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APPROXIMATING FIXED POINTS OF LIPSCHITZ PSEUDO-CONTRACTIVE SEMIGROUPS AND SOLUTIONS OF VARIATIONAL INEQUALITIES

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. Let C be a closed and convex subset of a real Hilbert space H, $\mathcal{T}$ be a pseudo-contractive semigroup on C and A be a Lipschitz monotone mapping from C into H. In this paper, we propose and investigate an iterative scheme for finding a common element of the set of common fixed points of $\mathcal{T}$ and the set of solutions of the variational inequality $VI(A, C)$. As a consequence, we prove that the scheme generated by the new method converges strongly under mild conditions on the parameters involved.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $T : C \to H$ is called *Lipschitz* if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y|| \quad \forall x, y \in C. \tag{1.1}$$

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If in (1.1) $L = 1$ the T is called *nonexpansive*. A mapping $T : C \to C$ is said to be *pseudo-contractive* if it satisfies the inequality

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2, \quad \forall x, y \in C.$$

It is not hard to show that T is a pseudo-contraction if and only if T satisfies the following property:

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2, \ \ \forall x, y \in C.$$

A mapping T is said to be *k-strictly pseudo-contractive* if there exists $k \in (0, 1)$ such that T satisfies the property

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \ \ \forall x, y \in C.$$

We note that the class of pseudo-contractive mappings includes the class of k-strictly pseudo-contractive and the class of nonexpansive mappings.

A one-parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$ is said to be a *pseudo-contractive semigroup* on C if the following conditions are satisfied:
(1) $T(0)x = x$ for all $x \in C$;
(2) $T(s + t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
(3) $T(t)$ is pseudo-contractive for each $t \geq 0$;
(4) for each $x \in C$, the mapping $T(\cdot)x$ from $[0, \infty)$ into C is continuous.

If the mapping $T(t)$ in condition (3) is replaced with $(3)'$ $T(t)$ is k-strictly pseudo-contractive for each $t \geq 0$, then $\mathcal{T} := \{T(t) : t \geq 0\}$ is said to be a *k-strictly pseud-contractive semigroup* on C. If the mapping $T(t)$ in condition (3) is replaced with $(3)''$ $T(t)$ is nonexpansive for each $t \geq 0$, then $\mathcal{T} := \{T(t) : t \geq 0\}$ is said to be a *nonexpansive semigroup* on C. We denote by $F(\mathcal{T})$ the set of common fixed points of pseudo-contractive semigroup $\mathcal{T}$, that is,

$$F(\mathcal{T}) = \cap_{t\geq 0} F(T(t)) = \{x \in C : T(t)x = x \text{ for each } t \geq 0\}.$$

A semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$ is said to be *Lipschitz* if there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that for any $x, y \in C$,

$$||T(t)x - T(t)y|| \leq L(t)||x - y||, \text{ for any } t \geq 0 \ . \tag{1.2}$$

In the sequel we denote the Lipschitz constant L, by $L = \sup\limits_{t\geq 0} L(t) < \infty$.

We remark that the class of pseudo-contractive semigroups includes the class of nonexpansive semigroups. The following example shows that the inclusion is proper.

Example 1.1. Let $\Omega := \{(a, 0, 0, b), a, b \in \mathbb{R}\}$ which is a subspace of $\mathbb{R}^4$. For $t \geq 0$, let $T(t) : \Omega \to \mathbb{R}^4$ be defined by

$$T(t)(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ -t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

Then one can easily show that $\mathcal{T} = \{T(t) : 0 \leq t \leq \beta\}$, for some $\beta > 0$, is Lipschitz pseudo-contractive semigroup which is not nonexpansive semigroup (see, [12]).

It is well known that the semigroup theory has applications in partial differential equations, evolutionary equations and fixed point theories. We note that the nonexpansive semigroup and pseudo-contractive semigroup have been studied by several authors (see, for example, [7, 21, 26] and the references therein) and are directly linked to solutions of differential equations.

Let C be a nonempty, closed and convex subset of H. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ be a semigroup from K into itself and let $f : K \to K$ be a contractive mapping. It follows from the Banach's fixed point theorem that the following implicit viscosity iteration process is well defined:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \forall n \geq 1, \tag{1.3}$$

where $\alpha_n \in (0, 1)$ and $T(t_n) \in \mathcal{T}$. Several authors studied the convergence of iteration process (1.3) for nonexpansive semigroups in certain Banach spaces (see, for example, [11, 32] and the references therein). In 2007, Chen and He [7] studied the convergence of (1.3) for nonexpansive semigroups in reflexive Banach spaces with weakly sequentially continuous duality mappings.

An interesting work is to extend the above results to the class of pseudo-contractive semigroup mappings. In [18], Qin and Cho proved the convergence of the following implicit iteration process for Lipschitz pseudo-contractive semigroup mappings under appropriate conditions:

$$x_0 \in K, x_n = \alpha_n x_{n-1} + \beta_n T(t_n)x_n + \gamma_n u_n, \forall n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $\{t_n\}$ is a sequence in $(0, \infty)$ and $\{u_n\}$ is a bounded sequence in K. Furthermore, Li *et al.* [12] discussed the convergence of (1.4) for pseudo-contractive semigroup and obtained convergence results under some mild conditions. Subsequent research related to pseudo-contractive semigroup can be referred in [8, 13, 18].

