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CONE METRIC SPACE WITH BANACH ALGEBRA AND FIXED POINT RESULTS FOR T-HARDY-ROGERS TYPE CONTRACTIONS

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. In this paper, we prove some common fixed and periodic point theorems for T-Hardy-Rogers type contraction of self mappings on cone b-metric spaces over Banach algebras with solid cone, by using properties of spectral radius. Our results improve and generalize the main results of Xu and Radenovic(*Fixed Point Theory and Applications, 2014:102*) and several well-known theorems in the literature of T-contraction mappings. Also we give examples as an application of the main result.

1. Introduction

Since Banach proved his famous fixed point theorem in 1922, fixed points of mappings satisfying certain contractive conditions has been studied at the center of strong research activity. In 2007, Huang and Zhang [4] introduced cone metric space and proved some fixed point theorems. Afterward, several fixed and common fixed point results on cone metric spaces have been proved in ([1],[8],[10]). Recently, A. Beiranvand [2], Filipovic et al. [3], Morales and Rojas [7] have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems.

In 2013, in order to generalize the Banach contraction principle to more general form, Liu and Xu [6] introduced the concept of cone metric spaces

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over Banach algebras, by replacing Banach spaces with Banach algebras as the underlying spaces of cone metric spaces, and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constants by means of spectral radius.

In this paper, we prove some common fixed and periodic point theorems for T-Hardy-Rogers type contraction of self mappings on cone b-metric spaces over Banach algebras with solid cone, by using properties of spectral radius. Our results improve and generalize Theorem 3.1, 3.2 and 3.3 of Xu and Radenovic [9], and Theorem 2.1, 2.2 and 2.3 of Liu and Xu [6] as well as several well-known theorems in the literature of T-contraction mappings. Also we give examples as an application of the main result.

We recall some definitions and other results that will be needed in the sequel.

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that A is a real Banach algebra with a unit (i.e., a multiplicative identity) e. An element $x \in A$ is said to be *invertible* if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Let A be a real Banach algebra with a unit e and θ the zero element of A. A nonempty closed subset P of Banach algebra A is called a *cone* if

(1) $\{\theta, e\} \subset P$;
(2) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β ;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a *partial ordering* $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \emptyset$ then P is called a *solid cone*. A cone P is called *normal* if there exists a number K such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number K satisfying condition (1.1) is called the *normal constant* of P.

In the following we always assume that P is a solid cone of A and $\preceq$ is the partial ordering with respect to P.

Definition 1.1. Let X be a nonempty set, $s \geq 1$ be a constant and A be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,y) \preceq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

Then d is called a *cone b-metric* on X, and (X,d) is called a *cone b-metric space* over the Banach algebra A.

Example 1.2. Let $A = C[a,b]$ be the set of continuous functions on $[a,b]$ with the supremum. Define multiplication in the usual way. Then A is a Banach algebra with a unit 1. Set $P = \{x \in A : x(t) \geq 0,\ t \in [a,b]\}$ and $X = \mathbb{R}$. We define a mapping $d : X \times X \to A$ by $d(x,y)(t) = |x-y|^p e^t$ for all $x, y \in X$ and for each $t \in [a,b]$, where $p > 1$ is a constant. This makes (X,d) into a cone b-metric space over Banach algebra with the coefficient $s = 2^{p-1}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Definition 1.3. Let (X,d) be a cone b-metric space over a Banach algebra A. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by
$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$
(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) A cone b-metric space (X,d) is said to be *complete* if every Cauchy sequence in X is convergent.
(4) A self mapping $T : X \to X$ is said to be *continuous at a point* $x \in X$ if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} Tx_n = Tx$ for every sequence $\{x_n\}$ in X.

Definition 1.4. Let E be a real Banach space with a solid cone P. A sequence $\{x_n\} \subset P$ is called a $c-$*sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $x_n \ll c$ for all $n \geq N$.

Let E be a real Banach space with a cone P. Then the following properties are often used, particularly when dealing with cone b-metric spaces in which the cone need not be normal (for details see ([8], [9]):

(p_1) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(p_2) If $a \preceq b$ and $b \ll c$, then $a \ll c$.
(p_3) If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.
(p_4) If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.
(p_5) If $\{x_n\}, \{y_n\}$ are sequences in E such that $x_n \to x$, $y_n \to y$ and $x_n \preceq y_n$ for all $n \geq 1$, then $x \preceq y$.

