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OPTIMALITY AND DUALITY FOR GENERALIZED APPROXIMATE SOLUTIONS IN SEMI-INFINITE MULTIOBJECTIVE OPTIMIZATION

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. This paper is dedicated to a generalized (α, ϵ)-quasi-efficient solution to semi-infinite multiobjective optimization problems (SMP). Relationships between the mentioned solution of (SMP) and the corresponding solution of the scalar problem due to Chankong–Haimes are established. Using this equivalence, ϵ-optimality conditions of Karush–Kuhn–Tucker (KKT) type are derived under the Farkas–Minkowski constraint qualification. In addition, we formulate dual problems of Wolfe and Mond–Weir types for (SMP), and prove weak and strong duality theorems.

1. Introduction

Multiobjective optimization problems aim to optimize the objective functions simultaneously and to find the best optimal compromise solution. However, frequently, it is not easy (or sometimes impossible) to find an optimal

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solution which is satisfied to all criteria at once. Hence, another important solution notion, namely efficient and properly efficient solution, should be taken into consideration. On the other hand, from a computational point of view it may be more meaningful to find not exact solutions but approximate ones. Indeed, one can consider approximate solutions with a small error while solving optimization problem by a numerical method; moreover, in some problems, if error value tends to zero, the limit of approximate solution is an exact solution, if it exists. First, approximate solutions were introduced by Kutateladze [10] and independently defined for multiobjective programming by Loridan [13]. In 1986 White [19] analyzed six different concepts of ϵ-solutions. Approximate solutions have got a keen interest by many researchers; see, for example, [18, 20, 14, 11] and references therein. In 2008 Beldiman *et al.* [1] suggested a unitary concept of approximate quasi efficient solutions which later was generalized by C. Gutiérrez *et al.* [8] and which is the main issue of this research.

To explore approximate solutions for multiobjective optimization problems, it is reasonable to use some scalarization methods, several of which are described, for example, in Engau and Wiecek [5]. In literature, one can find relationship theorems for approximate solutions of multiobjective optimization problems and related scalar problems for convex [11] and noncnovex [1] cases. One of the most well-known methods, namely weighted-sum scalarization method, is widely used for establishing relationship between properly efficient solutions for multiobjective optimization problems and optimal solutions of related scalar problems; see, for example [4, 11, 16]. But there is a strict restriction on choosing parameter vector, i.e. all components should be strictly positive and normalized so that sum of components are equal to one. On the other hand, the mentioned method can not be used for exploring efficient solutions. Due to this fact we consider scalarization method due to Chankong–Haimes [2] to find generalized quasi-(α, ϵ)-efficient solutions for (SMP).

Another hot topic is establishing ϵ-optimality conditions for approximate solutions; see, for example [8, 16] and references therein. There are also many papers dealing with ϵ-duality for approximate solutions; see, for example [12, 14, 15]. However, for the best of our knowledge, there are not so many papers considering ϵ-duality theorems and relationship between semi-infinite multiobjective optimization problem (SMP) and its dual problem for generalized approximate solutions. Our research is motivated by this fact.

This paper is organized as follows. In Section 2, the problem statement and main notions are described. Section 3 provides relationships between generalized (α, ϵ)-quasi-proper efficient solution to (SMP) and generalized (α_j, ϵ_j)-quasi-optimal solution of the corresponding scalar problem by using weighted-sum scalarization method. Using this equivalence, ϵ-optimality conditions of KKT type are established under Farkasw–Minkowski (FM) constraint qualification. Section 4 deals with the scalarization method due to Chankong–Haimes and aims to propose ϵ-optimality conditions of KKT type for (α, ϵ)-quasi-efficient solution to (SMP). Section 5 is devoted to duality relations which is meant to be our main result. Namely, both weak and strong ϵ-duality theorems for Wolfe type and Mond–Weir type dual problems are established. Finally, we provide conclusions in brief.

2. Preliminaries

Let us consider the following semi-infinite multiobjective optimization problem:

$$\begin{array}{lll} \text{(SMP)} & \text{Minimize} & f(x) := \big(f_1(x), f_2(x), ..., f_m(x)\big) \\ & \text{subject to} & g_t(x) \leqq 0, t \in T, \\ & & x \in C, \end{array}$$

where $f_i(x) : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, $i \in I := \{1, 2, ..., m\}$ and $g_t(x) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $t \in T$ (possibly infinite) are proper lower semicontinuous (l.s.c.) convex functions, and C is a closed convex subset of $\mathbb{R}^m$. The feasible set of (SMP) is denoted by $F_M := \{x \in C \mid g_t(x) \leqq 0, t \in T\}$.

Definition 2.1. Let ϵ and α be in $\mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in F_M$ is said to be

(1) an ***ϵ**-efficient solution* for (SMP), if there is no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \forall i \in I,$$

with at least one strict inequality;

(2) an ***α**-quasi-efficient solution* for (SMP), if there is no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i \|x - \bar{x}\|, \forall i \in I,$$

with at least one strict inequality.

Gutiérrez *et al.* [8] gave a notion of α-quasi-efficient solution with the help of function ϕ which is meant to be continuous. In our case we modifies this function, i.e. $\phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+$ is l.s.c convex function such that

$$\phi(x, \bar{x}) \begin{cases} = 0, & if \ x = \bar{x}, \\ > 0, & if \ x \neq \bar{x}. \end{cases}$$

Combining notions given by Beldiman *et al.*[1] and Gutiérrez *et al.* [8], we introduce a new concept of generalized approximate solutions.