Let C be a nonempty, closed and convex subset of E and let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find $x^* \in C$

such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{1.5}$$

We shall denote the solution set of the variational inequality (1.5) by $VI(C, A)$. It is known that the solution set $VI(C, A)$ is closed and convex whenever A is monotone and continuous. The theory of variational analysis has emerged as a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied mathematics, such as mechanics, physics, engineering, the theory of convex programming, and the theory of control: see, for instance, [14, 16, 23, 26, 28].

There are several iterative methods for solving $VI(C, A)$ (see, e.g., [1, 2, 5, 6, 10, 16, 29]). The basic idea consists of the projected gradient method $\{x_n\}$ given by

$$x_{n+1} = P_C[x_n - \alpha_n A x_n], n \geq 0, \tag{1.6}$$

where A is monotone mapping. A mapping $A : C \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \text{ for all } x, y \in C. \tag{1.7}$$

A is called γ-*inverse strongly monotone* if there exists a positive real number γ such that

$$\langle Ax - Ay, x - y \rangle \geq \gamma ||Ax - Ay||^2, \text{ for all } x, y \in C. \tag{1.8}$$

We observe that any γ-inverse strongly monotone A is Lipschitz with $L = \frac{1}{\gamma}$.

Clearly, the class of monotone mappings includes the class of γ-inverse strongly monotone mappings.

In [15], Nadezhkina1 and Takahashi suggested the following modified Korpelevich's method for a solution of a variational inequality $VI(C, A)$ for L-Lipschitz continuous monotone mapping A in infinite-dimensional Hilbert spaces. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n], \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C[x_n - \lambda_n A y_n], n \geq 0, \end{cases} \tag{1.9}$$

where P_C is a metric projection from H onto C, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, they proved that the sequences $\{x_n\}, \{y_n\}$ converge *weakly* to the minimum-norm point of $VI(C, A)$.

Several authors studied to obtain strong convergence by modifying the original method of Korpelevich's. For example, in [2, 9, 27], it is proved that some very interesting Korpelevich-type algorithms strongly converge to a solution of $VI(C, A)$.

Recently, the problem of finding a common point of element of common fixed point set of a nonexpansive semigroup mappings and a solution set of a variational inequality has been considered by many authors; see, for example, [4, 24] and the references therein.

In [4], Buong introduced a hybrid method and showed that under appropriate conditions, the sequences generated by the method converges strongly to an element of the set of common fixed points of a nonexpansive semigroup $\mathcal{T}$ and the solution $VI(A,C)$ for a monotone Lipschitz continuous mapping A. Moreover, in 2014, Thuy [24] studied another hybrid algorithm for finding a common element of the set of fixed points of a nonexpansive semigroup and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping in Hilbert spaces.

More recently, Boikanyo and Zegeye [3] studied the problem of finding a common element of the set of common fixed points of a nonexpansive semigroup $\mathcal{T}$ and the set of solutions to a variational inclusion for Lipschitz monotone mapping A by considering the following iterative algorithm:

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n] \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[a_n x_n + b_n P_C[x_n - \lambda_n A y_n] + c_n S(t_n) x_n], \end{cases} \tag{1.10}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences of reals in $(0,1)$, $\lambda_n > 0$ and $\alpha_n \in (0,1)$. They proved that under appropriate assumptions on these parameters, both sequences $\{y_n\}$ and $\{x_n\}$ converge strongly to a point $\hat{x} \in \mathfrak{F} = F(\mathcal{T}) \cap VI(A,C)$, which is the unique solution of the variational inequality $\langle (I - f)\hat{x}, x - \hat{x} \rangle \geq 0$ for all $x \in \mathfrak{F}$.

The above results naturally bring us to the following question.

Question. *Can we produce an iterative scheme that converges to a common point of the set of common fixed points of a pseudo-contractive semigroup and the set of solutions to a variational inequality for Lipschitz monotone mapping?*

It is our purpose in this paper to propose an iterative scheme which converges strongly to a point of common fixed point set of a pseudo-contractive semigroup and the set of solutions to a variational inequality for Lipschitz monotone mapping. As a consequence, we obtain a convergence theorem for approximating a common fixed point of a one-parameter pseudo-contractive semigroup. The results obtained in this paper improve and extend the results of Takahashi and Toyoda [23], Buong [4], Thuy [24] and Boikanyo and Zegeye [3] to more general class of pseudo-contractive semigroups.

2. Preliminaries

In the sequel, H represents a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. It is known that for any $x, y \in H$, the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle, \tag{2.1}$$

holds true. In addition, it can be proved easily that if α, β, γ are any real numbers in $(0,1)$ with $\alpha + \beta + \gamma = 1$, then for any $x, y, z \in H$, we have

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x-y\|^2 \\ &\quad - \alpha\gamma \|x-z\|^2 - \beta\gamma \|y-z\|^2, \end{aligned} \tag{2.2}$$

(see e.g., [17, 30]). If C is nonempty, closed and convex, then the nearest point projection of H onto C is denoted by P_C, that is, $||x - P_C x|| \leq ||x - y||$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that for any given $x \in H$, we have

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.3}$$

for all $y \in C$, (see e.g., [22]). Note that (2.3) implies

$$\|P_C x - y\|^2 \leq \|x-y\|^2 - \|x - P_C x\|^2, \tag{2.4}$$

for all $y \in C$. Moreover, the following equivalence holds:

$$x^* \in VI(A, C) \quad \Leftrightarrow \quad x^* = P_C(x^* - \lambda A x^*), \quad \lambda > 0.$$

A space X is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, then

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||, \text{ for all } y \in X \text{ with } y \neq x. \tag{2.5}$$

It is well known that every Hilbert space satisfies Opial's condition. We conclude this section by giving two lemmas that will be used in proving our main result.