Lemma 1.5. ([5], [9]) *Let A be a real Banach algebra with a unit e and P be a solid cone in A. We define the spectral radius $\rho(x)$ of $x \in A$ by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

(1) *If $0 \leq r(x) < 1$, then $e - x$ is invertible,*

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i \quad \text{and} \quad r((e-x)^{-1}) \leq \frac{1}{1-r(k)}.$$

(2) *If $r(x) < 1$, then $\|x^n\| \to 0$ as $n \to \infty$.*
(3) *If $x \in P$ and $r(x) < 1$, then $(e-x)^{-1} \in P$.*
(4) *If $k, u \in P$, $r(k) < 1$ and $u \preceq ku$, then $u = \theta$.*
(5) *$r(x) \leq \|x\|$ for all $x \in A$.*
(6) *If $x, y \in A$ and x, y commute, then we have the following inequalities:*
 (a) $r(xy) \leq r(x)r(y)$,
 (b) $r(x+y) \leq r(x) + r(y)$,
 (c) $|r(x) - r(y)| \leq r(x-y)$.

Lemma 1.6. ([8], [9]) *Let (X, d) be a complete cone b-metric space over a Banach algebra A and let P be a solid cone in A. Let $\{x_n\}$ be a sequence in X. Then we have the following statements:*

(1) *If $\|x_n\| \to 0$ as $n \to \infty$, then $\{x_n\}$ is a $c-$sequence.*
(2) *If $k \in P$ is any vector and $\{x_n\}$ is $c-$sequence in P, then $\{kx_n\}$ is a $c-$sequence.*
(3) *If $x, y \in A$, $a \in P$ and $x \preceq y$, then $ax \preceq ay$.*
(4) *If $\{x_n\}$ converges to $x \in X$, then $\{d(x_n, x)\}$, $\{d(x_n, x_{n+p})\}$ are c-sequences for any $p \in \mathbb{N}$.*

Definition 1.7. Let T and f be two self mappings of a cone b-metric space (X, d) over a Banach algebra A.

(1) f is said to be *T-contraction* if there exists $k \in P$ with $0 \leq r(k) < 1$ such that

$$d(Tfx, Tfy) \preceq kd(Tx, Ty) \tag{1.2}$$

for all $x, y \in X$.

(2) f is said to be T-*contractive*, if for every $x, y \in X$ with $Tx \neq Ty$,

$$d(Tfx, Tfy) \prec d(Tx, Ty).$$

If $T = I$, the identity mapping, then the Definition (1.2) reduces to Banach contraction mapping. It is obvious that every T-contraction mapping is T-contractive but the converse need not be true.

Definition 1.8. ([2], [3]) Let T be a self mapping of a cone b-metric space (X, d) over a Banach algebra A. Then

(1) T is said to be *sequentially convergent*, if the sequence $\{x_n\}$ in X is convergent whenever $\{Tx_n\}$ is convergent.
(2) T is said to be *subsequentially convergent*, if $\{x_n\}$ has a convergent subsequence whenever $\{Tx_n\}$ is convergent.

2. Common fixed point results

In this section, we prove a new common fixed point theorem for T-Hardy-Rogers type contraction on cone b-metric spaces over Banach algebras with solid cone, by using properties of spectral radius.

Theorem 2.1. *Let (X, d) be a complete cone b-metric space (X, d) over a Banach algebra A with coefficient $s \geq 1$ and P be a solid cone. Let $T : X \to X$ be a continuous and one to one mapping. Moreover, let f and g be two maps of X satisfying*

$$\begin{aligned} d(Tfx, Tgy) \preceq\ & a_1 d(Tx, Ty) + a_2[d(Tx, Tfx) + d(Ty, Tgy)] \\ & + a_3[d(Tx, Tgy) + d(Ty, Tfx)], \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where $a_i \in P$ commute for $i = 1, 2, 3$ and

$$sr(a_1) + (s^2 + 1)r(a_2) + (s^3 + s)r(a_3) < 1. \tag{2.2}$$

Then,

(1) *there exist $u_x \in X$ such that $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = u_x$.*
(2) *if T is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.*
(3) *there exist a unique $v_x \in X$ such that $fv_x = gv_x = v_x$, that is, f and g have a unique common fixed point.*
(4) *if T is sequentially convergent, then iterate sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to v_x.*

Proof. Let x_0 be any point of X. Define $\{x_n\}$ by

$$x_1 = fx_0,\ x_2 = gx_1, \cdots, x_{2n+1} = fx_{2n},\ x_{2n+2} = gx_{2n+1}$$

for $n = 0, 1, 2, \cdots$. First, we prove that $\{Tx_n\}$ is a Cauchy sequence. By (2.1),

