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COMMON FIXED POINTS OF FOUR MAPPINGS IN A FUZZY METRIC SPACE

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Abstract. In this paper, we prove some general fixed point theorems, which generalize the result of Pant [4] for mappings in fuzzy metric spaces. Here we give the existence of a contractive definition which generates a fixed point, but does not force the mapping to be continuous at the fixed point.

1. Introduction

The notion of a fuzzy set was introduced by Zadeh [8] in 1965. Following the concept of fuzzy sets, statistical metric spaces have been introduced by Sehweizer [6]. In 1975, fuzzy metric spaces have been developed by Kramosil and Michalck [3], and the fuzzy metric spaces have been further modified by George and Veeramani [1]. Recently several authors have proved fixed point theorems involving fuzzy sets ([2], [7]). Recently Vasuki [7] investigated some fixed point theorems in fuzzy metric spaces for R-weakly commuting maps and Pant [4] introduced the notion of reciprocal continuity of mappings in metric spaces and proved some common fixed point theorems. In this paper,

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as an application of the notion of reciprocal continuity, we obtain a fixed point theorem in a fuzzy metric space and show that in the setting of fuzzy metric spaces, the open problem (see e.g. Rhoades [5]) on the existence of a contractive definition which generates a fixed point but does not force the map to be continuous at the fixed point possesses an affirmative answer.

To prove the existence of fixed points in fuzzy metric spaces, the following preliminaries are needed.

Definition 1.1. [8] Let X be any set. A fuzzy set in X is a function with domain X and values in $[0, 1]$.

Definition 1.2. [6] A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous $t-$norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

Definition 1.3. [1] The 3$-$tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions.

1. $M(x, y, t) > 0$ for all $t > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$ for all $t > 0$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.

Definition 1.4. [1] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and $p > 0$, $(X, M, *)$ is complete if every cauchy sequence in X converges in X. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.

Definition 1.5. [4] Two self maps A and S of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(ASx_n,\ SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.6. Two self maps A and S of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n\to\infty} M(ASx_n,\ SAx_n,\ t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.7. [4] Two maps A and S of a metric space (X, d) are called $R-$weakly commuting at a point $x \in X$ if $d(ASx, SAx) \leq Rd(Ax, Sx)$.

Definition 1.8. Two mappings A and S of a fuzzy metric space $(X, M, *)$ into itself are said to be weakly commuting if $M(ASx, SAx, t) \geq M(Ax, Sx, t)$ for each $x \in X$ and $t > 0$.

Definition 1.9. Two mappings A and S of a fuzzy metric space $(X, M, *)$ into itself are $R-$weakly commuting provided there exists some real R such that $M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})$ for each $x \in X$ and $t > 0$.

Example 1.10. Let $X = \mathbb{R}$, the set of all real numbers. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all x, y in $X, t > 0$. Then $(X, M, *)$ is a fuzzy metric space. Define A and $B : X \to X$ by $Ax = x^3$ and $Bx = 3 - 2x$ respectively for all $x \in X$. Then, since $M(Ax_n, Bx_n, t) = \frac{t}{t+d(Ax_n,Bx_n)}$, $d(Ax_n, Bx_n) = |x_n - 1||x_n^2 + x_n + 3| \to 0$ if and only if $x_n \to 1$ as $n \to \infty$. Therefore $M(Ax_n, Bx_n, t) \to 1$ if and only if $x_n \to 1$ for any $t > 0$,

$$\begin{aligned}\lim_{n\to\infty} M(BAx_n, ABx_n, t) &= \lim_{n\to\infty} \frac{t}{t + d(ABx_n, BAx_n)} \\ &= \lim_{n\to\infty} \frac{t}{t + 6|x_n - 1|^2|x_n - 4|} \\ &= 1 \ as \ x_n \ \to \ 1 \text{ for any } t > 0.\end{aligned}$$

Thus A and B are compatible. Since $M(ABx, BAx, t) < M(Ax, Bx, t)$ for some x in X and $t > 0$, A and B are not weakly commuting mappings.