Lemma 2.1. ([25]) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the conditions:

(i) $\{a_n\} \subset [0,1]$, *with* $\sum_{n=0}^{\infty} a_n = \infty$,
(ii) $c_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} c_n < \infty$, *and*
(iii) $\limsup_{n\to\infty} b_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. ([14]) *Let $\{s_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $s_{k_j} < s_{k_j+1}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k\geq k_0}$ as*

$$m_k = \max\{k_0 \leq l \leq k : \; s_l < s_{l+1}\}.$$

Then $m_k \to \infty$ as $k \to \infty$ and for all $k \geq k_0$

$$\max\{s_{m_k}, s_k\} \leq s_{m_k+1}. \tag{2.6}$$

3. Main Results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a Lipschitz monotone mapping with Lipschitz constant L. Let $\mathcal{T} = \{T(t) : t > 0\}$ be a Lipschitz pseudo-contractive semigroup on C with $M := \sup_{t\geq 0}\{L(t)\} < \infty$, where $L(t)$ is the Lipschitz constant of the mapping $T(t)$. Assume that $\mathfrak{F} := F(\mathcal{T}) \cap VI(C, A)$ is not empty. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n] \\ z_n = (1-\beta_n)x_n + \beta_n T(t_n)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[a_n x_n + b_n P_C[x_n - \lambda_n A y_n] + c_n T(t_n) z_n], \; n \geq 1, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, L^{-1})$, and $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\} \subset (0, 1)$ with $a_n + b_n + c_n = 1$. Assume that $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \leq \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Then $\{x_n\}$ is bounded.

Proof. We begin by showing that for any $q \in \mathfrak{F}$ and for all $n \geq 0$,

$$\|x_{n+1} - q\|^2 \leq \alpha_n \|u - q\|^2 + (1-\alpha_n) \|x_n - q\|^2 . \tag{3.2}$$

Denote $u_n := P_C(x_n - \lambda_n A y_n)$ and $v_n := a_n x_n + b_n P_C(x_n - \lambda_n A y_n) + c_n T(t_n) z_n$. Then

$$v_n = a_n x_n + b_n u_n + c_n T(t_n) z_n.$$

For any $q \in \mathfrak{F}$, we have from (2.2)

$$\begin{aligned} \|v_n - q\|^2 &= \|a_n(x_n - q) + b_n(u_n - q) + c_n[T(t_n)z_n - q]\|^2 \\ &= a_n \|x_n - q\|^2 + b_n \|u_n - q\|^2 + c_n \|T(t_n)z_n - q\|^2 \\ &\quad - a_n b_n \|x_n - u_n\|^2 - a_n c_n \|x_n - T(t_n)z_n\|^2 \\ &\quad - b_n c_n \|u_n - T(t_n)z_n\|^2 . \end{aligned} \tag{3.3}$$

Again from (2.2), we have

$$\begin{aligned}\|z_n-q\|^2 &= \|(1-\beta_n)(x_n-q)+\beta_n(T(t_n)x_n-q)\|^2\\ &= (1-\beta_n)\|x_n-q\|^2+\beta_n\|T(t_n)x_n-q\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-T(t_n)x_n\|^2\\ &\leq (1-\beta_n)\|x_n-q\|^2+\beta_n[\|x_n-q\|^2+\|x_n-T(t_n)x_n\|^2]\\ &\quad -\beta_n(1-\beta_n)\|x_n-T(t_n)x_n\|^2\\ &= \|x_n-q\|^2+\beta_n^2\|x_n-T(t_n)x_n\|^2, \end{aligned} \tag{3.4}$$

where the inequality follows from the pseudo-contractive property of the semigroup. In addition,

$$\begin{aligned}\|z_n-T(t_n)z_n\|^2 &= \|(1-\beta_n)(x_n-T(t_n)z_n)+\beta_n(T(t_n)x_n-T(t_n)z_n)\|^2\\ &= (1-\beta_n)\|x_n-T(t_n)z_n\|^2+\beta_n\|T(t_n)x_n-T(t_n)z_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-T(t_n)x_n\|^2\\ &\leq (1-\beta_n)\|x_n-T(t_n)z_n\|^2+\beta_n M^2\|x_n-z_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-T(t_n)x_n\|^2, \end{aligned}$$

where the inequality follows from the fact that the semigroup is Lipschitzian. Using (3.1), we arrive at

$$\begin{aligned}\|z_n-T(t_n)z_n\|^2 &\leq (1-\beta_n)\|x_n-T(t_n)z_n\|^2\\ &\quad -\beta_n(1-\beta_n-\beta_n^2M^2)\|x_n-T(t_n)x_n\|^2. \end{aligned} \tag{3.5}$$

Using the pseudo-contractive property of the semigroup, we get

$$\begin{aligned}\|T(t_n)z_n-q\|^2 &\leq \|z_n-q\|^2+\|z_n-T(t_n)z_n\|^2\\ &\leq \|x_n-q\|^2+(1-\beta_n)\|x_n-T(t_n)z_n\|^2\\ &\quad -\beta_n(1-2\beta_n-\beta_n^2M^2)\|x_n-T(t_n)x_n\|^2, \end{aligned} \tag{3.6}$$

where the last inequality follows from (3.4) and (3.5).

