algorithm for finding a common fixed point of a family of countable Bregman quasi-strict pseudo-contractions in the framework of reflexive Banach spaces. The results presented in this paper improve and enrich the known corresponding results announced in the literature sources listed in this work.

2. Preliminaries

In this section, we collect some preliminaries, definitions, and lemmas which will be used to prove our main results. Throughout this paper, E is a real reflexive Banach space with norm $\|\cdot\|$ and E^* is the dual space of E. We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of positive integers and real numbers, respectively. $\to$ and $\rightharpoonup$ stand for strong convergence and weak convergence, respectively.

For $x \in$int dom$f$, the subdifferential of f at x is the convex set defined as

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \quad \forall y \in E\}.$$

For $x^* \in E^*$, the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

The function f is called essentially smooth if ∂f is both locally bounded and single-valued on its domain. It is called essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom ∂f. f is said to be a Legendre, if it is both essentially smooth and essentially strictly convex. When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$ (see [10]).

We note that for a reflexive Banach space E, the following conclusions hold:

(i) f is essentially smooth if and only if f^* is essentially strictly convex ([1]);

(ii) $(\partial f)^{-1} = \partial f^*$ ([2]);

(iii) f is Legendre if and only if f^* is Legendre ([1]);

(iv) If f is Legendre, then ∇f is bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $ran\nabla f = dom\nabla f^* = int\ dom f^*$ and $ran\nabla f^* = dom\nabla f = int\ dom f$ ([1]).

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that the functon f is said to be totally convex at a point $x \in$ int dom$f$ if its modulus of total convexity at x, that is, the function ν_f : int dom $f \times [0, +\infty) \to [0, +\infty)$, defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in int\ dom f, \|y - x\| = t\},$$

is positive whenever $t > 0$. The function f is said to be totally convex when it is totally convex at every point $x \in$int dom f. Moreover, the function f is said to be totally convex on bounded subset of E if $\nu_f(C, t) > 0$ for any bounded subset C of E and for any $t > 0$, where the modulus of total convexity of the function f on the set C is the function ν_f : int dom$f \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(C, t) := \inf\{\nu_f(x, t) : x \in C \cap int\ dom f\}.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.

Recall that the function f is called sequentially consistent [4] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.1}$$

Lemma 2.1. ([3]) *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous, and convex function. The domain of f is denoted by dom$f$, that is, dom$f := \{x \in E : f(x) < +\infty)\}$. For any $x \in$int dom$f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^{\circ}(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.2}$$

Next, we list some definitions about f by virtue of (2.2):

(1) The function f is said to be Gâteaux differentiable at x if $f^{\circ}(x, y)$ exists for any y. In this case, $f^{\circ}(x, y)$ coincides with $\nabla f(x)$, the value of the gradient $\nabla f(x)$ of f at x.

(2) The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int dom$f$.

(3) The function f is called Fréchet differentiable at x if the limit (2.2) is attained uniformly in $\|y\| = 1$.

(4) The function f is said to be uniformly Fréchet differentiable on a subset C of E if the limit (2.2) is attained uniformly for $x \in C$ and $\|y\| = 1$.

It is well known that if a continuous convex function $f : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇f is norm-to-weak* continuous [3]; and if f is Fréchet differentiable, then ∇f is norm-to-norm continuous [8].

The Bregman distance [5] with respect to f is the function D_f :dom$f\times$int dom$f \to [0, +\infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle.$$

With the function f we associate the bifunction $V_f : E \times E^* \to [0, +\infty)$ defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*.$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. Recall that the Bregman projection [12] of $x \in$int dom$f$ onto the nonempty closed and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Bregman projections with respect to totally convex and differentiable functions have the following variational characterizations.

Lemma 2.2. ([4]) *Suppose that f is Gâteaux differentiable and totally convex on int dom$f$. Let $x \in$ int dom$f$ and let $C \subset$ int dom$f$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*

(a) *The vector* $\hat{x}$ *is the Bregman projection of* x *onto* C *with respect to* f*, that is,* $\hat{x} = P_C^f(x)$*;*

(b) *The vector* $\hat{x}$ *is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(\hat{x}), \hat{x} - y \rangle \geq 0, \quad \forall\ y \in C;$$

(c) *The vector* $\hat{x}$ *is the unique solution of the inequality*

$$D_f(y, \hat{x}) + D_f(\hat{x}, x) \leq D_f(y, x), \quad \forall\ y \in C.$$

Lemma 2.3. ([13]) *Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_f(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is bounded too.*

Recall the following definitions of nonlinear mappings.

Definition 2.4. Let $C \subset$int dom$f$ be a nonempty, closed, and convex subset of E, $T : C \to C$ be a mapping with fixed point set $F(T) \neq \emptyset$, $\{T_n\}_{n\in\mathbb{N}} : C \to C$ be a sequence of mappings with common fixed point set $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$.