Definition 2.2. Let ϵ and α be in $\mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in F_M$ is said to be a *generalized* $(\boldsymbol{\alpha}, \boldsymbol{\epsilon})$*-quasi-efficient solution* for (SMP), if there is no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i \phi(x, \bar{x}) - \epsilon_i, \ for \ all \ i \in I,$$

with at least one strict inequality.

Remark 2.3. If $\epsilon = 0$, Def. 2.2 covers α-quasi-efficient solution given by Gutiérrez *et al.* [8]. If $\alpha\phi(x, \bar{x}) = 0$, the above definition reduces to an ϵ-efficient solution. If $\epsilon_i = \alpha_i = 0$ for all $i \in I$, we get the concept of efficient solution for (SMP). In special case, when $\phi(x, \bar{x}) = \|x - \bar{x}\|$, Def. 2.2 reduces to the unitary approximate notions given by Beldiman *et al.* [1].

Due to Chankong–Haimes method for $j \in I$ and $\bar{x} \in C$ we associated to (SMP) the following scalar problem,

$$\begin{array}{lll} (\mathrm{P_j(\bar{x})}) & \text{Minimize} & f_j(x) \\ & \text{subject to} & f_i(x) \leqq f_i(\bar{x}), i \in I^j := I \setminus \{j\}, \\ & & g_t(x) \leqq 0, t \in T \\ & & x \in C. \end{array}$$

For the problem

$$\min\{f_j(x) \mid x \in C, G_t(x) \leqq 0, t \in \overline{T}\}$$

we define G_t as follows (with the assumption that $T \cap I = \emptyset$):

$$G_t(\cdot) = \begin{cases} f_t(\cdot) - f_t(\bar{x}), & t \in I^j, \\ g_t(\cdot), & t \in T, \end{cases} \ and \ \overline{T} = T \cup I^j. \tag{2.1}$$

Similarly, generalized approximate solutions for $(P_j(\bar{x}))$ can be proposed as follows.

Let $\epsilon_j \geqq 0$ and $\alpha_j \geqq 0$. A point $\bar{x} \in F_j := \{x \in C \mid G_t(x) \leqq 0, t \in \overline{T}\}$ is said to be a *generalized* $(\boldsymbol{\alpha_j}, \boldsymbol{\epsilon_j})$*-quasi-optimal solution* for $(P_j(\bar{x}))$ if

$$f_j(\bar{x}) \leqq f_j(x) + \alpha_j \phi(x, \bar{x}) + \epsilon_j, \ \forall x \in F_j,$$

where F_j is a feasible set of $(P_j(\bar{x}))$.

Further on, we will consider $\epsilon > 0$ and $\alpha > 0$ case to deal with concept of generalized solutions. However, all theorems can be reduced to corresponding approximate solutions by putting ϵ or α equal to zero and still hold true.

Now, we give some basic concepts and notions. The following linear space shall be used for semi-infinite programming [6].

$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \ for\ all\ t \in T\ but\ only\ finitely\ many\ \lambda_t \neq 0\}.$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) = \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T.

The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}_+^{(T)} = \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geqq 0, t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$ and $g_t,\ t \in T$, we understand that

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t, & if \quad T(\lambda) \neq \emptyset, \\ 0, & if \quad T(\lambda) = \emptyset. \end{cases}$$

To establish ϵ-optimality conditions of KKT-type we need some notions related to ϵ-subdifferential concept.

Let $h : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper l.s.c convex function. *The* $\boldsymbol{\epsilon}$*-subdifferential* of h at $\bar{x} \in dom\ h$ is the set $\partial_\epsilon h(\bar{x})$ defined by

$$\partial_\epsilon h(\bar{x}) = \{x^* \in \mathbb{R}^n \mid h(y) \geqq h(\bar{x}) - \epsilon + \langle x^*, y - \bar{x}\rangle,\ \forall y \in dom\ h\}.$$

Consider a function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. *The conjugate of* h, $h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$h^*(x^*) = \sup_{x^*\in\mathbb{R}^n} \{\langle x^*, x\rangle - h(x)\}.$$

The ϵ-subdifferential definition in term of *conjugate function* h^* of h is as follows:

$$\partial_\epsilon h(\bar{x}) = \{x^* \in \mathbb{R}^n \mid h^*(x^*) + h(\bar{x}) \leqq \langle x^*, \bar{x}\rangle + \epsilon\}.$$

The indicator function δ_K of a subset $K \subset \mathbb{R}^n$ is the function defined as follows:

$$\delta_K = \begin{cases} 0, & if\ x \in K, \\ +\infty, & if\ x \in \mathbb{R}^n \backslash K. \end{cases}$$

Note that if K is convex, then δ_K is also convex.

Let C be a nonempty closed convex subset of $\mathbb{R}^n$. *The* $\boldsymbol{\epsilon}$*-normal set* of C at $\bar{x}$ is the set

$$N_\epsilon(C; \bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - \bar{x}\rangle \leqq \epsilon,\ \forall y \in C\}.$$

where $\epsilon > 0$ and $\bar{x} \in C$.

If $\epsilon = 0$, the ϵ-normal set reduces to the normal cone $N(C;\bar{x})$ to C at $\bar{x}$ that is

$$N(C;\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y-\bar{x}\rangle \leqq 0, \ \forall y \in C\}.$$

It is easy to check that

$$\partial_\epsilon \delta_C(\bar{x}) = N_\epsilon(C;\bar{x}) = \{x^* \in \mathbb{R}^n \mid \delta_C^*(x^*) \leqq \langle x^*, \bar{x}\rangle + \epsilon\}.$$

For ϵ-subdifferential calculus the following propositions (see [[3], Theorem 2.115 and Theorem 2.117]) are very useful.