On the other hand, from property (2.4) of projections, we have

$$\begin{aligned}\|u_n-q\|^2 &\leq \|x_n-\lambda_nAy_n-q\|^2-\|x_n-\lambda_nAy_n-u_n\|^2\\ &= \|x_n-q\|^2-\|x_n-u_n\|^2+2\lambda_n\langle Ay_n,q-u_n\rangle\\ &= \|x_n-q\|^2-\|x_n-u_n\|^2+2\lambda_n\langle Ay_n-Aq,q-y_n\rangle\\ &\quad +2\lambda_n[\langle Aq,q-y_n\rangle+\langle Ay_n,y_n-u_n\rangle]. \end{aligned}$$

From the conditions $y_n \in C$ and $q \in \mathfrak{F}$, we conclude that $\langle Aq, q - y_n\rangle \leq 0$. Also, by the monotonicity of A, we have $\langle Ay_n - Aq, q - y_n\rangle \leq 0$. Therefore,

$$\begin{aligned}\|u_n - q\|^2 &\leq \|x_n - q\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\langle Ay_n, y_n - u_n\rangle \\ &= \|x_n - q\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 \\ &\quad -2\langle x_n - y_n, y_n - u_n\rangle + 2\lambda_n\langle Ay_n, y_n - u_n\rangle \\ &= \|x_n - q\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 \\ &\quad -2\langle x_n - y_n - \lambda_n Ay_n, y_n - u_n\rangle.\end{aligned}$$

Note that from the property (2.3) of projections, we have

$$\langle x_n - \lambda_n Ax_n - y_n, u_n - y_n\rangle \leq 0.$$

Therefore, using the L-Lipschitz continuity of A, we have

$$\begin{aligned}2\langle x_n - y_n - \lambda_n Ay_n, u_n - y_n\rangle &= 2\langle x_n - \lambda_n Ax_n - y_n, u_n - y_n\rangle \\ &\quad +2\lambda_n\langle Ax_n - Ay_n, u_n - y_n\rangle \\ &\leq 2\lambda_n \|Ax_n - Ay_n\| \|y_n - u_n\| \\ &\leq 2\lambda_n L \|x_n - y_n\| \|y_n - u_n\| \\ &\leq \lambda_n L(\|x_n - y_n\|^2 + \|y_n - u_n\|^2).\end{aligned}$$

Thus,

$$\|u_n - q\|^2 \leq \|x_n - q\|^2 - (1 - \lambda_n L)\left(\|x_n - y_n\|^2 + \|y_n - u_n\|^2\right).$$

Using this last inequality and (3.6) into (3.3), we get

$$\begin{aligned}\|v_n - q\|^2 &\leq \|x_n - q\|^2 - a_n b_n \|x_n - u_n\|^2 \\ &\quad -c_n(\beta_n - b_n - c_n)\|x_n - T(t_n)z_n\|^2 - b_n c_n \|u_n - T(t_n)z_n\|^2 \\ &\quad -b_n(1 - \lambda_n L)\left(\|x_n - y_n\|^2 + \|y_n - u_n\|^2\right) \\ &\quad -c_n\beta_n(1 - 2\beta_n - \beta_n^2 M^2)\|x_n - T(t_n)x_n\|^2 .\end{aligned} \tag{3.7}$$

Since $\beta_n < \beta$ and $\beta(\sqrt{1 + M^2} + 1) < 1$, it follows that

$$1 - 2\beta_n - \beta_n^2 M^2 \geq 1 - 2\beta - \beta^2 M^2 > 0.$$

On using the assumptions $\lambda_n L < 1$ and $b_n + c_n < \beta_n$, together with this last inequality, we obtain from (3.7) that

$$\|v_n - q\| \leq \|x_n - q\|.$$

Furthermore, from (3.1) and (2.2), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(u - q) + (1 - \alpha_n)(v_n - q)\|^2 \\ &= \alpha_n \|u - q\|^2 + (1 - \alpha_n)\|v_n - q\|^2 - \alpha_n(1 - \alpha_n)\|u - v_n\|^2 \\ &\leq \alpha_n \|u - q\|^2 + (1 - \alpha_n)\|x_n - q\|^2 .\end{aligned}$$

This last inequality is exactly (3.2).

We next show that the sequence $\{x_n\}$ is bounded. For this, it suffices to show that for any $q \in \mathfrak{F}$,

$$\|x_n - q\|^2 \le K := \|u - q\|^2 + \|x_0 - q\|^2 , \tag{3.8}$$

holds for all $n \ge 0$. We show by induction that the sequence $\{x_n - q\}$ is bounded. If $n = 0$, then (3.8) holds trivially. We assume that (3.8) holds for some $n = k > 0$. Then from (3.2), we have

$$\begin{aligned} \|x_{k+1} - q\|^2 &\le \alpha_k \|u - q\|^2 + (1 - \alpha_k) \|x_k - q\|^2 . \\ &\le \alpha_k K + (1 - \alpha_k) K = K. \end{aligned}$$

