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THE 3-VARIABLE HERMITE POLY-BERNOULLI POLYNOMIALS OF THE SECOND KIND

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. Recently a remarkably large number of polynomials and numbers and their generalizations have been introduced and investigated, due mainly to their usefulness. In this sequel, we aim to introduce the 3-variable Hermite poly-Bernoulli polynomials of the second kind and investigate some of their properties and formulas such as implicit summation formulas and symmetric identities. The results presented here are sure to be new and

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potentially useful. They, being general, can be specialized to yield some known and new formulas.

1. Introduction and preliminaries

The 2-variable Hermite Kampé de Fériet polynomials $H_n(x, y)$ [2, 4] are defined by

$$H_n(x, y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

It is easy to find from (1.1) that

$$H_n(2x, -1) = H_n(x) \quad \text{and} \quad H_n\left(x, -1/2\right) = He_n(x),$$

where $H_n(x)$ and $He_n(x)$ are ordinary Hermite polynomials (see, e.g., [1]). Also

$$H_n(x, 0) = x^n.$$

The 2-variable Hermite polynomials $H_n(x, y)$ are generated by the following function (see, e.g., [1, 15])

$$\mathrm{e}^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \tag{1.2}$$

The alternating sums $T_k(n)$ ($n \in \mathbb{N}$, $k \in \mathbb{N}_0$) are defined by

$$T_k(n) = \sum_{r=0}^{n-1} (-1)^r r^k = 0^k - 1^k + 2^k - \cdots + (-1)^{n-1}(n-1)^k,$$

which are generated by the following function

$$\frac{1-(-\mathrm{e}^t)^n}{1+\mathrm{e}^t} = \sum_{k=0}^{\infty} T_k(n) \frac{t^k}{k!}. \tag{1.3}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The polylogarithm functions $\mathrm{Li}_k(z)$ ($k \in \mathbb{Z}$) are defined by (see, e.g., [21, p. 185])

$$\begin{aligned} \mathrm{Li}_k(z) :&= \sum_{n=1}^{\infty} \frac{z^n}{n^k} \quad (|z| \leq 1;\ k \in \mathbb{N} \setminus \{1\}) \\ &= \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t}\, dt \quad (k \in \mathbb{N} \setminus \{1\}) \end{aligned} \tag{1.4}$$

and

$$\mathrm{Li}_1(z) = -\mathrm{Log}(1-z), \quad \mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \cdots .$$

Kaneko [6] used the polylogarithm function $\mathrm{Li}_k(z)$ to introduce and investigate poly-Bernoulli numbers

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{1-\mathrm{e}^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \quad (k \in \mathbb{Z}). \tag{1.5}$$

The particular case $k=1$ in (1.5) yields the classical Bernoulli numbers B_n as follows: $B_n^{(1)} = (-1)^n B_n$ $(n \in \mathbb{N}_0)$ (see, e.g., [21, Section 1.7]).

Kim et al. [11] introduced the poly-Bernoulli polynomials of the second kind defined by the generating function

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x) \frac{t^n}{n!}. \tag{1.6}$$

It is noted that

$$\sum_{n=0}^{\infty} b_n^{(1)}(x) \frac{t^n}{n!} = \frac{\mathrm{Li}_1(1-\mathrm{e}^{-t})}{\log(1+t)}(1+t)^x = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \tag{1.7}$$

where $b_n(x)$ are called the Bernoulli numbers of the second kind (see [12, 13]).