Proposition 2.4. [Sum Rule] *Consider two proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}, i = 1, 2$ such that ri dom $\phi_1 \cap$ ri dom $\phi_2 \neq \emptyset$, where ri denotes the relative interior (see [[3], Definition 2.1.13]). Then for $\epsilon > 0$,*

$$\partial_\epsilon(\phi_1+\phi_2)(\bar{x}) = \bigcup_{\epsilon_1 \geqq 0, \epsilon_2 \geqq 0, \epsilon_1+\epsilon_2=\epsilon} \big(\partial_{\epsilon_1}\phi_1(\bar{x}) + \partial_{\epsilon_2}\phi_2(\bar{x})\big)$$

for every $\bar{x} \in$ dom $\phi_1 \cap$ dom ϕ_2.

Proposition 2.5. [Scalar Product Rule] *For a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and any $\epsilon \geq 0$,*

$$\partial_\epsilon(\lambda\phi)(\bar{x}) = \lambda \partial_{\epsilon/\lambda}\phi(\bar{x}), \ \forall \lambda > 0.$$

3. ϵ-optimality for (SMP)

In this section, we study the relationships between corresponding approximate solutions of (SMP) and $(P_j(\bar{x}))$ and establish ϵ-optimality conditions.

Theorem 3.1. *Let $\bar{x} \in C$ and ϵ, $\alpha \in \mathbb{R}^m_+\backslash\{0\}$. A feasible point $\bar{x}$ is a generalized (α,ϵ)-quasi-efficient solution for (SMP) if and only of $\bar{x}$ is a generalized (α_j,ϵ_j)-quasi-optimal solution for $(P_j(\bar{x}))$ for each $j \in I$.*

Proof. Let $\bar{x}$ be a generalized (α_j,ϵ_j)-quasi-optimal solution for $(P_j(\bar{x}))$ for each $j \in I$. Hence,

$$f_j(\bar{x}) \leqq f_j(x) + \alpha_j\phi(x,\bar{x}) + \epsilon_j, \ \textit{for all } j \in I.$$

If $\bar{x}$ is not a generalized (α,ϵ)-quasi-efficient solution for (SMP) then there exists $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i\phi(x,\bar{x}) - \epsilon_i, \ \textit{for all } i \in I,$$

with at least one strict inequality. Suppose that the strict inequality takes place at k. We get $f_k(x) < f_k(\bar{x}) - \alpha_k\phi(x,\bar{x}) - \epsilon_k$, i.e., $f_k(x) + \alpha_k\phi(x,\bar{x}) + \epsilon_k < f_k(\bar{x})$. Hence, there exists $k \in I$ such that $\bar{x}$ is not a generalized (α_k,ϵ_k)-quasi-optimal solution for $(P_k(\bar{x}))$ that is a contradiction. Conversely, let $\bar{x}$ be a generalized (α, ϵ)-quasi-efficient solution for (SMP). Hence, there exists no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i \phi(x, \bar{x}) - \epsilon_i, \ \textit{for all } i \in I,$$

with at least one strict inequality. If there exists $j \in I$ such that $\bar{x}$ is not a generalized (α_j, ϵ_j)-quasi-optimal solution for $(P_j(\bar{x}))$ then there exists $x \in F_j(\bar{x})$ such that

$$f_j(x) + \alpha_j \phi(x, \bar{x}) - \epsilon_j < f_j(\bar{x}),$$

which is a contradiction. □

To obtain ϵ-optimality conditions, let us define the following sets:

$$\begin{aligned} S_i &= \{x \in \mathbb{R}^n \mid f_i - f_i(\bar{x}) \leqq 0\} \ \textit{for } i \in I^j, \\ S_t &= \{x \in \mathbb{R}^n \mid g_t(x) \leqq 0\} \ \textit{for } t \in T. \end{aligned}$$

Since G_t is defined by (2.1), with the help of Proposition 2.2. in Strodiot et al. [17], it is possible to establish the following lemma:

Lemma 3.2. *Let $\epsilon \geqq 0$. Let $\bar{x} \in S = \left(\bigcap_{t \in T(v)} S_t\right) \bigcap \left(\bigcap_{i \in I^j} S_i\right)$ and the following constraint qualification of the Slater type holds:*

(CQ) $\qquad \exists x_0 \in C : \ G(x_0) < 0,$

where $G = \sup_{t \in \overline{T}} G_t$. Then $x^* \in N_\epsilon(S; \bar{x})$, iff there exist $v \geqq 0$ and $\bar{\epsilon} \geqq 0$ such that

$$x^* \in \partial_{\bar{\epsilon}}(vG)(\bar{x}) \quad \textit{and} \quad \bar{\epsilon} - \epsilon \leqq (vG)(\bar{x}) \leqq 0.$$

It is worth mentioning that Slater type (CQ) should be replaced by another one suitable for semi-infinite programming (see [7]).

Definition 3.3. The convex semi-infinite programming problem is said to satisfy the Farkas–Minkowski (FM) qualification if

$$\{v_t g_t(x), t \in T(v), x \in C\}$$

is a (FM) system, i.e. its characteristic cone $K := cone\{\bigcup_{t \in T(v)} epi(v_t g_t)^* + epi\delta_C^*\}$ is closed.

Remark 3.4. According to [[3], Proposition 11.16] if (CQ) holds then (FM) is also satisfied.