This shows that (3.8) also holds for $n = k + 1$. Hence, $\{x_n\}$ is bounded. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a Lipschitz monotone mapping with constant L. Let $\mathcal{T} := \{T(t) : t > 0\}$ be a Lipschitz pseudo-contractive semigroup on C with $M := \sup_{t\ge0}\{L(t)\} < \infty$, where $L(t)$ is the Lipschitz constant of the mapping $T(t)$. Assume that $\mathfrak{F} := F(\mathcal{T}) \cap VI(C, A)$ is not empty. Let $\{x_n\}$ be the sequence generated by (3.1) satisfying $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \le \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Assume that $t_n > 0$ (for all $n \ge 0$) satisfies $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$, and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \mathfrak{F}$, which is the unique point in the set $\mathfrak{F}$ that is nearest to u.*

Proof. From Theorem 3.1 we have that the sequence $\{x_n\}$ is bounded. Now using inequality (2.1), we get

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &= \|\alpha_n(u - \hat{x}) + (1 - \alpha_n)(v_n - \hat{x})\|^2 \\ &\le (1 - \alpha_n) \|v_n - \hat{x}\|^2 + 2\alpha_n \langle u - \hat{x}, x_{n+1} - \hat{x}\rangle \end{aligned}$$

Substituting (3.7) (with $\hat{x}$ instead of q) into this last inequality, we obtain

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 \le\ & (1-\alpha_n) \Big[\|x_n-\hat{x}\|^2 - b_n(1-\lambda_n L) \Big(\|x_n-y_n\|^2 + \|y_n-u_n\|^2\Big)\Big] \\ & -(1 - \alpha_n) b_n \Big[a_n \|x_n - u_n\|^2 + c_n \|u_n - T(t_n) z_n\|^2\Big] \\ & -(1 - \alpha_n) c_n \beta_n (1 - 2\beta_n - \beta_n^2 M^2) \|x_n - T(t_n) x_n\|^2 \\ & -(1 - \alpha_n) c_n (\beta_n - b_n - c_n) \|x_n - T(t_n) z_n\|^2 \\ & +2\alpha_n \langle u - \hat{x}, x_{n+1} - \hat{x}\rangle . \end{aligned} \tag{3.9}$$

Now, from this last inequality we show that the sequence $\{x_n\}$ converges strongly to $\hat{x} = P_{\mathfrak{F}} u$. It suffices to show that $\{x_n - \hat{x}\}$ converges strongly to zero. For this, we consider two possible cases on $\{x_n - \hat{x}\}$. Denote $s_n := x_n - \hat{x}$.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers $\{s_n\}$ is decreasing for all $n \geq n_0$. It then follows that $\{s_n\}$ is convergent. Since the sequences $\{b_n\}$ and $\{c_n\}$ are bounded below away from zero, $b_n + c_n \leq \beta_n < \beta$ and $1 - 2\beta_n - \beta_n^2 M^2 \geq 1 - 2\beta - \beta^2 M^2 > 0$ for all $n \geq 0$, it follows from (3.9) and the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = 0 = \lim_{n\to\infty} \|x_n - T(t_n)z_n\|. \tag{3.10}$$

Since, the sequence $\{x_n\} \subset H$ is bounded we can extract a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to $z \in C$ and satisfying

$$\limsup_{n\to\infty} \langle u - \hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty} \langle u - \hat{x}, x_{n_j} - \hat{x}\rangle. \tag{3.11}$$

Without loss of generality, as in [20], let

$$\lim_{j\to\infty} t_{n_j} = \lim_{j\to\infty} \frac{\left\|x_{n_j} - T(t_{n_j})x_{n_j}\right\|}{t_{n_j}} = 0. \tag{3.12}$$

Our aim is to show that $z \in F(\mathcal{T}) \cap VI(C, A) = \mathfrak{F}$. We first show that $z \in F(\mathcal{T})$, that is, $z = T(t)z$ for a fixed $t > 0$. It is easy to see that

$$\begin{aligned}
\left\|x_{n_j} - T(t)x_{n_j}\right\| &\leq \sum_{k=0}^{[t/t_{n_j}]-1} \left\|T(kt_{n_j})x_{n_j} - T((k+1)t_{n_j})x_{n_j}\right\| \\
&\quad + \left\|T\left(\left[\frac{t}{t_{n_j}}\right] t_{n_j}\right) x_{n_j} - T(t)x_{n_j}\right\| \\
&\leq \left[\frac{t}{t_{n_j}}\right] \left\|x_{n_j} - T(t_{n_j})x_{n_j}\right\| M \\
&\quad + \left\|T\left(t - \left[\frac{t}{t_{n_j}}\right] t_{n_j}\right) x_{n_j} - x_{n_j}\right\| M \\
&\leq Mt \cdot \frac{\left\|x_{n_j} - T(t_{n_j})x_{n_j}\right\|}{t_{n_j}} + M \sup_{0\leq v<t_{n_j}} \left\|T(v)x_{n_j} - x_{n_j}\right\|,
\end{aligned}$$

where the second inequality follows from the fact that the semigroup is Lipschitzian. Passing to the limit as $j \to \infty$ in the above inequality and making use of (3.12) as well as the continuity of the semigroup, we arrive at

$$\lim_{j\to\infty} \left\|x_{n_j} - T(t)x_{n_j}\right\| = 0. \tag{3.13}$$

Moreover, from the Lipschitz and pseudo-contractive property of the semigroup, we have