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(Tfx_{2n}, Tgx_{2n+1}) \\ &\preceq a_1 d(Tx_{2n}, Tx_{2n+1}) + a_2[d(Tx_{2n}, Tfx_{2n}) \\ &\quad + d(Tx_{2n+1}, Tgx_{2n+1})] \\ &\quad + a_3[d(Tx_{2n}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n})] \\ &= a_1 d(Tx_{2n}, Tx_{2n+1}) + a_2[d(Tx_{2n}, Tx_{2n+1}) \\ &\quad + d(Tx_{2n+1}, Tx_{2n+2})] \\ &\quad + a_3[d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})] \\ &\preceq (a_1 + a_2 + sa_3) d(Tx_{2n}, Tx_{2n+1}) \\ &\quad + (a_2 + sa_3) d(Tx_{2n+1}, Tx_{2n+2}). \end{aligned}$$

Since $r(a_2) + sr(a_3) < 1$ by hypothesis (2.2), $e - a_2 - sa_3$ is invertible. Thus

$$d(Tx_{2n+1}, Tx_{2n+2}) \preceq kd(Tx_{2n}, Tx_{2n+1}),$$

where $k = (e - a_2 - sa_3)^{-1}(a_1 + a_2 + sa_3)$ and $r(k) < 1$ by hypothesis (2.2).

Similarly, we get

$$d(Tx_{2n+3}, Tx_{2n+2}) \preceq kd(Tx_{2n+2}, Tx_{2n+1}).$$

Thus, for all n

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\preceq kd(Tx_{n-1}, Tx_n) \preceq k^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\preceq \cdots \preceq k^n d(Tx_0, Tx_1). \end{aligned} \tag{2.3}$$

If $m, n \in \mathbb{N}$ such that $m > n$, then we have, since $r(sk) < 1$,

$$\begin{aligned} d(Tx_n, Tx_m) &\preceq s[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_m)] \\ &\preceq sd(Tx_n, Tx_{n+1}) + s^2[d(Tx_{n+1}, Tx_{n+2}) + d(Tx_{n+2}, Tx_m)] \\ &\vdots \\ &\preceq sd(Tx_n, Tx_{n+1}) + s^2 d(Tx_{n+1}, Tx_{n+2}) + \cdots \\ &\quad + s^{m-n-1} d(Tx_{m-2}, Tx_{m-1}) + s^{m-n} d(Tx_{m-1}, Tx_m) \\ &\preceq (sk^n + s^2 k^{n+1} + \cdots + s^{m-n} k^{m-1}) d(Tx_0, Tx_1) \\ &\preceq sk^n[e + sk + (sk)^2 + \cdots] d(Tx_0, Tx_1) \\ &= sk^n (e - sk)^{-1} d(Tx_0, Tx_1). \end{aligned}$$

Since $r(k) < 1$, it follows that $\|k^n\| \to 0$ as $n \to \infty$. By Lemma 1.6, it follows that for $\theta \ll c$ and large n, $sk^n(e - sk)^{-1} d(Tx_0, Tx_1) \ll c$. Thus, according to (p_2), $d(Tx_n, Tx_m) \ll c$. Hence, it follows that $\{Tx_n\}$ is a Cauchy sequence in X by Definition. Since X is a complete cone b-metric space, there exist $u_x \in X$ such that $Tx_n \to u_x$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} Tfx_{2n} = u_x, \quad \lim_{n\to\infty} Tgx_{2n+1} = u_x. \tag{2.4}$$

Now, if T is subsequentially convergent, then by definition $\{fx_{2n}\}$ (resp. $\{gx_{2n+1}\}$) has a convergent subsequence. Thus, there exist $v_{x_1} \in X$ and $\{fx_{2n_i}\}$ (resp. $v_{x_2} \in X$ and $\{gx_{2n_i+1}\}$) such that

$$\lim_{n\to\infty} fx_{2n_i} = v_{x_1}, \quad \lim_{n\to\infty} gx_{2n_i+1} = v_{x_2}. \tag{2.5}$$

Because of continuity T and by (2.5), we have

$$\lim_{n\to\infty} Tfx_{2n_i} = Tv_{x_1}, \quad \lim_{n\to\infty} Tgx_{2n_i+1} = Tv_{x_2}. \tag{2.6}$$

Now, by (2.4) and (2.6) and because of injectivity of T, there exist $v_x \in X$ (set $v_x = v_{x_1} = v_{x_2}$) such that $Tv_x = u_x$.