Pathan and Khan [15] introduced and investigated the following generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$

$$\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha} \mathrm{e}^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y) \frac{t^n}{n!}. \tag{1.8}$$

The particular case of (1.8) when $\alpha = 1$ reduces to the known polynomials ${}_HB_n(x,y)$ (see [4, p. 386, Eq. (1.6)])

$$\frac{t}{\mathrm{e}^t-1} \mathrm{e}^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y) \frac{t^n}{n!}. \tag{1.9}$$

The Stirling numbers $s(n,k)$ and $S(n,k)$ of the first kind and the second kind, respectively, are defined by the following generating functions (see, e.g., [21, Section 1.6])

$$z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k)\, z^k \quad (n \in \mathbb{N}_0) \tag{1.10}$$

and

$$(e^z-1)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{z^n}{n!} \quad (k \in \mathbb{N}_0)\,. \tag{1.11}$$

For easier use, we recall some formal manipulations of double series as in the following lemma (see, e.g., [3], [10], [20, pp. 56-57], and [22, p. 52]).

Lemma 1.1. *The following identities hold true:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n-pk} \quad (p \in \mathbb{N}); \tag{1.12}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n+pk} \quad (p \in \mathbb{N}); \tag{1.13}$$

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n}; \tag{1.14}$$

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{1.15}$$

Here, the $A_{k,n}$ and $f(N)$ ($k,\ n,\ N \in \mathbb{N}_0$) are real or complex valued functions of the k, n and N, respectively, and x and y are real or complex numbers. Also, in order to verify rearrangements of the involved series, all the associated series should be absolutely convergent.

In this paper, we introduce 3-variable Hermite poly-Bernoulli polynomials of the second kind ${}_H b_n^{(k)}(x,y,z)$ and investigate some of their interesting properties. The results presented here are also shown to be specialized to yield some known formulas and identities which are given in Kim et al. [11]-[13], Qi et al. [19], Dattoli et al. [4], Khan [7]-[9], and Pathan and Khan [14]-[18].

2. A new class of Hermite poly-Bernoulli numbers and polynomials of the second kind

We begin by defining the 3-variable Hermite poly-Bernoulli polynomials ${}_H b_n^{(k)}(x,y,z)$ of the second kind

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\log(1+t)}(1+t)^x\,\mathrm{e}^{yt+zt^2} = \sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y,z)\,\frac{t^n}{n!} \quad (k \in \mathbb{Z}), \tag{2.1}$$

whose special case $k=1$ is simply expressed as follows:

$$\frac{t}{\log(1+t)}(1+t)^x\,\mathrm{e}^{yt+zt^2} = \sum_{n=0}^{\infty} {}_H b_n(x,y,z)\,\frac{t^n}{n!}. \tag{2.2}$$

It is remarked that the special case of (2.1) when $y = z = 0$ reduces to the poly-Bernoulli polynomials of the second kind together with the generating function in (1.6).

By combining (1.2) and (1.6), it is easy to see that the 3-variable Hermite poly-Bernoulli polynomials ${}_Hb_n^{(k)}(x, y, z)$ in (2.1) can be expressed in terms of the 2-variable Hermite polynomials $H_n(x, y)$ in (1.2) and the poly-Bernoulli polynomials of the second kind $b_n^{(k)}(x)$ in (1.6), which is asserted by the following theorem.

Theorem 2.1. *The following formula holds.*

$$ {}_Hb_n^{(k)}(x, y, z) = \sum_{m=0}^{n} \binom{n}{m} b_{n-m}^{(k)}(x)\, H_m(y, z) \quad (k \in \mathbb{Z};\ n \in \mathbb{N}_0)\,. \tag{2.3} $$

Proof. We can prove the desired identity by using (1.12) with $p = 1$. We omit the details. □

Theorem 2.2. *The following identity holds.*

$$ {}_Hb_n^{(2)}(x, y, z) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m}{m+1}\, {}_Hb_{n-m}(x, y, z) \quad (n \in \mathbb{N}_0)\,, \tag{2.4} $$

where B_m are Bernoulli numbers (see, e.g., [21, Section 1.7]*).*

Proof. Setting $k = 2$ in (2.1), we have

$$ \sum_{n=0}^{\infty} {}_Hb_n^{(2)}(x, y, z) \frac{t^n}{n!} = \frac{\mathrm{Li}_2(1 - \mathrm{e}^{-t})}{\log(1+t)} (1+t)^x\, e^{yt+zt^2}. \tag{2.5} $$