Definition 1.11. Two mappings A and S are called pointwise $R-$weakly commuting on X if given x in $(X, M, *)$ there exists $R > 0$ such that

$$M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R}).$$

Definition 1.12. Let A and S be self mappings of a fuzzy metric space $(X, M, *)$. We will call A and S to be reciprocally continuous if $\lim_{n\to\infty} ASx_n = Ap$ and $\lim_{n\to\infty} SAx_n = Sp$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X. If A and S are both continuous then they are obviously reciprocally continuous. But the converse need not be true.

2. Main Results

If A, B, S and T are self mappings of a fuzzy metric space $(X, M, *)$ in the sequal we shall denote

$$\begin{aligned}m(x, y, t) = \min\{&M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ &M(Ax, Ty, 2t) * (By, Sx, 2t)\}.\end{aligned}$$

Theorem 2.1. Let (A, S) and (B, T) be pointwise $R-$weakly commuting pairs of self-mappings of a complete fuzzy metric space $(X, M, *)$ with $r * r \geq r$ satisfying

(i) $AX \subseteq TX, BX \subseteq SX$

(ii) $M(Ax, By, t) > m(x, y, \frac{t}{h}), 0 < h < 1, x, y \in X$ and $t > 0$.

Suppose that (A, S) or (B, T) is compatible pair of reciprocal continuous mappings. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. The sequences $\{x_n\}$ and $\{y_n\}$ in X are defined by the way

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

This can be done by virtue of (i). Then using (ii), we obtain

$$\begin{aligned}
&M(y_{2n}, y_{2n+1}, t)\\
&\quad = M(Ax_{2n}, Bx_{2n+1}, t) > m(x_{2n}, x_{2n+1}, \frac{t}{h})\\
&\quad = \min\{M(Sx_{2n}, Tx_{2n+1}, \frac{t}{h}), M(Ax_{2n}, Sx_{2n}, \frac{t}{h}),\\
&\qquad\quad M(Bx_{2n+1}, Tx_{2n+1}, \frac{t}{h}),\\
&\qquad\quad M(Ax_{2n}, Tx_{2n+1}, \frac{2t}{h}) * M(Bx_{2n+1}, Sx_{2n}, \frac{2t}{h})\}\\
&\quad = \min\{M(y_{2n-1}, y_{2n}, \frac{t}{h}), M(y_{2n}, y_{2n-1}, \frac{t}{h}), M(y_{2n+1}, y_{2n}, \frac{t}{h}),\\
&\qquad\quad M(y_{2n}, y_{2n}, \frac{2t}{h}) * M(y_{2n+1}, y_{2n-1}, \frac{2t}{h})\}\\
&\quad = \min\{M(y_{2n-1}, y_{2n}, \frac{t}{h}), M(y_{2n+1}, y_{2n-1}, \frac{2t}{h})\}\\
&\quad \geq \min\{M(y_{2n-1}, y_{2n}, \frac{t}{h}), M(y_{2n+1}, y_{2n}, \frac{t}{h}) * M(y_{2n}, y_{2n-1}, \frac{t}{h})\}\\
&\quad = M(y_{2n-1}, y_{2n}, \frac{t}{h}) * M(y_{2n}, y_{2n+1}, \frac{t}{h}).
\end{aligned}$$

Hence

$$\begin{aligned}
M(y_{2n}, y_{2n+1}, t) &> M(y_{2n-1}, y_{2n}, \frac{t}{h}) * M(y_{2n}, y_{2n+1}, \frac{t}{h})\\
&> M(y_{2n-1}, y_{2n}, \frac{t}{h}) * M(y_{2n-1}, y_{2n}, \frac{t}{h^2}) * M(y_{2n}, y_{2n+1}, \frac{t}{h^2})\\
&> M(y_{2n-1}, y_{2n}, \frac{t}{h}) * M(y_{2n}, y_{2n+1}, \frac{t}{h^2})\\
&\quad \cdots\\
&\geq M(y_{2n-1}, y_{2n}, \frac{t}{h}) * M(y_{2n}, y_{2n+1}, \frac{t}{h^k})\\
&\to M(y_{2n-1}, y_{2n}, \frac{t}{h}) \text{ as } k \to \infty.
\end{aligned}$$