$$\begin{aligned}
\left\|x_{n_j} - T(t)z\right\|^2 &= \left\|x_{n_j} - T(t)x_{n_j} + T(t)x_{n_j} - T(t)z\right\|^2 \\
&= \left\|x_{n_j} - T(t)x_{n_j}\right\|^2 + \left\|T(t)x_{n_j} - T(t)z\right\|^2 \\
&\quad +2\left\langle x_{n_j} - T(t)x_{n_j}, T(t)x_{n_j} - T(t)z\right\rangle \\
&\leq \left\|x_{n_j} - T(t)x_{n_j}\right\|^2 + \left[\left\|x_{n_j} - z\right\|^2 + \left\|x_{n_j} - T(t)x_{n_j}\right\|^2\right] \\
&\quad +2\left\|x_{n_j} - T(t)x_{n_j}\right\|\left\|T(t)x_{n_j} - T(t)z\right\| \\
&\leq 2\left\|x_{n_j} - T(t)x_{n_j}\right\|^2 + \left\|x_{n_j} - z\right\|^2 \\
&\quad +2M\left\|x_{n_j} - T(t)x_{n_j}\right\|\left\|x_{n_j} - z\right\|.
\end{aligned}$$

The above inequality together with (3.13) imply that

$$\limsup_{j\to\infty}\left\|x_{n_j} - T(t)z\right\| \leq \limsup_{j\to\infty}\left\|x_{n_j} - z\right\|.$$

Since every Hilbert space satisfies Opial's condition, we conclude that $z = T(t)z$. Therefore, $z \in F(\mathcal{T})$.

It remains to show that $z \in VI(C, A)$. To this end, we start by observing that from (3.9) and our assumption that $\{s_n\}$ is convergent, we deduce

$$\lim_{n\to\infty}\|x_n - y_n\| = 0 = \lim_{n\to\infty}\|y_n - u_n\|. \tag{3.14}$$

Therefore, from the inequality

$$\|x_n - u_n\| \leq \|x_n - y_n\| + \|y_n - u_n\|,$$

we obtain the limit

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.15}$$

Moreover, from (3.14) and the Lipschitz continuity of A, we have

$$\lim_{n\to\infty}\|Ay_n - Au_n\| = 0 \tag{3.16}$$

Now define (as is the case in [1]) a monotone operator T on H by

$$S(x) = \begin{cases} Ax + N_C(x), & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C, \end{cases}$$

where $N_C(x) := \{w \in H | \langle x - y, w\rangle \geq 0 \text{ for all } y \in C\}$ is the normal cone to C at $x \in C$. The map S is maximal monotone (see [19]) and $0 \in S(x)$ if and only if $x \in VI(C, A)$. Note that if $v \in C$ and $w \in S(v)$, then $w - Av \in N_C(v)$. Therefore, by the definition of $N_C(v)$, we have $\langle v - y, w - Av\rangle \geq 0$ for all $y \in C$. On the other hand, since $u_{n_j} = P_C(x_{n_j} - \lambda_{n_j}Ay_{n_j})$, we have from (2.3) that $\langle x_{n_j} - \lambda_{n_j} Ay_{n_j} - u_{n_j}, u_{n_j} - v\rangle \geq 0$, or equivalently $0 \geq \langle v - u_{n_j}, (x_{n_j} - u_{n_j})/\lambda_{n_j} - Ay_{n_j}\rangle$. Therefore, from $u_{n_j} \in C$ and $w - Av \in N_C(v)$, we have

$$\begin{aligned}
\langle v - u_{n_j}, w\rangle &\geq \langle v - u_{n_j}, Av\rangle \\
&\geq \langle v - u_{n_j}, Av\rangle + \langle v - u_{n_j}, (x_{n_j} - u_{n_j})/\lambda_{n_j} - Ay_{n_j}\rangle \\
&= \langle v - u_{n_j}, Av - Au_{n_j}\rangle + \langle v - u_{n_j}, Au_{n_j} - Ay_{n_j}\rangle \\
&\quad + \langle v - u_{n_j}, (x_{n_j} - u_{n_j})/\lambda_{n_j}\rangle \\
&\geq \langle v - u_{n_j}, Au_{n_j} - Ay_{n_j}\rangle + \langle v - u_{n_j}, (x_{n_j} - u_{n_j})/\lambda_{n_j}\rangle.
\end{aligned}$$

Passing to the limit in the above inequality and using the limits in (3.15) and (3.16), we obtain $\langle v - z, w\rangle \geq 0$. The maximality of S implies that $0 \in S(z)$. That is, $z \in VI(C, A)$. Since $z \in F(\mathcal{T})$ and $z \in VI(C, A)$, it follows that $z \in \mathfrak{F}$. Therefore from (2.3) and (3.11) we derive

$$\limsup_{n\to\infty} \langle u - \hat{x}, x_n - \hat{x}\rangle = \langle u - \hat{x}, z - \hat{x}\rangle \leq 0. \tag{3.17}$$

Moreover, from (3.1), (3.10) and (3.15), we have

$$\|x_{n+1} - x_n\| \leq \alpha_n \|u - x_n\| + (1 - \alpha_n)\{b_n \|u_n - x_n\| + c_n \|T(t_n)z_n - x_n\|\},$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From this last limit and (3.17), we deduce the inequality

$$\limsup_{n\to\infty} \langle u - \hat{x}, x_{n+1} - \hat{x}\rangle \leq 0.$$

Finally, we observe that (3.9) reduces to

$$\|x_{n+1} - \hat{x}\|^2 \leq (1 - \alpha_n) \|x_n - \hat{x}\|^2 + 2\alpha_n \langle u - \hat{x}, x_{n+1} - \hat{x}\rangle,$$

and the conclusion that $\{s_n\}$ converges to zero follows at once from Lemma 2.1. Thus $\{x_n\}$ converges strongly to $\hat{x} = P_{\mathfrak{F}} u$.