We find from (1.4) (see also [6, Eq. (2)]) that

$$ \mathrm{Li}_2(1 - \mathrm{e}^{-t}) = \int_0^t \frac{u}{\mathrm{e}^u - 1}\, du. \tag{2.6} $$

Applying the generating function of the Bernoulli numbers (see, e.g., [21, p. 81, Eq. (2)]) to the integrand of (2.6), we have

$$ \mathrm{Li}_2(1 - \mathrm{e}^{-t}) = \sum_{m=0}^{\infty} \frac{B_m}{m!\,(m+1)}\, t^{m+1}. \tag{2.7} $$

Using (2.2) and (2.7) into (2.5), with the help of (1.12) when $p = 1$, we obtain

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_Hb_n^{(2)}(x, y, z) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_Hb_n(x, y, z) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{t^m B_m m!}{(m+1)} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{B_m}{m+1}\, {}_Hb_{n-m}(x, y, z) \frac{t^n}{n!}, \end{aligned} $$

which, upon equating the coefficients of t^n, yields the desired identity (2.4). □

Setting $y = z = 0$ in (2.4) reduces to the known result in [11, Theorem 2.1], which is recalled in the following corollary.

Corollary 2.3. *The following formula holds.*

$$b_n^{(2)}(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m}{m+1}\, b_{n-m}(x) \quad (n \in \mathbb{N}_0)\,. \tag{2.8}$$

Theorem 2.4. *The following identity holds.*

$$\begin{aligned} {}_H b_n^{(k)}(x,y,z) = \sum_{j=0}^{n} \frac{(-1)^j}{j+1} \binom{n}{j}\, {}_H b_{n-j}(x,y,z) \\ \times \sum_{\ell=0}^{j} \frac{(-1)^{\ell}\,(\ell+1)!\, S(j+1,\ell+1)}{(\ell+1)^k} \end{aligned} \tag{2.9}$$

for $k \in \mathbb{N} \setminus \{1\}$; $n \in \mathbb{N}_0$.

Proof. By using (1.4) and (1.11), we have, for $k \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned} \mathrm{Li}_k\left(1 - \mathrm{e}^{-t}\right) &= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}\left(\mathrm{e}^{-t} - 1\right)^{\ell}}{\ell^k} \\ &= \sum_{\ell=1}^{\infty} \sum_{j=\ell}^{\infty} \frac{(-1)^{\ell+j}\,\ell!\, S(j,\ell)}{\ell^k\, j!}\, t^j \\ &= \sum_{\ell=0}^{\infty} \sum_{j=\ell}^{\infty} \frac{(-1)^{\ell+j}\,(\ell+1)!\, S(j+1,\ell+1)}{(\ell+1)^k\,(j+1)!}\, t^{j+1}. \end{aligned}$$

Applying (1.14) to the last double summation, we obtain

$$\mathrm{Li}_k\left(1 - \mathrm{e}^{-t}\right) = t \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \frac{(-1)^{\ell+j}\,(\ell+1)!\, S(j+1,\ell+1)}{(\ell+1)^k\,(j+1)!}\, t^j. \tag{2.10}$$

Using (2.10) and (2.2) in (2.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y,z)\,\frac{t^n}{n!}\\
&= \frac{t}{\log(1+t)}(1+t)^x\,\mathrm{e}^{yt+zt^2}\sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\frac{(-1)^{\ell+j}\,(\ell+1)!\,S(j+1,\ell+1)}{(\ell+1)^k\,(j+1)!}\,t^j\\
&= \sum_{n=0}^{\infty} {}_H b_n(x,y,z)\,\frac{t^n}{n!}\sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\frac{(-1)^{\ell+j}\,(\ell+1)!\,S(j+1,\ell+1)}{(\ell+1)^k\,(j+1)!}\,t^j.
\end{aligned}$$

Applying the identity (1.12) with $p=1$ to the last double summation, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y,z)\,\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} {}_H b_{n-j}(x,y,z)\,\frac{1}{(n-j)!}\sum_{\ell=0}^{j}\frac{(-1)^{\ell+j}\,(\ell+1)!\,S(j+1,\ell+1)}{(\ell+1)^k\,(j+1)!}\right\} t^n,
\end{aligned}$$

which, upon equating the coefficients of t^n, yields the desired identity. □

Setting $y = z = 0$ in (2.9) reduces to yield the known result (see [11, Theorem 2.2]) which is recalled in the following corollary.