On the other hand, by hypothesis (2.2), we have

$$\begin{aligned}
&d(Tv_x, Tgv_x)\\
&\preceq sd(Tv_x, Tgx_{2n_i+1}) + s^2 d(Tgx_{2n_i+1}, Tfx_{2n_i})\\
&\quad + s^2 d(Tfx_{2n_i}, Tgv_x)\\
&\preceq sd(Tv_x, Tx_{2n_i+2}) + s^2 d(Tx_{2n_i+2}, Tx_{2n_i+1})\\
&\quad + s^2 a_1 d(Tx_{2n_i}, Tv_x)\\
&\quad + s^2 a_2 [d(Tx_{2n_i}, Tx_{2n_i+1}) + d(Tv_x, Tgv_x)]\\
&\quad + s^2 a_3 [d(Tx_{2n_i}, Tgv_x) + d(Tv_x, Tx_{2n_i+1})]\\
&\preceq sd(Tv_x, Tx_{2n_i+2}) + s^2 k^{2n_i+1} d(Tx_0, Tx_1) + s^2 a_1 d(Tx_{2n_i}, Tv_x)\\
&\quad + s^2 a_2 d(Tv_x, Tgv_x) + s^2 k^{2n_i} a_2 d(Tx_0, Tx_1)\\
&\quad + s^2 a_3 d(Tv_x, Tx_{2n_i+1}) + s^2 a_3 d(Tx_{2n_i}, Tgv_x)\\
&\preceq sd(Tv_x, Tx_{2n_i+2}) + s^2 (k^{2n_i+1} + k^{2n_i} a_2) d(Tx_0, Tx_1)\\
&\quad + s^2 (a_1 + sa_3) d(Tx_{2n_i}, Tv_x) + s^2 a_3 d(Tv_x, Tx_{2n_i+1})\\
&\quad + s^2 (a_2 + sa_3) d(Tv_x, Tgv_x).
\end{aligned}$$

Since $r(k) < 1$, $\{k^{2n_i+1}\}, \{k^{2n_i}\}$ are c-sequences. Also by (2.6) and Lemma 1.6, $\{d(Tv_x, Tx_{2n_i+2})\}$, $\{d(Tx_{2n_i}, Tv_x)\}$, $\{d(Tv_x, Tx_{2n_i+1})\}$ are c-sequences in cone P. Thus the above inequality implies

$$d(Tv_x, Tgv_x) \preceq s^2(a_2 + sa_3) d(Tv_x, Tgv_x) + z_n$$

where $\{z_n\}$ is a c-sequence in cone P. Since for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $z_n \ll c$ for $n \geq n_0$, we have

$$d(Tv_x, Tgv_x) \preceq s^2(a_2 + sa_3) d(Tv_x, Tgv_x) + c$$

for each $n \geq n_0$ and thus, by (p_3).

$$d(Tv_x, Tgv_x) \preceq s^2(a_2 + sa_3) d(Tv_x, Tgv_x).$$

Since $r(s^2a_2 + s^3a_3) < 1$ and $s^2a_2 + s^3a_3 \in P$, by Lemma 1.5, we have $d(Tv_x, Tgv_x) = \theta$, that is, $Tv_x = Tgv_x$. Since T is one to one, then $gv_x = v_x$.

Now, we shall show that $fv_x = v_x$.

$$\begin{aligned} d(Tfv_x, Tv_x) &= d(Tfv_x, Tgv_x) \\ &\preceq a_1 d(Tv_x, Tv_x) + a_2[d(Tv_x, Tfv_x) + d(Tv_x, Tgv_x)] \\ &\quad + a_3[d(Tv_x, Tgv_x) + d(Tv_x, Tfv_x)] \\ &= (a_2 + a_3) d(Tv_x, Tfv_x). \end{aligned}$$

Since $r(a_2+a_3) < 1$ by hypothesis (2.2), using the definition of partial ordering on P and properties of cone P, we have $d(Tfv_x, Tv_x) = \theta$. and so $Tfv_x = Tv_x$. Since T is one to one, then $fv_x = v_x$. Thus, $fv_x = gv_x = v_x$, that is, v_x is a common fixed point of f and g.