Up to now we are ready to establish ϵ-optimality conditions for $(P_j(\bar{x}))$.

Theorem 3.5. *Let $\epsilon_j \geqq 0$, $\alpha_j > 0$ and $\bar{x}$ be a feasible point of $(P_j(\bar{x}))$. Suppose that* (FM) *constraint qualification holds. Then $\bar{x}$ is a generalized (α_j, ϵ_j)-quasi-optimal solution to $(P_j(\bar{x}))$ if and only if there exist $\bar{\epsilon}_{0j} \geqq 0$, $\bar{\epsilon}_{0i} \geqq 0$ for $i \in I^j$, $\bar{\epsilon}_t \geqq 0$ for $t \in T(\bar{v})$, $\bar{\epsilon}_b$, $\bar{\epsilon}_q \geqq 0$, $\bar{\lambda}_i \geqq 0$ for $i \in I^j$ and $\bar{v}_t \in \mathbb{R}_+^{(T)}$, such that*

$$
\begin{aligned}
0 \in \partial_{\bar{\epsilon}_{0j}} f_j(\bar{x}) + \textstyle\sum_{i\in I^j} \partial_{\bar{\epsilon}_{0i}}(\bar{\lambda}_i f_i)(\bar{x}) + \sum_{t\in T(\bar{v})} \partial_{\bar{\epsilon}_t}(\bar{v}_t g_t)(\bar{x}) \\
+\alpha_j \partial_{\beta_b}\phi(\cdot,\bar{x}) + N_{\bar{\epsilon}_q}(C;\bar{x}), \qquad (3.1) \\
\bar{\epsilon}_{0j} + \textstyle\sum_{i\in I^j} \bar{\lambda}_i \bar{\epsilon}_{0i} + \sum_{t\in T(\bar{v})} \bar{v}_t \bar{\epsilon}_t + \bar{\beta}_b + \bar{\epsilon}_q - \epsilon_j \leqq \sum_{t\in \overline{T}(\bar{v})} \bar{v}_t G_t(\bar{x}) \leqq 0, \quad (3.2)
\end{aligned}
$$

where $\bar{\beta}_b = \dfrac{\bar{\epsilon}_b}{\alpha_j}$.

Proof. If $\bar{x}$ is a generalized (α_j, ϵ_j)-quasi-optimal solution, then

$$f_j(\bar{x}) \leqq f_j(x) + \alpha_j \phi(x,\bar{x}) + \epsilon_j.$$

We can rewrite it as follows:

$$f_j(\bar{x}) + \alpha_j\phi(\bar{x},\bar{x}) \leqq f_j(x) + \alpha_j \phi(x,\bar{x}) + \epsilon_j.$$

Hence, $\bar{x}$ is an ϵ_j-optimal solution of the following problem:

$$
\begin{array}{ll}
\text{Minimize} & f_j(\cdot) + \alpha_j\phi(\cdot,\bar{x}) \\
\text{subject to} & f_i(x) \leqq f_i(\bar{x}), i \in I^j, \\
 & g_t(x) \leqq 0, t \in T, \\
 & x \in C.
\end{array}
$$

By using indicator functions we can obtain the following equivalent unconstrained problem:

$$
\begin{array}{ll}
\text{Minimize} & f_j(\cdot) + \sum_{i\in I^j} \delta_{S_i}(\cdot) + \sum_{t\in T(v)} \delta_{S_t}(\cdot) + \alpha_j\phi(\cdot,\bar{x}) + \delta_C(\cdot) \\
x \in \mathbb{R}^m &
\end{array}
$$

So, $\bar{x}$ is an ϵ_j-optimal solution of the above problem if and only if

$$0 \in \partial_{\epsilon_j}\big(f_j + \sum_{i\in I^j} \delta_{S_i} + \sum_{t\in T(v)} \delta_{S_t} + \alpha_j\phi(\cdot,\bar{x}) + \delta_C\big)(\bar{x}).$$

Since there is at least one point $x_0 \in int\ S_i \cap int\ S_t \cap int\ C$ and (FM) holds, by using the Proposition 2.4 we have

$$\begin{aligned}
&\partial_{\epsilon_j}\Big(f_j + \sum_{i\in I^j} \delta_{S_i} + \sum_{t\in T(v)} \delta_{S_t} + \alpha_j \phi(\cdot,\bar{x}) + \delta_C\Big)(\bar{x}) \\
&= \bigcup_{\substack{\epsilon_{0j}\geqq 0,\epsilon_{0i}\geqq 0,\epsilon_t\geqq 0,\epsilon_b\geqq 0,\epsilon_q\geqq 0\\ \epsilon_{0j}+\sum_{i\in I^j}\epsilon_{0i}+\sum_{t\in T(v)}\epsilon_t+\epsilon_b+\epsilon_q=\epsilon_j}} \Big\{\partial_{\epsilon_{0j}} f_i(\bar{x}) + \sum_{i\in I^j} \partial_{\epsilon_{0i}} \delta_{S_i}(\bar{x}) \\
&\quad + \sum_{t\in T(v)} \partial_{\epsilon_t} \delta_{S_t}(\bar{x}) + \partial_{\alpha_j\epsilon_b} \phi(\cdot,\bar{x})(\bar{x}) + \partial_{\epsilon_q} \delta_C(\bar{x})\Big\}.
\end{aligned}$$