Case 2. Assume that there exists a subsequence $\{s_{k_i}\}$ of $\{s_k\}$ such that $s_{k_i} < s_{k_i+1}$ for all $i \geq 0$. Then in view of Lemma 2.2, we can define a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and $\max\{s_{m_k}, s_k\} \leq s_{m_k+1}$ for all $k \in \mathbb{N}$. Since $\lambda_k \leq b < L^{-1}$ for all $k \geq 0$ and the sequences $\{b_k\}$ and $\{c_k\}$ are bounded from below away from zero, it follows from (3.9) and the boundedness of $\{x_k\}$ that

$$\lim_{k\to\infty} \|x_{m_k} - T(t_{m_k})x_{m_k}\| = 0 = \lim_{k\to\infty} \|x_{m_k} - T(t_{m_k})z_{m_k}\|. \tag{3.18}$$

If we take any subsequence of $\{x_{m_k}\}$ that converges weakly to p, then using similar arguments as in Case 1 above, we arrive at $p \in F(\mathcal{T})$.

On the other hand, from (3.9) and our assumption that $\{s_k\}$ is convergent, we deduce

$$\lim_{k\to\infty} \|x_{m_k} - y_{m_k}\| = 0 = \lim_{k\to\infty} \|y_{m_k} - u_{m_k}\| . \tag{3.19}$$

Therefore, from the inequality

$$\|x_{m_k} - u_{m_k}\| \leq \|x_{m_k} - y_{m_k}\| + \|y_{m_k} - u_{m_k}\| ,$$

we obtain the limit

$$\lim_{k\to\infty} \|x_{m_k} - u_{m_k}\| = 0. \tag{3.20}$$

Moreover, from (3.19) and the Lipschitz continuity of A, we have

$$\lim_{k\to\infty} \|Ay_{m_k} - Au_{m_k}\| = 0.$$

Again using similar arguments as in Case 1 above we can derive $p \in VI(C, A)$. Therefore, for any subsequence of $\{x_{m_k}\}$ converging weakly to p, we have $p \in \mathfrak{F}$. Consequently,

$$\limsup_{k\to\infty} \langle u - \hat{x}, x_{m_k} - \hat{x}\rangle \leq 0. \tag{3.21}$$

Moreover, from (3.1), (3.18) and (3.20), we have

$$\begin{aligned} \|x_{m_k+1} - x_{m_k}\| \quad \leq \quad & \alpha_{m_k} \|u - x_{m_k}\| + (1 - \alpha_{m_k})\{b_{m_k} \|u_{m_k} - x_{m_k}\| \\ & +c_{m_k} \|S(t_{m_k})z_{m_k} - x_{m_k}\|\}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0.$$

From this last limit and (3.21), we deduce the inequality

$$\limsup_{k\to\infty} \langle u - \hat{x}, x_{m_k+1} - \hat{x}\rangle \leq 0.$$

Now making use of $s_{m_k} \leq s_{m_k+1}$ for all $k \in \mathbb{N}$ and rearranging terms in (3.9), we derive

$$\alpha_{m_k} \|x_{m_k+1} - \hat{x}\|^2 \leq 2\alpha_{m_k} \langle u - \hat{x}, x_{m_k+1} - \hat{x}\rangle .$$

Diving throughout by α_{m_k} and passing to the limit as $k \to \infty$ in the resulting inequality, we obtain $\|x_{m_k+1} - \hat{x}\| \to 0$ as $k \to \infty$. Since $\|x_k - \hat{x}\| \leq \|x_{m_k+1} - \hat{x}\|$, it follows that $\|x_k - \hat{x}\| \to 0$ as $k \to \infty$. Thus $x_k \to \hat{x}$ as $k \to \infty$.

We have shown in both cases that the sequence $\{x_n\}$ generated by (3.1) converges strongly to $\hat{x} = P_{\mathfrak{F}}u$. The proof is complete. □

If, in Theorem 3.3, we assume that $A = 0$, then we get the following corollary.

Corollary 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\mathcal{T} := \{T(t) : t > 0\}$ be a Lipschitz pseudo-contractive semigroup on C with $M := \sup_{t\geq 0}\{L(t)\} < \infty$, where $L(t)$ is the Lipschitz constant of the mapping $T(t)$. Assume that $F(\mathcal{T})$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T(t_n)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[(a_n+b_n)x_n + c_n T(t_n)z_n], \ n \geq 1, \end{cases} \tag{3.22}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ with $a_n+b_n+c_n = 1$ and satisfying $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \leq \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Assume that $t_n > 0$ (for all $n \geq 0$) satisfies $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$, and $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in F(\mathcal{T})$, which is the unique point in the set $F(\mathcal{T})$ that is nearest to u.

If, in Theorem 3.3, we assume that $\mathcal{T} = \{T(t) : t \geq 0\}$, is a k-strictly pseudo-contractive semigroup on C, then we get the following corollary. Recall that if T is k-strictly pseudo-contractive semigroup, then T is Lipschitz (see, eg, [31]).