Now, we shall show that v_x is a unique common fixed point. Suppose that v'_x be another common fixed point of f and g. Then

$$\begin{aligned} d(Tv_x, Tv'_x) &= d(Tfv_x, Tgv'_x) \\ &\preceq a_1 d(Tv_x, Tv'x) + a_2[d(Tv_x, Tfv_x) + d(Tv'_x, Tgv'_x)] \\ &\quad + a_3[d(Tv_x, Tgv'_x) + d(Tv'_x, Tfv_x)] \\ &= (a_1 + 2a_3) d(Tv_x, Tv'_x). \end{aligned}$$

Since $r(a_1+2a_3) < 1$ by hypothesis (2.2), by the same arguments as above, we conclude that $d(Tv_x, Tv'_x) = \theta$, which implies the equality $Tv_x = Tv'_x$. Since T is one to one, then $v_x = v'_x$. Thus f and g have a unique common fixed point.

Ultimately, if T is sequentially convergent, then we replace n for n_i. Thus, we have

$$\lim_{n\to\infty} fx_{2n} = v_x, \quad \lim_{n\to\infty} gx_{2n+1} = v_x.$$

Therefore if T is sequentially convergent, then iterate sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to v_x. □

The following results are obtained from Theorem 2.1.

Corollary 2.2. *Let (X, d) be a complete cone b-metric space (X, d) over a Banach algebra A with coefficient $s \geq 1$ and P be a solid cone. Let f and g be two maps of X satisfying*

$$d(fx, gy) \preceq a_1 d(x, y) + a_2[d(x, fx) + d(y, gy)] + a_3[d(x, gy) + d(y, fx)]$$

for all $x, y \in X$ where $a_i \in P$ commute for $i = 1, 2, 3$ and

$$sr(a_1) + (s^2 + 1)r(a_2) + (s^3 + s)r(a_3) < 1.$$

Then f and g have a unique common fixed point.

Proof. The proof follows by taking $T = I$ in Theorem 2.1. □

Corollary 2.3. *Let (X,d) be a complete cone b-metric space (X,d) over a Banach algebra A with coefficient $s \geq 1$ and P be a solid cone. Let $T : X \to X$ be a continuous and one to one mapping. Moreover, let f be a map of X satisfying*

$$\begin{aligned} d(Tfx, Tfy) &\preceq a_1 d(Tx, Ty) + a_2[d(Tx, Tfx) + d(Ty, Tfy)] \\ &+ a_3[d(Tx, Tfy) + d(Ty, Tfx)], \end{aligned}$$

for all $x, y \in X$ where $a_i \in P$ commute for $i = 1, 2, 3$ and

$$sr(a_1) + (s^2 + 1)r(a_2) + (s^3 + s)r(a_3) < 1.$$

Then,

(1) *for each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence.*
(2) *there exist $u_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = u_{x_0}$.*
(3) *if T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *there exist a unique $v_{x_0} \in X$ such that $fv_{x_0} = v_{x_0}$, that is, f has a unique fixed point.*
(5) *if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{f^n x_0\}$ converges to v_{x_0}.*

Proof. The proof follows by taking $f = g$ in Theorem 2.1. □

Corollary 2.4. *Let (X,d) be a complete cone b-metric space (X,d) over a Banach algebra A with coefficient $s \geq 1$ and P be a solid cone. Let f be a map of X satisfying*

$$d(fx, fy) \preceq a_1 d(x, y) + a_2[d(x, fx) + d(y, fy)] + a_3[d(x, fy) + d(y, fx)]$$

for all $x, y \in X$ where $a_i \in P$ commute for $i = 1, 2, 3$ and

$$sr(a_1) + (s^2 + 1)r(a_2) + (s^3 + s)r(a_3) < 1.$$

Then f has a unique fixed point.

Proof. The proof follows by taking $T = I$ and $f = g$ in Theorem 2.1. □

The following Corollary extends Theorem 3.3 of Xu and Radenovic [9].

Corollary 2.5. *Let (X,d) be a complete cone b-metric space (X,d) over a Banach algebra A with coefficient $s \geq 1$ and P be a solid cone. Let $T : X \to X$ be a continuous and one to one mapping. Moreover, let f be a map of X satisfying*

$$d(Tfx, Tfy) \preceq k[d(Tx, Tfx) + d(Ty, Tfy)]$$

for all $x, y \in X$ where $k \in P$ and $(s^2 + 1)r(k) < 1$. Then,

(1) *for each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence.*

(2) *there exist* $u_{x_0} \in X$ *such that* $\lim_{n\to\infty} Tf^n x_0 = u_{x_0}$.
(3) *if* T *is subsequentially convergent, then* $\{f^n x_0\}$ *has a convergent subsequence.*
(4) *there exist a unique* $v_{x_0} \in X$ *such that* $fv_{x_0} = v_{x_0}$, *that is,* f *has a unique fixed point.*
(5) *if* T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{f^n x_0\}$ *converges to* v_{x_0}.