By using Proposition 2.5 we can move α_j outside the ∂_{ϵ_b} and set $\dfrac{\epsilon_b}{\alpha_j} = \beta_b$. Hence, there exist $\bar{\lambda}_i \geqq 0$, $\bar{v} \in \mathbb{R}_+^{(T)}$, $\bar{\epsilon}_{0j} \geqq 0$, $\bar{\epsilon}_{0i} \geqq 0$ for $i \in I^j$, $\bar{\epsilon}_t \geqq 0$ for $t \in T(\bar{v})$, $\bar{\beta}_b$, $\bar{\epsilon}_q \geqq 0$ such that

$$0 \in \partial_{\bar{\epsilon}_{0j}} f_j(\bar{x}) + \sum_{i\in I^j} \partial_{\bar{\epsilon}_{0i}} (\bar{\lambda}_i f_i)(\bar{x}) + \sum_{t\in T(\bar{v})} \partial_{\bar{\epsilon}_t} (\bar{v}_t g_t)(\bar{x}) + \alpha_j \partial_{\beta_b} \phi(\cdot,\bar{x}) + N_{\bar{\epsilon}_q}(C;\bar{x}).$$

Condition (3.2) follows from Lemma 3.2 by summing over $t \in T(\bar{v})$ and replacing (CQ) by (FM) constraint qualification. $\square$

Remark 3.6. In practice, it is more meaningful to consider the special case $\phi(x,\bar{x}) = \|x - \bar{x}\|$. It is not difficult to check that $\alpha\partial_{\epsilon_b/\alpha}\|\cdot - \bar{x}\|(\bar{x}) = \alpha B$, where B denotes a unit ball. Then condition (3.1) reduces to

$$\in \partial_{\bar{\epsilon}_{0j}} f_j(\bar{x}) + \sum_{i\in I^j} \partial_{\bar{\epsilon}_{0i}} (\bar{\lambda}_i f_i)(\bar{x}) + \sum_{t\in T(\bar{v})} \partial_{\bar{\epsilon}_t} (\bar{v}_t g_t)(\bar{x}) + \alpha_j B + N_{\bar{\epsilon}_q}(C;\bar{x}).$$

Further on, we will consider $\phi(x,\bar{x}) = \|x - \bar{x}\|$ to be more significant.

Up to now we are ready to establish ϵ-optimality conditions for a generalized (α,ϵ)-quasi-efficient solution for (SMP) with the help of Theorem 3.1 and Theorem 3.5.

Theorem 3.7. *Let $\bar{x} \in C$, ϵ and α be in $\mathbb{R}_+^m\backslash\{0\}$ and* (FM) *constraint qualification hold. Then $\bar{x}$ is a generalized (α,ϵ)-quasi-efficient solution for* (SMP) *if and only if there exist $\lambda_i \geqq 0$, $\bar{\beta}_i$, $i \in I$, $\sum_{i\in I} \lambda_i = 1$, $\bar{\beta}_t \geqq 0$, $t \in T(v)$, $\bar{\beta}_b \geqq$*

$0,\ \bar{\beta}_q \geqq 0$ *and* $v \in \mathbb{R}_+^{(T)}$ *such that*

$$0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(\bar{x}) + \sum_{t\in T(\bar{v})} \partial_{\bar{\beta}_t}(v_t g_t)(\bar{x}) + \lambda_j \alpha_j B + N_{\bar{\beta}_q}(C;\bar{x}), \tag{3.3}$$

$$\sum_{i\in I} \lambda_i \bar{\beta}_{0i} + \sum_{t\in T(v)} v_t \bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \lambda_j \epsilon_j \leqq \sum_{t\in T(v)} v_t g_t(\bar{x}) \leqq 0. \tag{3.4}$$

Proof. By Theorem 3.1, $\bar{x}$ is a generalized (α, ϵ)-quasi-efficient solution for (SMP) if and only if $\bar{x}$ is a generalized (α_j, ϵ_j)-quasi-optimal solution for $(P_j(\bar{x}))$ for all $j \in I$. According to Lemma 3.5 there exist $\bar{\epsilon}_{0j} \geqq 0$, $\bar{\epsilon}_{0i} \geqq 0$ for $i \in I^j$, $\bar{\epsilon}_t \geqq 0$ for $t \in T(\bar{v})$, $\bar{\epsilon}_b$, $\bar{\epsilon}_q \geqq 0$, $\bar{\lambda}_i \geqq 0$ for $i \in I^j$ and $\bar{v}_t \in \mathbb{R}_+^{(T)}$ such that (3.1) and (3.2) holds. First, let us focus on (3.1). It implies that

$$\begin{aligned} 0 \in \frac{1}{1+\sum_{i\in I^j}\bar{\lambda}_i}\Bigg(&\partial_{\bar{\epsilon}_{0j}} f_j(\bar{x}) + \sum_{i\in I^j} \partial_{\bar{\epsilon}_{0i}} \bar{\lambda}_i f_i(\bar{x}) + \sum_{t\in T(\bar{v})} \partial_{\bar{\epsilon}_t} \bar{v}_t g_t(\bar{x}) \\ &+ \alpha_j B + N_{\bar{\epsilon}_q}(C,\bar{x})\Bigg). \end{aligned} \tag{3.5}$$