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a Lipschitz monotone mapping with Lipschitz constant L. Let $\mathcal{T} = \{T(t) : t > 0\}$ be a k-strictly pseudo-contractive semigroup on C with $M := \sup_{t\geq 0}\{L(t)\} < \infty$, where $L(t)$ is the Lipschitz constant of the mapping $T(t)$. Assume that $\mathfrak{F} := F(\mathcal{T}) \cap VI(C,A)$ is not empty. Let $\{x_n\}$ be the sequence generated by (3.1) satisfying $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \leq \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Assume that $t_n > 0$ (for all $n \geq 0$) satisfies $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$, and $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \mathfrak{F}$, which is the unique point in the set $\mathfrak{F}$ that is nearest to u.*

If, in Theorem 3.3, we assume that $\mathcal{T} = \{T(t) : t \geq 0\}$, is a nonexpansive semigroup on C, then we get Theorem 3.1 of [3].

If, in Theorem 3.3, we assume that A, is a γ-inverse strongly monotone mapping, then A is L-Lipschitz with constant $L = \frac{1}{\gamma}$ and hence we get the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a γ-inverse strongly monotone mapping. Let $\mathcal{T} = \{T(t) : t > 0\}$ be a strongly continuous semigroup of Lipschitz pseudo-contractive mappings on C with $M := \sup_{t\geq 0}\{L(t)\} < \infty$, where $L(t)$ is the*

Lipschitz constant of the mapping $T(t)$. Assume that $\mathfrak{F} := F(\mathcal{T}) \cap VI(C, A)$ is not empty. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n] \\ z_n = (1-\beta_n)x_n + \beta_n T(t_n)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[a_n x_n + b_n P_C[x_n - \lambda_n A y_n] + c_n T(t_n) z_n], \ n \geq 1, \end{cases} \tag{3.23}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, \gamma)$, and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$ with $a_n + b_n + c_n = 1$ and satisfying $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \leq \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Assume that $t_n > 0$ (for all $n \geq 0$) satisfies $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$, and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \mathfrak{F}$, which is the unique point in the set $\mathfrak{F}$ that is nearest to u.

If, in Theorem 3.3, we assume that $C = H$, then the projection mapping P_C is reduced to the identity mapping in H and $VI(C, A) = A^{-1}(0)$. Thus, we get the following corollary.

Corollary 3.3. *Let C be a real Hilbert space H and $A : H \to H$ be a Lipschitz monotone mapping with Lipschitz constant L. Let $\mathcal{T} = \{T(t) : t > 0\}$ be a strongly continuous semigroup of Lipschitz pseudo-contractive mappings on H with $M := \sup_{t\geq 0}\{L(t)\} < \infty$, where $L(t)$ is the Lipschitz constant of the mapping $T(t)$. Assume that $\mathfrak{F} := F(\mathcal{T}) \cap A^{-1}(0)$ is not empty. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = x_n - \lambda_n A x_n \\ z_n = (1-\beta_n)x_n + \beta_n T(t_n)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[a_n x_n + b_n(x_n - \lambda_n A y_n) + c_n T(t_n) z_n], \ n \geq 1, \end{cases} \tag{3.24}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, L^{-1})$, and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$ with $a_n + b_n + c_n = 1$ and satisfying $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \leq \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Assume that $t_n > 0$ (for all $n \geq 0$) satisfies $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1}-t_n) = 0$, and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \mathfrak{F}$, which is the unique point in the set $\mathfrak{F}$ that is nearest to u.

We note that the method of proof of Theorem 3.3 provides the following theorem for approximating the minimum norm point of the set of common fixed points of $\mathcal{T}$ and the set of solutions of the variational inequality $VI(C, A)$.

Theorem 3.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a Lipschitz monotone mapping with Lipschitz constant L. Let $\{T(t) : t > 0\}$ be a Lipschitz pseudo-contractive semigroup on C with $M := \sup_{t\geq 0}\{L(t)\} < \infty$, where $L(t)$ is the Lipschitz constant of the* *mapping $T(t)$. Assume that $\mathfrak{F} := F(\mathcal{T}) \cap VI(C, A)$ is not empty. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n] \\ z_n = (1-\beta_n)x_n + \beta_n T(t_n)x_n \\ x_{n+1} = P_C\Big((1-\alpha_n)[a_n x_n + b_n P_C[x_n - \lambda_n A y_n] + c_n T(t_n)z_n]\Big), \ n \geq 1, \end{cases} \tag{3.25}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, L^{-1})$, and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$ with $a_n + b_n + c_n = 1$ and satisfying $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$ and $b_n + c_n \leq \beta_n < \beta$ with $\beta < \frac{1}{\sqrt{1+M^2}+1}$. Assume that $t_n > 0$ (for all $n \geq 0$) satisfies $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$, and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a minimum norm point x^* of $\mathcal{F}$.

Remark 3.1. Theorem 3.3 provides convergence sequence to a common point of solution set of a variational inequality problem for Lipschitz monotone mapping and common fixed point set of pseudo-contractive semigroup in Hilbert spaces.

Remark 3.2. Theorem 3.3 extends results of Takahashi and Toyoda [23] and Iiduka and Takahashi [10] to more general class of pseud-contractive semigroup and more general class of monotone mappings. In addition, Theorem 3.3 extends results of Buong [4], Thuy [24] and Boikanyo and Zegeye [3] to more general class of pseud-contractive semigroups.

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