(1) T is called Bregman quasi-nonexpansive if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C,\ p \in F(T).$$

(2) T is said to be Bregman quasi-strictly pseudo-contractive [4] if there exists a constant $k \in [0, 1)$ and $F(T) \neq \emptyset$ such that

$$D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx), \quad \forall\ x \in C,\ p \in F(T).$$

(3) $\{T_n\}_{n\in\mathbb{N}} : C \to C$ is called uniformly closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $\|T_n x_n - x_n\| \to 0$ as $n \to \infty$, then the limit of $\{x_n\}$ belongs to $\mathcal{F}$.

Now, we give some examples of Bregman quasi-strict pseudo-contractions.

Example 2.5. ([11]) Let E be a real reflexive Banach space, $A : E \to 2^{E^*}$ be a maximal monotone mapping and $f : E \to (-\infty, +\infty]$ be a uniformly Fréchet differentiable and bounded on bounded subsets of E such that $A^{-1}(0^*) \neq \emptyset$, then the resolvent

$$Res_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x)$$

is closed and Bregman relatively nonexpansive from E onto dom A, so is a closed Bregman quasi-strict pseudo-contraction.

Example 2.6. ([15]) Let E be a smooth Banach space, and define $f(x) = \|x\|^2$ for all $x \in E$. Let $x_0 \neq 0$ be any element of E, the mapping $T : E \to E$ be defined as follows:

$$T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0, & if\ x = (\frac{1}{2} + \frac{1}{2^n})x_0; \\ -x, & if\ x \neq (\frac{1}{2} + \frac{1}{2^n})x_0 \end{cases}$$

for all $n \geq 1$. Then T is a Bregman quasi-strict pseudo-contraction.

The following lemma is also useful in the next section.

Lemma 2.7. ([14]) *Let $f : E \to R$ be a Legendre function enjoying uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to C$ be a Bregman quasi-strictly pseudo-contractive mapping with respect to f. Then $F(T)$ is closed and convex.*

3. Main Results

In this section, we state and prove our main theorem.

Theorem 3.1. *Let E be a real reflexive Banach space, C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E, and $\{T_n\}_{n=1}^{\infty}$ be a uniformly closed family of countable Bregman quasi-strict pseudo-contractions such that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = proj_{C_1}^f(x_0), \\ C_{n+1} = \{z \in C_n : D_f(z_n, T_n z_n) \\ \qquad\qquad \leq \frac{1}{1-\kappa}\langle z_n - z, \nabla f(z_n) - \nabla f(T_n z_n)\rangle\}, \\ x_{n+1} = proj_{C_{n+1}}^f(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.1}$$

where $\kappa \in [0, 1)$, $z_n = x_n + e_n$, $\{e_n\} \subset C$ is a sequence of errors which satisfies $\lim_{n\to\infty} e_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $p = proj_{\mathcal{F}}^f(x_0)$, where $proj_{\mathcal{F}}^f$ is the Bregman projection of E onto $\mathcal{F}$.

Proof. From Lemma 2.7, we know $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n)$ is closed and convex. Furthermore, $proj_{\mathcal{F}}^f$ is well defined for any $x_0 \in C$. Indeed, it is not difficult to check that the sets C_n are closed and convex for all $n \in \mathbb{N}$. Hence, $proj_{C_n}^f$ is also well defined for any $x_0 \in C$. On the other hand, it is obvious that $\mathcal{F} \subset C = C_1$. Suppose that $\mathcal{F} \subset C_m$ for some $m \in \mathbb{N}$. For any $p \in \mathcal{F}$, we have $p \in C_m$. From the definition of T_n, we obtain

$$D_f(p, T_n(x_m + e_m)) \leq D_f(p, x_m + e_m) + \kappa D_f(x_m + e_m, T_n(x_m + e_m)).$$

And, in view of the three point identity of the Bregman distance, we get

$$\begin{aligned}D_f(p, T_n(x_m + e_m)) &= D_f(p, x_m + e_m) + D_f(x_m + e_m, T_n(x_m + e_m)) \\ &\quad + \langle \nabla f(x_m + e_m) - \nabla f(T_n(x_m + e_m)), p - (x_m + e_m)\rangle.\end{aligned}$$

Therefore, from the two formula above, we have

$$D_f(z_m, T_n(z_m)) \le \frac{1}{1-\kappa}\langle z_m - p, \nabla f(z_m) - \nabla f(T_n(z_m))\rangle,$$

where $z_m = x_m + e_m$. It follows that $p \in C_{m+1}$. From the arbitrariness of p, we learn that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$.