Corollary 2.5. *The following identity holds.*

$$b_n^{(k)}(x) = \sum_{j=0}^{n}\frac{(-1)^j}{j+1}\binom{n}{j}\,b_{n-j}(x)\sum_{\ell=0}^{j}\frac{(-1)^{\ell}\,(\ell+1)!\,S(j+1,\ell+1)}{(\ell+1)^k} \tag{2.11}$$

for $k \in \mathbb{N}\setminus\{1\}$; $n \in \mathbb{N}_0$.

Theorem 2.6. *The following formula holds.*

$$\begin{aligned}
&{}_H b_{n+1}^{(k)}(x+1,y,z) - {}_H b_{n+1}^{(k)}(x,y,z)\\
&= \sum_{j=0}^{n}\frac{n+1}{j+1}\binom{n}{j}\,{}_H b_{n-j}(x,y,z)\sum_{\ell=0}^{j}\frac{(-1)^{\ell+j}\,(\ell+1)!\,S(j+1,\ell+1)}{(\ell+1)^k}
\end{aligned} \tag{2.12}$$

for $k \in \mathbb{N}\setminus\{1\}$; $n \in \mathbb{N}_0$.

Proof. From (2.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x+1,y,z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y,z)\frac{t^n}{n!}\\&= \sum_{n=1}^{\infty} \left\{ {}_H b_n^{(k)}(x+1,y,z) - {}_H b_n^{(k)}(x,y,z)\right\} \frac{t^n}{n!}\\&= \frac{t}{\log(1+t)}(1+t)^x \,\mathrm{e}^{yt+zt^2}\,\mathrm{Li}_k(1-e^{-t}),\end{aligned}$$

where the first equality follows from

$${}_H b_0^{(k)}(x,y,z) = 1 \quad (x,\, y,\, z \in \mathbb{C};\, k \in \mathbb{N}\setminus\{1\}).$$

Using (2.2) and (2.10), with the aid of (1.12) when $p = 1$, similarly as above, we can prove the desired identity. We omit the details. □

Remark 2.7. Setting $y = z = 0$ in (2.12) yields a known result (cf., [11, Theorem 2.3].

3. Implicit summation formulae involving the Hermite poly-Bernoulli numbers and polynomials of the second kind

Khan [5] and Pathan and Khan [14]-[18] have established ceratin interesting implicit summation formulae for the ordinary Hermite and related polynomials and Hermite-Bernoulli polynomials, respectively. Here, we present some implicit summation formulae for the Hermite poly-Bernoulli polynomials ${}_H b_n^{(k)}(x,y,z)$ in (2.1). We begin by stating the following implicit summation formula for the Hermite poly-Bernoulli polynomials ${}_H b_n^{(k)}(x,y,z)$.

Theorem 3.1. *The following formula holds: For $n,\, \ell \in \mathbb{N}_0$,*

$${}_H b_{n+\ell}^{(k)}(x,w,z) = \sum_{p=0}^{n}\sum_{q=0}^{\ell} \binom{n}{p}\binom{\ell}{q}(w-y)^{p+q}\,{}_H b_{n+\ell-p-q}^{(k)}(x,y,z). \tag{3.1}$$