Set

$$\begin{aligned} N_{\bar{\beta}_q}(C;\bar{x}) &= \frac{1}{1+\sum_{i\in I^j}\bar{\lambda}_i} N_{\bar{\epsilon}_q}(C;\bar{x}); \\ \lambda_j &= \frac{1}{1+\sum_{i\in I^j}\bar{\lambda}_i}; \\ \lambda_i &= \frac{\bar{\lambda}_i}{1+\sum_{i\in I^j}\bar{\lambda}_i}, && i \in I^j; \\ v_t &= \frac{\bar{v}_t}{1+\sum_{i\in I^j}\bar{\lambda}_i}, && t \in T(v); \\ \beta_{0j} &= \frac{1}{1+\sum_{i\in I^j}\bar{\lambda}_i} \bar{\epsilon_{0j}}; \\ \beta_{0i} &= \frac{1}{1+\sum_{i\in I^j}\bar{\lambda}_i} \bar{\epsilon_{0i}}, && i \in I^j; \\ \beta_t &= \frac{1}{1+\sum_{i\in I^j}\bar{\lambda}_i} \bar{\epsilon}_t, && t \in T \end{aligned} \tag{3.6}$$

and note that $N_{\bar{\beta}_q}(C;\bar{x}) \subset N_{\bar{\epsilon}_q}(C;\bar{x})$ from (3.5) and by using Proposition 2.5 we deduce

$$0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(\bar{x}) + \sum_{t\in T(v)} \partial_{\bar{\beta}_t}(v_t g_t)(\bar{x}) + \lambda_j \alpha_j B + N_{\bar{\beta}_q}(C;\bar{x}).$$

It is easy to check that $\sum_{i\in I}\lambda_i = 1$. Since the feasible set of (SMP) is $F_M := \{x \in C \mid g_t(x) \leqq 0, t \in T\}$, without loss of generality, we can reduce $\sum_{t\in\bar{T}(\bar{v})}\bar{v}_t G_t$ to $\sum_{t\in T(\bar{v})}\bar{v}_t g_t$. Hence, using the same method, we have

$$\sum_{i\in I}\lambda_i\bar{\beta}_{0i} + \sum_{t\in T(v)} v_t\bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \lambda_j\epsilon_j \leqq \sum_{t\in T(v)} v_t g_t(\bar{x}) \leqq 0,$$

where $\bar{\beta}_q = \dfrac{\bar{\epsilon}_q}{1+\sum_{i\in I^j}\bar{\lambda}_i}$. □

4. ϵ-Duality

In this section, we discuss about weak and strong ϵ-duality theorems. First, we propose Wolfe type dual problem due to (SMP) as follows:

$$\begin{array}{lll}(\mathrm{MD})_{\mathrm{W}} & \text{Maximize} & f(y) + \sum_{t\in T} v_t g_t(y) e \\ & \text{subject to} & 0 \in \sum_{i\in I}\partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(y) + \sum_{t\in T(v)}\partial_{\bar{\beta}_t} v_t g_t(y) \\ & & +\lambda^T\alpha B + N_{\bar{\beta}_q}(C; y), \\ & & \sum_{i\in I}\lambda_i\bar{\beta}_{0i} + \sum_{t\in T(v)} v_t\bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \lambda^T\epsilon \leqq 0, \\ & & \lambda > 0, \lambda^T e = 1, e = (1, ..., 1) \in \mathbb{R}^m, \\ & & (y,\lambda,v) \in C \times \mathbb{R}^m \times \mathbb{R}^{(T)}_{+}.\end{array}$$

Now we derive ϵ-weak Duality theorem.

Theorem 4.1. [ϵ-Weak Duality] *Let x and (y, λ, v) be feasible solutions to* (SMP) *and* $(\mathrm{MD})_{\mathrm{W}}$*, respectively. Then the following cannot hold:*

$$f_i(x) \leqq f_i(y) + \sum_{t\in T(v)} v_t g_t(y) - \alpha_i\|x-y\| - \epsilon_i, \ \textit{for all}\ i \in I$$

with at least one strict inequality.

Proof. Suppose contrary to result that it holds. Multiplying by corresponding $\lambda_i > 0$ and summing for $i \in I$ with $\lambda^T e = 1$, we have

$$\sum_{i\in I}\lambda_i f(x) < \sum_{i\in I}\lambda_i f_i(y) + \sum_{t\in T(v)} v_t g_t(y) - \sum_{i\in I}\lambda_i\alpha_i\|x-y\| - \sum_{i\in I}\lambda_i\epsilon_i.$$

Hence $x \in C$ and $\sum_{t\in T(v)} v_t g_t(x) \leqq 0$ and $g_t(y) \geqq 0$, we obtain $x \neq y$ and,

$$\sum_{i\in I}\lambda_i f_i(x) + \sum_{t\in T(v)} v_t g_t(x) < \sum_{i\in I}\lambda_i f_i(y) + \sum_{t\in T(v)} v_t g_t(y) - \lambda^T\alpha\|x-y\| - \lambda^T\epsilon.$$

Since (y, λ, v) is a feasible solution to $(MD)_W$, there exist $u_i \in \partial_{\bar{\beta}_{0i}} f_i(y)$, $i \in I$, $\mu_t \in \partial_{\bar{\beta}_t} g_t,\ t \in T(v)$, $l \in B$ and $w \in N_{\bar{\beta}_q}(C; y)$ such that

$$\sum_{i\in I} \lambda_i u_i(x-y) + \sum_{t\in T(v)} v_t \mu_t(x-y) + \lambda^T \alpha l(x-y) + w(x-y) = 0.$$