Next, we show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Indeed, since $x_n = proj^f_{C_n}(x_0)$, from Lemma 2.2 (c), we have

$$\begin{aligned}D_f(x_n, x_0) &= D_f(proj^f_{C_n}(x_0), x_0) \\ &\le D_f(p, x_0) - D_f(p, proj^f_{C_n}(x_0)) \\ &\le D_f(p, x_0),\end{aligned}$$

for each $p \in \mathcal{F}$ and for each $n \in \mathbb{N}$. So $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is bounded. Using Lemma 2.3, we get $\{x_n\}$ is bounded too.

On the other hand, note that $x_n = proj^f_{C_n}(x_0)$ and $x_{n+1} = proj^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we obtain that $D_f(x_n, x_0) \le D_f(x_{m+1}, x_0)$ for all $n \in \mathbb{N}$. This implies that $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is a nondecreasing sequence. So $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{p} \in C = C_1$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $\widehat{p} \in C_n$ for all $n \in \mathbb{N}$. From $x_{n_i} = proj^f_{C_{n_i}}(x_0)$, we obtain that $D_f(x_{n_i}, x_0) \le D_f(\widehat{p}, x_0)$ for each $n_i \in \mathbb{N}$. Since f is a lower semi-continuous function on convex set C, it is weakly lower semi-continuous on C. Then, we get

$$\begin{aligned}\liminf_{i\to\infty} D_f(x_{n_i}, x_0) &= \liminf_{i\to\infty}\{f(x_{n_i}) - f(x_0) - \langle x_{n_i} - x_0, \nabla f(x_0)\rangle\} \\ &\ge f(\widehat{p}) - f(x_0) - \langle \widehat{p} - x_0, \nabla f(x_0)\rangle \\ &= D_f(\widehat{p}, x_0).\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}D_f(\widehat{p}, x_0) &\le \liminf_{i\to\infty} D_f(x_{n_i}, x_0) \\ &\le \limsup_{i\to\infty} D_f(x_{n_i}, x_0) \\ &\le D_f(\widehat{p}, x_0),\end{aligned}$$

which implies that

$$\lim_{i\to\infty} D_f(x_{n_i}, x_0) = D_f(\widehat{p}, x_0). \tag{3.2}$$

Using Lemma 2.2 (c), we have

$$D_f(p, x_{n_i}) \leq D_f(p, x_0) - D_f(x_{n_i}, x_0). \tag{3.3}$$

Taking $i \to \infty$ in the above inequality and using (3.2), we get

$$\lim_{i\to\infty} D_f(\widehat{p}, x_{n_i}) = 0. \tag{3.4}$$

Note that f is totally convex on bounded sets, So f is sequentially consistent and we have $\lim_{i\to\infty} x_{n_i} = \widehat{p}$. On the other hand, note that $\{D_f(x_n, x_0)\}$ is convergent. This together with (3.2) implies that

$$\lim_{i\to\infty} D_f(x_n, x_0) = D_f(\widehat{p}, x_0).$$

If we do the similar work as (3.3) and (3.4), we also can obtain that

$$\lim_{n\to\infty} x_n = \widehat{p}. \tag{3.5}$$

In view of $\lim_{n\to\infty} e_n = 0$ and (3.5), we have

$$\lim_{n\to\infty} (x_n + e_n) = \widehat{p}. \tag{3.6}$$

and

$$\lim_{n\to\infty} (x_n + e_n - x_{n+1}) = 0. \tag{3.7}$$

Next, we show that the limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n)$. Since $x_{n+1} = proj^f_{C_{n+1}}(x_0) \in C_{n+1}$, we have from (3.1) that

$$D_f(z_n, T_n z_n) \leq \frac{1}{1-\kappa} \langle z_n - x_{n+1}, \nabla f(z_n) - \nabla f(T_n z_n)\rangle,$$

which together with (3.7) implies that

$$\lim_{n\to\infty} D_f(z_n, T_n z_n) = 0.$$

Noticing that f is totally convex on bounded subsets of E, f is sequentially consistent. It follows that

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0.$$

Since the uniform closedness of T_n and (3.6), we have $\widehat{p} \in \bigcap_{n=1}^{\infty} F(T_n) = \mathcal{F}$.