Proof. Replacing t by $t+u$ in (2.1), we obtain

$$\begin{aligned}&\frac{\mathrm{Li}_k(1-\mathrm{e}^{-(t+u)})}{\log(1+t+u)}(1+t+u)^x\,\mathrm{e}^{z(t+u)^2}\\&= \mathrm{e}^{-y(t+u)}\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y,z)\,\frac{(t+u)^n}{n!}.\end{aligned} \tag{3.2}$$

Using binomial theorem to expand $(t+u)^n$ and then (1.13) when $p=1$ in the right side of (3.2), we get

$$\begin{aligned}&\frac{\mathrm{Li}_k(1-\mathrm{e}^{-(t+u)})}{\log(1+t+u)}(1+t+u)^x\,\mathrm{e}^{z(t+u)^2}\\&=\mathrm{e}^{-y(t+u)}\sum_{n,\ell=0}^{\infty}{}_Hb_{n+\ell}^{(k)}(x,y,z)\,\frac{t^n}{n!}\,\frac{u^\ell}{\ell!}.\end{aligned}\tag{3.3}$$

It is noted that the left side of (3.3) is independent of the variable y. Replacing y by w in the right side of (3.3) and equating the two expressions, we have

$$\mathrm{e}^{(w-y)(t+u)}\sum_{n,\ell=0}^{\infty}{}_Hb_{n+\ell}^{(k)}(x,y,z)\,\frac{t^n}{n!}\,\frac{u^\ell}{\ell!}=\sum_{n,\ell=0}^{\infty}{}_Hb_{n+\ell}^{(k)}(x,w,z)\,\frac{t^n}{n!}\,\frac{u^\ell}{\ell!}.\tag{3.4}$$

Expanding $\mathrm{e}^{(w-y)(t+u)}$ and applying (1.15), we get

$$\mathrm{e}^{(w-y)(t+u)}=\sum_{N=0}^{\infty}\frac{(w-y)^N\,(t+u)^N}{N!}=\sum_{p,q=0}^{\infty}(w-y)^{p+q}\,\frac{t^p}{p!}\,\frac{u^q}{q!}\tag{3.5}$$

Substituting (3.5) in (3.4) and using (1.12) when $p=1$ in the resulting 4-ple series, we obtain

$$\begin{aligned}&\sum_{n,\ell=0}^{\infty}\left\{\sum_{p=0}^{n}\sum_{q=0}^{\ell}\frac{1}{(n-p)!p!\,(\ell-q)!q!}\,(w-y)^{p+q}\,{}_Hb_{n+\ell-p-q}^{(k)}(x,y,z)\right\}t^nu^\ell\\&=\sum_{n,\ell=0}^{\infty}{}_Hb_{n+\ell}^{(k)}(x,w,z)\,\frac{t^n}{n!}\,\frac{u^\ell}{\ell!},\end{aligned}$$

which, upon equating the coefficients of t^n and u^ℓ, yields the desired identity. □

Theorem 3.2. *The following formula holds.*

$${}_Hb_n^{(k)}(x+u,y,z)=\sum_{j=0}^{n}\binom{n}{j}(-1)^j(-u)_j\,{}_Hb_{n-j}^{(k)}(x,y,z)\quad(n\in\mathbb{N}_0).\tag{3.6}$$

Proof. Recall the generalized binomial formula

$$(1-z)^{-\alpha}=\sum_{n=0}^{\infty}(-\alpha)_n\,\frac{z^n}{n!}\quad(\alpha\in\mathbb{C};\ |z|<1),\tag{3.7}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{3.8}$$

We find from (2.1), using (3.7) and (1.12) when $p = 1$, that

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_Hb_n^{(k)}(x+u,y,z)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\log(1+t)}(1+t)^x\,\mathrm{e}^{yt+zt^2}\,(1+t)^u \\ &= \sum_{n=0}^{\infty} {}_Hb_n^{(k)}(x,y,z)\frac{t^n}{n!}\sum_{j=0}^{\infty}(-1)^j\,(-u)_j\,\frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{1}{(n-j)!j!}(-1)^j\,(-u)_j\,{}_Hb_{n-j}^{(k)}(x,y,z)\,t^n, \end{aligned}$$

which, equating the coefficients of t^n, yields the desired identity. □

Theorem 3.3. *The following identity holds.*

$${}_Hb_n^{(k)}(x,y+u,z+w) = \sum_{m=0}^{n}\binom{n}{j}\,{}_Hb_{n-j}^{(k)}(x,y,z)\,H_j(u,w) \quad (n \in \mathbb{N}_0)\,. \tag{3.9}$$