So, using the convexity of f_i, $i \in I$ and $g_t,\ t \in T(v)$, we can obtain

$$\begin{aligned}
&\sum_{i\in I} \lambda_i f_i(x) + \sum_{t\in T(v)} v_t g_t(x) - \Bigg(\sum_{i\in I} \lambda_i f_i(y) + \sum_{t\in T(v)} v_t g_t(y) \\
&\quad - \lambda^T \alpha \|x-y\| - \lambda^T \epsilon \Bigg) \\
&= \sum_{i\in I} \lambda_i \Big(f_i(x) - f_i(y) \Big) + \sum_{t\in T(v)} v_t \Big(g_t(x) - g_t(y) \Big) \\
&\quad + \lambda^T \alpha \|x-y\| + \lambda^T \epsilon \\
&\geqq \sum_{i\in I} \lambda_i u_i(x-y) + \sum_{t\in T(v)} v_t \mu_t(x-y) + \lambda^T \alpha l(y-x) + \lambda^T \epsilon \\
&\quad - \sum_{i\in I} \lambda_i \bar{\beta}_{0i} - \sum_{t\in T(v)} v_t \bar{\beta}_t \\
&= -w(x-y) + \lambda^T \epsilon - \sum_{i\in I} \lambda_i \bar{\beta}_{0i} - \sum_{t\in T(v)} v_t \bar{\beta}_t - \bar{\beta}_b \\
&\geqq \lambda^T \epsilon - \sum_{i\in I} \lambda_i \bar{\beta}_{0i} - \sum_{t\in T(v)} v_t \bar{\beta}_t - \bar{\beta}_b - \bar{\beta}_q \\
&\geqq 0,
\end{aligned}$$

that is a contradiction. This completes the proof. □

Using Theorem 3.7 and Theorem 4.1, we establish ϵ-strong duality theorem for a generalized (α, ϵ)-efficient solution.

Theorem 4.2. [ϵ-Strong Duality] *Let $\epsilon, \alpha \in \mathbb{R}^m_+ \backslash \{0\}$. Assume that* (FM) *and ϵ-weak duality hold. If $\bar{x} \in C$ is a generalized (α, ϵ)-efficient solution for* (SMP) *then there exist $\lambda \in \mathbb{R}^m$ and $v \in \mathbb{R}^{(T)}_+$ such that $(\bar{x}, \lambda, v)$ is a generalized $(\alpha, 2\epsilon)$-efficient solution for $(MD)_W$.*

Proof. Since $\bar{x}$ is generalized (α, ϵ)-efficient solution for (SMP), by Theorem 3.7, there exist $\lambda_i,\ i \in I$ and $v \in \mathbb{R}_+^{(T)}$ such that

$$\begin{aligned}
&0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(\bar{x}) + \sum_{t\in T(v)} \partial_{\bar{\beta}_t}(v_t g_t)(\bar{x}) + \lambda_j \alpha_j B + N_{\bar{\beta}_q}(C;\bar{x}) \\
&\subset 0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(\bar{x}) + \sum_{t\in T(v)} \partial_{\bar{\beta}_t} v_t g_t(\bar{x}) + \lambda^T \alpha B + N_{\bar{\beta}_q}(C;\bar{x})
\end{aligned}$$

holds, and then $(\bar{x}, \lambda, v)$ is feasible for $(MD)_W$. Suppose that $(\bar{x}, \lambda, v)$ is not a generalized $(\alpha, 2\epsilon)$-efficient solution for $(MD)_W$. Then there exists (x^*, λ^*, v^*) such that the following cannot hold:

$$f_i(x^*) + \sum_{t\in T(v)} v_t^* g_t(x^*)e - \alpha_i\|x^* - \bar{x}\| - 2\epsilon_i \leqq f_i(\bar{x}) + \sum_{t\in T(v)} v_t g_t(\bar{x})e,$$

with at least on strict inequality. Taking strict inequality at j^{th} term, we get

$$f_j(x^*) + \sum_{t\in T(v)} v_t^* g_t(x^*) - \alpha_j\|x^* - \bar{x}\| - 2\epsilon_j > f_j(\bar{x}) + \sum_{t\in T(v)} v_t g_t(\bar{x}).$$

It implies that

$$f_j(\bar{x}) + \sum_{t\in T(v)} v_t g_t(\bar{x}) - f_j(x^*) - \sum_{t\in T(v)} v_t^* g_t(x^*) + \alpha_j\|x^* - \bar{x}\| + 2\epsilon_j < 0.$$

On the other hand, by ϵ-weak duality (Theorem 4.1) and since

$$\bar{\beta}_{0i} + \sum_{t\in T(v)} v_t \bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \epsilon_j \leqq \sum_{t\in T(v)} v_t g_t(\bar{x}) \leqq 0,$$

the following cannot hold

$$\begin{aligned}
&f_j(\bar{x}) - \left(f_j(x^*) + \sum_{t\in T(v)} v_t^* g_t(x^*)\right) + \sum_{t\in T(v)} v_t g_t(\bar{x}) + \alpha_j\|x^* - \bar{x}\| + 2\epsilon_j \\
&\leqq f_j(\bar{x}) - \left(f_j(x^*) + \sum_{t\in T(v)} v_t^* g_t(x^*)\right) + \alpha_j\|x^* - \bar{x}\| + \epsilon_j \\
&\quad - \bar{\beta}_{0i} - \sum_{t\in T(v)} v_t \bar{\beta}_t - \bar{\beta}_b - \bar{\beta}_q - \epsilon_j \\
&\leqq f_j(\bar{x}) - f_j(x^*) - \sum_{t\in T(v)} v_t^* g_t(x^*) + \alpha_j\|x^* - \bar{x}\| + \epsilon_j \\
&\leqq 0.
\end{aligned}$$