Proof. The proof follows by taking $f = g$ and $a_2 = k, a_1 = a_3 = \theta$ in Theorem 2.1. □

The following Corollary extends Theorem 3.2 of Xu and Radenovic [9].

Corollary 2.6. *Let* (X, d) *be a complete cone b-metric space* (X, d) *over a Banach algebra* A *with coefficient* $s \geq 1$ *and* P *be a solid cone. Let* $T : X \to X$ *be a continuous and one to one mapping. Moreover, let* f *be a map of* X *satisfying*

$$d(Tfx, Tfy) \preceq k[d(Tx, Tfy) + d(Ty, Tfx)]$$

for all $x, y \in X$ *where* $k \in P$ *and* $(s^3 + s)r(k) < 1$. *Then,*

(1) *for each* $x_0 \in X$, $\{Tf^n x_0\}$ *is a Cauchy sequence.*
(2) *there exist* $u_{x_0} \in X$ *such that* $\lim_{n\to\infty} Tf^n x_0 = u_{x_0}$.
(3) *if* T *is subsequentially convergent, then* $\{f^n x_0\}$ *has a convergent subsequence.*
(4) *there exist a unique* $v_{x_0} \in X$ *such that* $fv_{x_0} = v_{x_0}$, *that is,* f *has a unique fixed point.*
(5) *if* T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{f^n x_0\}$ *converges to* v_{x_0}.

Proof. The proof follows by taking $f = g$ and $a_1 = a_2 = \theta, a_3 = k$ in Theorem 2.1. □

Corollary 2.7. *Let* (X, d) *be a complete cone b-metric space* (X, d) *over a Banach algebra* A *with coefficient* $s \geq 1$ *and* P *be a solid cone. Let* $T : X \to X$ *be a continuous and one to one mapping. Moreover, let* f *be a map of* X *satisfying*

$$d(Tfx, Tfy) \preceq kd(Tx, Ty)$$

for all $x, y \in X$ *where* $k \in P$ *and* $r(k) < \frac{1}{s}$. *That is,* f *be a* T*-contraction. Then,*

(1) *for each* $x_0 \in X$, $\{Tf^n x_0\}$ *is a Cauchy sequence.*
(2) *there exist* $u_{x_0} \in X$ *such that* $\lim_{n\to\infty} Tf^n x_0 = u_{x_0}$.
(3) *if* T *is subsequentially convergent, then* $\{f^n x_0\}$ *has a convergent subsequence.*

(4) *there exist a unique* $v_{x_0} \in X$ *such that* $fv_{x_0} = v_{x_0}$*, that is,* f *has a unique fixed point.*
(5) *if* T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{f^n x_0\}$ *converges to* v_{x_0}*.*

Proof. The proof follows by taking $f = g$ and $a_1 = k, a_2 = a_3 = \theta$ in Theorem 2.1. □

The following corollary extends Theorem 3.1 of [9] and is the Banach-type version of a fixed point results for contractive mappings.

Corollary 2.8. *Let* (X, d) *be a complete cone b-metric space* (X, d) *over a Banach algebra* A *with coefficient* $s \geq 1$ *and* P *be a solid cone. Let* f *be a map of* X *satisfying*

$$d(fx, fy) \preceq kd(x, y)$$

for all $x, y \in X$*, where* $k \in P$ *and* $r(k) < \frac{1}{s}$*. Then* f *has a unique fixed point.*

Proof. The proof follows by taking $T = I$ in Corollary 2.7. □

Corollary 2.9. *Let* (X, d) *be a complete cone b-metric space* (X, d) *over a Banach algebra* A *with coefficient* $s \geq 1$ *and* P *be a solid cone. Let* f *be a map of* X *satisfying*

$$d(f^n x, f^n y) \preceq kd(x, y)$$

for all $x, y \in X$*, where* $k \in P$ *and* $r(k) < \frac{1}{s}$*. Then* f *has a unique fixed point.*

Proof. From Corollary 2.8, f^n has a unique fixed point x^*. But $f^n(fx^*) = f(f^n x^*) = fx^*$, So fx^* is also a fixed point of f^n. Hence $fx^* = x^*, x^*$ is a fixed point of f. Since the fixed point of f is also fixed point of f^n, then fixed point of f is unique, □