Proof. Using (2.1) and (1.2), similarly as above, we can prove the desired identity. We omit the details. □

Theorem 3.4. *The following identity holds.*

$${}_Hb_n^{(k)}(x,y,z) = \sum_{j=0}^{[n/2]}\sum_{m=0}^{n-2j}\frac{n!}{m!j!(n-2j-m)!}\,y^{n-m-2j}\,z^j \quad (n \in \mathbb{N}_0)\,. \tag{3.10}$$

Proof. We find from (2.1) and (1.6) that

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_Hb_n^{(k)}(x,y,z) \\ &= \sum_{n=0}^{\infty}\frac{y^n\,t^n}{n!}\sum_{m=0}^{\infty}b_m^{(k)}(x)\frac{t^m}{m!}\sum_{j=0}^{\infty}\frac{z^j\,t^{2j}}{j!} \\ &= \sum_{j=0}^{\infty}\left\{\sum_{j=0}^{[n/2]}\sum_{m=0}^{n-2j}\frac{1}{m!j!(n-2j-m)!}\,y^{n-m-2j}\,z^j\right\}t^n, \end{aligned} \tag{3.11}$$

which, upon equating the coefficients of t^n, proves the desired identity. For the second equality in (3.11), we have used (1.12) when $p = 1$ and $p = 2$, successively. □

Theorem 3.5. *The following identity holds.*

$$
{}_Hb_n^{(k)}(x,y,z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j (-u)_j \, {}_Hb_{n-j}^{(k)}(x-u,y,z) \qquad (n \in \mathbb{N}_0)\,. \quad (3.12)
$$

Proof. Expressing

$$
\sum_{n=0}^{\infty} {}_Hb_n^{(k)}(x,y,z)\,\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\log(1+t)}(1+t)^{x-u}\,\mathrm{e}^{(yt+zt^2)} \cdot (1+t)^u
$$

and using (2.1) and (3.7) to expand the right side, similarly as above, we can prove the desired identity. We omit the details. □

4. Symmetric Identities

Here, we give two symmetric identities for the polynomials ${}_Hb_n^{(k)}(x,y,z)$ in (2.1). The first one is expressed in terms of the 2-variable Hermite polynomials $H_n(x,y)$ in (1.2) and the poly-Bernoulli polynomials of the second kind $b_n^{(k)}(x)$ in (1.6), which is asserted by Theorem 4.1. The other one is expressed in terms of the 2-variable Hermite polynomials $H_n(x,y)$ in (1.2) and the alternating sums in (1.3) as given in Theorem 4.2. For other known symmetric identities, we refer, for example, to Khan [7]-[9] and Pathan et al. [14]-[18].

Theorem 4.1. *Let c, $d \in \mathbb{R}^+$ with $c \neq d$ and m, $n \in \mathbb{N}_0$. Then*

$$
\begin{aligned}
&\sum_{j=0}^{n} \frac{c^{n-j}\,d^j}{(n-j)!\,j!}\, {}_Hb_{n-j}^{(k)}(cx,dy,d^2z)\, {}_Hb_j^{(k)}(dx,cy,c^2z) \\
&= \sum_{j=0}^{n-m} \sum_{\ell=0}^{m} \frac{c^{n-m}\,d^m}{(n-m-j)!j!(m-\ell)!\ell!} \\
&\quad \times b_{n-m-j}^{(k)}(cx)\, b_{n-\ell}^{(k)}(dx)\, H_j(dy,d^2z)\, H_\ell(cy,c^2z).
\end{aligned} \tag{4.1}
$$