So we get a contradiction. $\square$

Let us consider Mond–Weir type dual problem which is denoted as follows:

$$
\begin{array}{lll}
(\mathrm{MD})_{\mathrm{M}} & \text{Maximize} & f(y) \\
 & \text{subject to} & 0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(y) + \sum_{t\in T(v)} \partial_{\bar{\beta}_t} v_t g_t(y) \\
 & & +\lambda^T \alpha B + N_{\bar{\beta}_q}(C; y), \\
 & & \sum_{i\in I} \lambda_i \bar{\beta}_{0i} + \sum_{t\in T(v)} v_t \bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \lambda^T \epsilon \leqq 0, \\
 & & \lambda > 0, \lambda^T e = 1, e = (1, ..., 1) \in \mathbb{R}^m, \\
 & & (y, \lambda, v) \in C \times \mathbb{R}^m \times \mathbb{R}_+^{(T)}.
\end{array}
$$

Theorem 4.3. [ϵ-Weak Duality] *Let x and (y, λ, v) be feasible solutions to* (SMP) *and $(MD)_M$, respectively. Then the following cannot hold*

$$
f_i(x) \leqq f_i(y) - \alpha_i \|x - y\| - \epsilon_i, \ \textit{for all } i \in I,
$$

with at least one strict inequality.

Proof. Suppose contrary to result that it holds. Multiplying by corresponding $\lambda_i > 0$ and summing for $i \in I$ with $\lambda^T e = 1$, we have

$$
\sum_{i\in I} \lambda_i f_i(x) < \sum_{i\in I} \lambda_i f_i(y) - \sum_{i\in I} \lambda_i \alpha_i \|x - y\| - \sum_{i\in I} \lambda_i \epsilon_i.
$$

Hence $x \in C$ and $g_t(x) \leqq 0$ and $v_t g_t(y) \geqq 0$, we obtain

$$
\sum_{i\in I} \lambda_i f_i(x) + \sum_{t\in T(v)} v_t g_t(x) < \sum_{i\in I} \lambda_i f_i(y) + \sum_{t\in T(v)} v_t g_t(y) - \lambda^T \alpha \|x - y\| - \lambda^T \epsilon.
$$

Since (y, λ, v) is a feasible solution to $(MD)_M$, there exist $u_i \in \partial_{\bar{\epsilon}_{0i}} f_i(y)$, $i \in I$, $\mu_t \in \partial_{\bar{\beta}_t} g_t$, $t \in T(v)$, $l \in B$ and $w \in N_{\bar{\beta}_q}(C; y)$ such that

$$
\sum_{i\in I} \lambda_i u_i(x - y) + \sum_{t\in T(v)} v_t \mu_t(x - y) + \lambda^T \alpha l(x - y) + w(x - y) = 0.
$$

Following the same method like in the proof of the Theorem 4.1 we can obtain a contradiction. □

Using Theorem 3.7 and Theorem 4.3, we can establish ϵ-strong duality. It should be mentioned that in contrast to ϵ-strong duality of Wolfe type, $\bar{x}$ is a generalized (α, ϵ)-efficient solution for $(MD)_M$, not 2ϵ-efficient.

Theorem 4.4. [ϵ-Strong Duality] *Let $\epsilon, \alpha \in \mathbb{R}_+^m \backslash \{0\}$. Assume that* (FM) *and ϵ-weak duality hold. If $\bar{x} \in C$ is a generalized (α, ϵ)-efficient solution for* (SMP) *then there exist $\lambda \in \mathbb{R}^m$ and $v \in \mathbb{R}_+^{(T)}$ such that $(\bar{x}, \lambda, v)$ is generalized (α, ϵ)-efficient solution for $(MD)_M$.*

Proof. Since $\bar{x}$ is generalized (α, ϵ)-efficient solution for (SMP), by Theorem 3.7, there exist $\lambda_i,\ i \in I$ and $v \in \mathbb{R}_+^{(T)}$ such that

$$0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(\bar{x}) + \sum_{t\in T(v)} \partial_{\bar{\beta}_t}(v_t g_t)(\bar{x}) + \lambda_j \alpha_j B + N_{\bar{\beta}_q}(C;\bar{x})$$
$$\subset 0 \in \sum_{i\in I} \partial_{\bar{\beta}_{0i}}(\lambda_i f_i)(\bar{x}) + \sum_{t\in T(v)} \partial_{\bar{\beta}_t} v_t g_t(\bar{x}) + \lambda^T \alpha B + N_{\bar{\beta}_q}(C;\bar{x})$$

holds, and then $(\bar{x}, \lambda, v)$ is feasible for $(MD)_M$. Suppose that $(\bar{x}, \lambda, v)$ is not generalized (α, ϵ)-efficient solution for $(MD)_M$, there exists (x^*, λ^*, v^*) such that the following cannot hold:

$$f_i(x^*) - \alpha_i\|x^* - \bar{x}\| - \epsilon_i \leqq f_i(\bar{x}),$$

with at least on strict inequality, which contradicts ϵ-weak duality Theorem 4.3. □

5. Conclusion

In this paper we discussed about generalized approximate solutions to semi-infinite multiobjective optimization problem. Relationships between generalized (α, ϵ)-quasi-efficient solution for (SMP) and generalized (α_j, ϵ_j)-quasi-optimal solution for $(P_j(\bar{x}))$ were established. Using this equivalences, ϵ-optimality conditions for (SMP) were derived under FM constraint qualification due to Goberna *et al.* [7]. In addition, we established both weak and strong ϵ-duality theorem for Wolfe type and Mond–Weir type dual problems.

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