Corollary 2.10. *Let* (X, d) *be a complete cone metric space* (X, d) *over a Banach algebra* A *and* P *be a solid cone. Let* $T : X \to X$ *be a continuous and one to one mapping. Moreover, let* f *and* g *be two maps of* X *satisfying*

$$\begin{aligned} d(Tfx, Tgy) \;\; \preceq \;\; & a_1 d(Tx, Ty) + a_2[d(Tx, Tfx) + d(Ty, Tgy)] \\ & + a_3[d(Tx, Tgy) + d(Ty, Tfx)] \end{aligned}$$

for all $x, y \in X$*, where* $a_i \in P$ *commute for* $i = 1, 2, 3$ *and*

$$r(a_1) + 2r(a_2) + 2r(a_3) < 1.$$

That is, f *and* g *be a* T*-contraction. Then,*

(1) *there exist* $u_x \in X$ *such that* $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = u_x$*.*
(2) *if* T *is subsequentially convergent, then* $\{fx_{2n}\}$ *and* $\{gx_{2n+1}\}$ *have a convergent subsequence.*

(3) *there exist a unique* $v_x \in X$ *such that* $fv_x = gv_x = v_x$*, that is,* f *and* g *have a unique common fixed point.*

(4) *if* T *is sequentially convergent, then iterate sequences* $\{fx_{2n}\}$ *and* $\{gx_{2n+1}\}$ *converge to* v_x*.*

Proof. The proof follows by taking $s = 1$ in Theorem 2.1. □

Corollary 2.11. *Let* (X, d) *be a complete cone metric space* (X, d) *over a Banach algebra* A *and* P *be a solid cone. Let* f *be a map of* X *satisfying*

$$\begin{aligned} d(fx, fy) &\preceq a_1 d(x, y) + a_2[d(x, fx) + d(y, fy)] \\ &\quad + a_3[d(x, fy) + d(y, fx)] \end{aligned}$$

for all $x, y \in X$*, where* $a_i \in P$ *commute for* $i = 1, 2, 3$ *and*

$$r(a_1) + 2r(a_2) + 2r(a_3) < 1.$$

Then f *has a unique common fixed point.*

Proof. The proof follows by taking $s = 1$, $T = I$ and $f = g$ in Theorem 2.1. □

Remark 2.12. In Corollary 2.11, if we suppose that (X, d) is a complete cone metric space over a Banach algebra A and P is a normal cone with normal constant K, then we obtain Theorem 2.1, 2.2 and 2.3 that were given by Liu and Xu [6]. So Corollary 2.11 is a generalization of Theorem 2.1, 2.2 and 2.3 in [6].

As an application of the main result, we give the following examples:

Example 2.13. Let $X = [0, 1]$ and let (X, d) be a complete cone b-metric space over a Banach algebra $A = C[0, 1]$ as defined in Example 1.2, where $d(x, y)(t) = |x - y|^2 e^t$ for all $x, y \in X$ and for each $t \in [0, 1]$. Then the set $P = \{x \in A : x \geq 0\}$ is a normal cone in A. Define two mappings $T, f : X \to X$ by $Tx = x^2$ and $fx = \frac{x}{2}$. Then, we have

(1) T and f are continuous on X. Also T is one to one and subsequentially convergent.

(2) f is a contraction.

(3) Take the constant functions $a_1 = a_3 = \theta, a_2 = \frac{1}{9}$ in P. Then $r(a_1) = r(a_3) = 0, r(a_2) = \frac{1}{9}$ and $2sr(a_1) + (s^2 + 1)r(a_2) + (s^3 + s)r(a_3) = \frac{5}{9} < 1$. Also for each $t \in [0, 1]$,

$$\begin{aligned} d(Tfx, Tfy)(t) &= |\frac{x^2}{4} - \frac{y^2}{4}|^2 e^t = \frac{1}{16}|x^2 - y^2|^2 e^t \\ &\preceq \frac{9}{16}(x^4 + y^4)e^t \\ &= a_2[d(Tx, Tfx) + d(Ty, Tfy)](t). \end{aligned}$$

Thus all the conditions of Theorem 2.1 are fulfilled. So by Theorem 2.1, f has a unique fixed point $x = 0$.

(4) But there does not exist $a \in P$ with $r(a) < 1$ such that

$$d(fx, fy) \preceq a[d(x, fx) + d(y, fy)].$$

(5) Also if we take the constant functions $a_1 = k, a_2 = a_3 = 0(\frac{1}{16} \leq r(k) < \frac{1}{2})$ in P, then

$$d(Tfx, Tfy)(t) = \frac{1}{16}|x^2 - y^2|^2 e^t \preceq kd(Tx, Ty)$$

and so all the conditions of Theorem 2.1 are fulfilled. Thus by Theorem 2.1, f has a unique fixed point $x = 0$.