Proof. We start to define a function

$$
f(t) := \frac{\mathrm{Li}_k\left(1-\mathrm{e}^{-ct}\right)\,\mathrm{Li}_k\left(1-\mathrm{e}^{-dt}\right)}{\log(1+ct)\,\log(1+dt)}\,(1+ct)^{cx}\,(1+dt)^{dx}\,\mathrm{e}^{2cdyt}\,\mathrm{e}^{2c^2d^2zt^2}. \tag{4.2}
$$

It is easy to see that $f(t)$ is symmetric in c and d. Then we write

$f(t) := f_1(t)\, f_2(t)$, where

$$f_1(t) = \frac{\mathrm{Li}_k\left(1 - \mathrm{e}^{-ct}\right)}{\log(1+ct)}\,(1+ct)^{cx}\,\mathrm{e}^{cdyt}\,\mathrm{e}^{c^2d^2zt^2}$$

and

$$f_2(t) = \frac{\mathrm{Li}_k\left(1 - \mathrm{e}^{-dt}\right)}{\log(1+dt)}\,(1+dt)^{dx}\,\mathrm{e}^{cdyt}\,\mathrm{e}^{c^2d^2zt^2}.$$

We use (2.1) to obtain

$$\begin{aligned} f(t) &= f_1(t)\, f_2(t) \\ &= \left\{\sum_{n=0}^{\infty} {}_Hb_n^{(k)}(cx, dy, d^2z)\,\frac{(ct)^n}{n!}\right\}\left\{\sum_{j=0}^{\infty} {}_Hb_j^{(k)}(dx, cy, c^2z)\,\frac{(dt)^j}{j!}\right\}, \end{aligned}$$

which, upon applying (1.12) with $p = 1$ to combine the two series, yields

$$f(t) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{c^{n-j}\,d^j}{(n-j)!j!}\,{}_Hb_{n-j}^{(k)}(cx, dy, d^2z)\,{}_Hb_j^{(k)}(dx, cy, c^2z)\,t^n. \tag{4.3}$$

We use (1.2) and (1.6), with the aid of (1.12) with $p = 1$, to find

$$\begin{aligned} f_1(t) &= \left\{\sum_{n=0}^{\infty} b_n^{(k)}(cx)\,\frac{(ct)^n}{n!}\right\}\left\{\sum_{j=0}^{\infty} H_j(dy, d^2z)\,\frac{(ct)^j}{j!}\right\} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} b_{n-j}^{(k)}(cx)\,H_j(dy, d^2z)\,\frac{(ct)^n}{(n-j)!j!}. \end{aligned}$$

Similarly, we have

$$f_2(t) = \sum_{m=0}^{\infty}\sum_{\ell=0}^{m} b_{m-\ell}^{(k)}(dx)\,H_\ell(cy, c^2z)\,\frac{(dt)^m}{(m-\ell)!\ell!}.$$

We apply (1.12) with $p = 1$ to combine the two series expressions of $f_1(t)$ and $f_1(t)2$ and get

$$\begin{aligned} f(t) = \sum_{n=0}^{\infty} c^{n-m}\,d^m \sum_{j=0}^{n-m}\sum_{\ell=0}^{m} & \frac{1}{(n-m-j)!j!(m-\ell)!\ell!} \\ & \times b_{n-m-j}^{(k)}(cx)\,b_{m-\ell}^{(k)}(dx)\,H_j(dy, d^2z)\,H_\ell(cy, c^2z)\;t^n. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we equate the coefficients of t^n to yield the desired identity (4.1). □