Finally, we show that $\widehat{p} = Proj^f_{\mathcal{F}}(x_0)$. From $x_n = proj^f_{C_n} x_0$, we get

$$\langle y - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \leq 0, \quad \forall y \in C_n.$$

Since $\mathcal{F} \subset C_n$ for each $n \in \mathbb{N}$, we have

$$\langle y - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \leq 0, \quad \forall y \in \mathcal{F}.$$

By taking limit of the above inequality , we have

$$\langle y - \widehat{p}, \nabla f(x_0) - \nabla f(\widehat{p})\rangle \leq 0, \quad \forall y \in \mathcal{F}.$$

In view of Lemma 2.2 (a) and Lemma 2.2 (b), It is obvious that $p = proj^f_{\mathcal{F}}(x_0)$ holds. This completes the proof of Theorem 3.1. □

4. Applications

In this section we consider the problem for seeking the common solution of a system of generalized mixed equilibrium problems. Let C be a nonempty, closed, and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\{g_n\}_{n\in\mathbb{N}}$ be a family of bifunctions from $C\times C$ into $\mathbb{R}$, $\{A_n\}_{n\in\mathbb{N}} : C \to E^*$ be a sequence of nonlinear mappings, and $\{\varphi_n\}_{n\in\mathbb{N}} : C \times \mathbb{R}$ be a sequence of real-valued functions. The "so-called" system of generalized mixed equilibrium problems is to find $x \in C$, for arbitrary $y \in C,\ n \in \mathbb{N}$, such that

$$f_n(x, y) + \langle y - x, A_n x\rangle + \varphi_n(y) - \varphi_n(x) \geq 0. \tag{4.1}$$

The set of solutions of the system of generalized mixed equilibrium problems is denoted by $SGMEP(g_n, A_n, \varphi_n)$, where $n \in \mathbb{N}$.

A mapping $A : C \to E^*$ is said to be monotone if

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping A is said to be Lipschitz continuous if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

For solving the generalized mixed equilibrium problem, let us assume that $g : C \times C \to \mathbb{R}$ is a bifunction satisfying the following conditions:

(A1) $g(x, x) = 0$ for all $x \in C$;
(A2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y);$$

(A4) for each $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

The resolvent operator of a bifunction $g : C \times C \to \mathbb{R}$, $Res : E \to C$ is defined as follows: for all $x \in E$,

$$Res(x) = \{G(z, y) + \langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C\},$$

where $G(x, y) = g(z, y) + \langle y - z, Az\rangle + \varphi(y) - \varphi(z)$. The set of solutions of a generalized mixed equilibrium problem is denoted by $GMEP$. If $f : E \to \mathbb{R}$ is a Legendre function, $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), it is not hard to know that Res has the following properties:

(a) Res is single-valued;
(b) the set of fixed points of Res is the solution set of the corresponding generalized mixed equilibrium problem, i.e., $F(Res) = GMEP$;

(c) $GMEP$ is closed and convex;

(d) for any $p \in GMEP$, $x \in E$,

$$D_f(p, Res(x)) + D_f(Res(x), x) \leq D_f(p, x).$$

Theorem 4.1. *Let E be a real reflexive Banach space, C be a nonempty, closed, and convex subset of E, $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\{f_n\}_{n\in\mathbb{N}} : C \times C \to \mathbb{R}$ be a sequence of bifunctions satisfying the conditions* (A1)-(A2), *$\{A_n\}_{n\in\mathbb{N}} : C \to E^*$ be a sequence of continuous and monotone mappings, $\{\varphi_n\}_{n\in\mathbb{N}} : C \to \mathbb{R}$ be a sequence of lower semi-continuous and convex functions. Assume that $\mathcal{F} = \bigcap_{n=1}^{\infty} SGMEP(f_n, A_n, \varphi_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = proj^f_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : D_f(z_n, Res_n z_n) \\ \leq \frac{1}{1-\kappa}\langle z_n - z, \nabla f(z_n) - \nabla f(Res_n z_n)\rangle\}, \\ x_{n+1} = proj^f_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where Res_n is the sequence of resolvents of the system of generalized mixed equilibrium problems (4.1), *$\kappa \in [0, 1)$, $z_n = x_n + e_n$, $\{e_n\} \subset C$ is a sequence of errors which satisfies $\lim_{n\to\infty} e_n = 0$. Suppose that the resolvent operator Res_n is uniformly closed. Then the sequence $\{x_n\}$ converges strongly to $p = proj^f_{\mathcal{F}}(x_0)$, where $proj^f_{\mathcal{F}}$ is the Bregman projection of E onto $\mathcal{F}$.*

Proof. From the properties (d) of the resolvent operator Res, we learn that Res_n is a Bregman quasi-strict pseudo-contraction for each $n \in \mathbb{N}$. By applying Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $p = proj^f_{\mathcal{F}}(x_0)$. □

5. Conclusions

In this article, using a simple hybrid Bregman projection iterative algorithm, we investigate a convex feasibility problem based on a family of countable Bregman quasi-strict pseudo-contractions and give a strong convergence theorem of the proposed algorithm in a reflexive Banach space. As a application, we solve a system of generalized mixed equilibrium problems via the proposed algorithm (3.1).

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