Example 2.14. Let $X = [0, \frac{1}{2}]$ and let (X, d) be a complete cone b-metric space over a Banach algebra $A = C[0, 1]$ as defined in Example 1.2, where $d(x, y)(t) = |x - y|^2 e^t$ for all $x, y \in X$ and for each $t \in [0, 1]$. Then the set $P = \{x \in A : x \geq 0\}$ is a normal cone in A.

Define two mappings $T, f : X \to X$ by $Tx = x^2$ and $fx = \frac{x^2}{\sqrt{2}}$. Take $k = \frac{1}{16} \in P$. Then f is not contractive. But f is T-contraction, because for all $t \in [0, 1]$,

$$\begin{aligned} d(Tfx, Tfy)(t) &= |\frac{x^4}{2} - \frac{y^4}{2}|^2 e^t = \frac{1}{4}|x^2 + y^2|^2 |x^2 - y^2|^2 e^t \\ &\preceq \frac{1}{16}|x^2 - y^2|^2 e^t \\ &= \frac{1}{16} d(Tx, Ty)(t). \end{aligned}$$

So, by Theorem 2.1, f has a unique fixed point $x = 0$.

The following examples show that we can not omit the conditions of Theorem 2.1. In the following note we have two examples which show that we can not omit the one-to-one of T in Theorem 2.1. In first example f has more than one fixed point and in the second example f has not a fixed point.

Example 2.15. Let $X = \{0, \frac{1}{2}, 1\}$ and let (X, d) be a complete cone b-metric space over a Banach algebra $A = \mathbb{R}$, where $d(x, y) = |x - y|^2$.

Case 1. Define two functions $T_1, f_1 : X \to X$ defined by

$$T_1 x = \begin{cases} 0 & x = 0, 1 \\ \frac{1}{2} & x = \frac{1}{2} \end{cases} \quad \text{and} \quad f_1 x = \begin{cases} 0 & x = 0, \frac{1}{2} \\ 1 & x = 1. \end{cases}$$

Then T_1 is subsequentially convergent and since for any $k \in (0, 1)$

$$d(T_1 f_1 x, T_1 f_1 y) = |T_1 f_1 x - T_1 f_1 y|^2 \preceq kd|T_1 x, T_1 y) \quad (x, y \in X),$$

f_1 is T_1-contraction. But T_1 is not one-to-one and f_1 has two fixed points.

Case 2. If we define the functions $T_2, f_2 : X \to X$ by

$$T_2 x = \begin{cases} 0 & x = 0, 1 \\ \frac{1}{2} & x = \frac{1}{2} \end{cases} \qquad \text{and} \quad f_2 x = \begin{cases} 1 & x = 0, \frac{1}{2} \\ 0 & x = 1, \end{cases}$$

then T_2 is subsequentially convergent and since

$$d(T_2 f_2 x, T_2 f_2 y) = |T_2 f_2 x - T_2 f_2 y|^2 \preceq \frac{1}{2}|T_2 x - T_2 y|^2 \quad (x, y \in X)$$

and $k = \frac{1}{2} \in P$, f_2 is T_2-contraction. But T_2 is not one-to-one and f_2 has not a fixed point.

The following example shows that we can not omit the subsequentially convergent of T in Theorem 2.1.

Example 2.16. Let $X = [0, \infty)$ and let (X, d) be a complete cone b-metric space over a Banach algebra $A = \mathbb{R}$ as defined in Example 2.15. We define two mapping $T, f : X \to X$ by $Tx = e^{-x}$ and $fx = 2x + 1$ and take $k = \frac{4}{e^2} \in P$. Then T is one-to-one and f is T-contraction since

$$\begin{aligned} d(Tfx, Tfy) &= |Tfx - Tfy|^2 = |e^{-2x-1} - e^{-2y-1}|^2 \\ &= e^{-2}|e^{-x} + e^{-y}|^2 |e^{-x} - e^{-y}|^2 \\ &\preceq 4e^{-2}|e^{-x} - e^{-y}|^2 \\ &= \frac{4}{e^2} d(Tx, Ty). \end{aligned}$$

But T is not subsequentially convergent ($Tn \to 0$ as $n \to \infty$ but $\{n\}_1^\infty$ has not any convergence subsequence) and f has not a fixed point.

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