Theorem 4.2. *Let c, $d \in \mathbb{R}^+$ with $c \neq d$, $\alpha \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned} &\sum_{p=0}^{n} \sum_{m=0}^{n-p} \sum_{q=0}^{p} \frac{c^{n-m-q}\, d^{m+q}}{(n-p-m)!m!(p-q)!q!} \\ &\quad \times {}_H b_{n-p-m}^{(k)}(dx, dy, d^2 z)\, {}_H b_m^{(k)}(cx, cy, c^2 z)\, T_{p-q}(\alpha)\, T_q(\alpha) \\ &= \sum_{m=0}^{n} \sum_{q=0}^{n-m} \sum_{p=0}^{m} \frac{c^{n+p-m-q}\, d^{m-p+q}}{(n-m-q)!q!(m-p)!p!} \\ &\quad \times {}_H b_{n-m-q}^{(k)}(dx, dy, d^2 z)\, {}_H b_{m-p}^{(k)}(cx, cy, c^2 z)\, T_p(\alpha)\, T_q(\alpha). \end{aligned} \tag{4.5}$$

Proof. Consider a function

$$\begin{aligned} g(t) :=& \frac{\mathrm{Li}_k\left(1-\mathrm{e}^{-ct}\right)\, \mathrm{Li}_k\left(1-\mathrm{e}^{-dt}\right)}{\log(1+ct)\, \log(1+dt)}\, (1+ct)^{dx}\, (1+dt)^{cx}\, \mathrm{e}^{2cdyt}\, \mathrm{e}^{2c^2d^2zt^2} \\ &\times \frac{1-\left(-\mathrm{e}^{-ct}\right)^{\alpha}}{1+\mathrm{e}^{ct}}\, \frac{1-\left(-\mathrm{e}^{-dt}\right)^{\alpha}}{1+\mathrm{e}^{dt}}. \end{aligned} \tag{4.6}$$

It is noted that $g(t)$ is symmetric in c and d. By applying (1.3) and (2.1) to the function $g(t)$, we obtain

$$\begin{aligned} g(t) =& \left\{\sum_{n=0}^{\infty} {}_H b_n^{(k)}(dx, dy, d^2 z)\, \frac{(ct)^n}{n!}\right\} \left\{\sum_{m=0}^{\infty} {}_H b_m^{(k)}(cx, cy, c^2 z)\, \frac{(dt)^m}{m!}\right\} \\ &\times \left\{\sum_{p=0}^{\infty} T_p(\alpha) \frac{(ct)^p}{p!}\right\} \left\{\sum_{q=0}^{\infty} T_q(\alpha) \frac{(dt)^q}{q!}\right\}. \end{aligned} \tag{4.7}$$

First, we apply (1.12) with $p = 1$ to combine the first two series and the last two series in (4.7). Then, we apply (1.12) with $p = 1$ to combine the resulting two series to get

$$\begin{aligned} g(t) =& \sum_{n=0}^{\infty} \left\{ \sum_{p=0}^{n} \sum_{m=0}^{n-p} \sum_{q=0}^{p} \frac{c^{n-m-q}\, d^{m+q}}{(n-p-m)!m!(p-q)!q!} \right. \\ &\left. \times {}_H b_{n-p-m}^{(k)}(dx, dy, d^2 z)\, {}_H b_m^{(k)}(cx, cy, c^2 z)\, T_{p-q}(\alpha)\, T_q(\alpha) \right\} t^n. \end{aligned} \tag{4.8}$$

Secondly, we apply (1.12) with $p = 1$ to combine the first and fourth series and the second and third series. Then, we apply (1.12) with $p = 1$ to combine the resulting two series to find

$$g(t) = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \sum_{q=0}^{n-m} \sum_{p=0}^{m} \frac{c^{n+p-m-q}\, d^{m-p+q}}{(n-m-q)!q!(m-p)!p!} \right. \\ \left. \times\, {}_H b_{n-m-q}^{(k)}(dx, dy, d^2 z)\, {}_H b_{m-p}^{(k)}(cx, cy, c^2 z)\, T_p(\alpha)\, T_q(\alpha) \right\} t^n. \tag{4.9}$$

Finally, equating the coefficients of t^n in (4.8) and (4.9), we obtain the desired identity (4.5). □

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