Hence $M(y_{2n+1}, y_{2n}, t) > M(y_{2n}, y_{2n-1}, \frac{t}{h})$.
Similarly we can prove that $M(y_{2n+2}, y_{2n+1}, t) > M(y_{2n+1}, y_{2n}, \frac{t}{h})$ for all $n > 0$

and $t > 0$. Hence we have

$$\begin{aligned} M(y_{n+1}, y_n, t) &> M(y_n, y_{n-1}, \frac{t}{h}) \text{ for all } n > 0, t > 0 \text{ and } 0 < h < 1 \\ &\cdots \\ &> M(y_1, y_0, \frac{t}{h^n}) \text{ for all } n > 0, t > 0 \text{ and } 0 < h < 1 \\ &\to 1 \text{ as } n \to \infty. \end{aligned}$$

Moreover for every integer $p > 0$, we get

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_{n+1}, y_{n+2}, \frac{t}{p}) * \cdots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &> M(y_0, y_1, \frac{t}{ph^n}) * M(y_0, y_1, \frac{t}{ph^{n+1}}) * \cdots * M(y_0, y_1, \frac{t}{ph^{n+p-1}}) \\ &\to 1 * 1 * \cdots * 1 = 1 \text{ as } n \to \infty. \end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X such that $y_n \to z$ as $n \to \infty$. Thus $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$. Suppose A and S are compatible and reciprocally continuous mappings, then $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$. Compatibility of A and S yields

$$\lim_{n\to\infty} M(ASx_{2n}, SAx_{2n}, t) = 1.$$

That is $M(Az, Sz, t) \to 1$. Hence $Az = Sz$. Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. Using (ii), we get that

$$\begin{aligned} M(Az, Bw, t) &> m(z, w, \frac{t}{h}) \\ &= \min\{M(Sz, Tw, \frac{t}{h}), M(Az, Sz, \frac{t}{h}), M(Bw, Tw, \frac{t}{h}), \\ &\qquad M(Az, Tw, \frac{2t}{h}) * M(Bw, Sz, \frac{2t}{h})\} \\ &= \min\{1, 1, M(Bw, Tw, \frac{t}{h}), M(Bw, Sz, \frac{2t}{h})\} \\ &\geq \min\{M(Bw, Az, \frac{t}{h}) * M(Sz, Az, \frac{t}{h}), M(Bw, Az, \frac{t}{h})\} \\ &= M(Bw, Az, \frac{t}{h}) \\ &\cdots \\ &= M(Bw, Az, \frac{t}{h^n}) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence $Az = Bw$. Thus $Az = Sz = Bw = Tw$. Pointwise R-weakly continuity of A and S implies that there exists $R > 0$ such that

$$M(ASz, SAz, t,) \geq M(Az, Sz, \frac{t}{R}) = 1.$$

That is $ASz = SAz$ and $AAz = ASz = SAz = SSz$.

Similarly pointwise R-commutative of B and T implies that

$$BBw = BTw = TBw = TTw.$$

Using (ii), we now get that

$$\begin{aligned} M(Az, AAz, t) = M(Bw, AAz, t) > m(w, Az, \frac{t}{h}) \\ = \min\{M(SAz, Tw, \frac{t}{h}), M(AAz, SAz, \frac{t}{h}), M(Bw, Tw, \frac{t}{h}), \\ M(AAz, Tw, \frac{2t}{h}) * M(Bw, SAz, \frac{2t}{h})\}. \end{aligned}$$

Thus $M(Az, AAz, t) > M\left(AAz, Az, \frac{t}{h}\right) = \cdots = M\left(AAz, Az, \frac{t}{h^n}\right) \to 1$ as $n \to \infty$. Hence $Az = AAz = SAz$. Thus Az is a common fixed point of S and A.

Similarly we can prove that $Bw = BBw = TBw$. Thus Bw is a common fixed point of B and T. But $Bw = Az$. Thus Az is common fixed point of A, S, B and T.

Uniqueness: Suppose x and y be two common fixed points of A, S, B and T. Then $Ax = Bx = Sx = Tx = x$ and $Ay = By = Sy = Ty = y$.

Now

$$\begin{aligned} M(x, y, t) &= M(Ax, By, t) > m\left(x, y, \frac{t}{h}\right) \\ &= \min\{M(Sx,\, Ty, \frac{t}{h}), M(Ax, Sx, \frac{t}{h}), M(By,\, Ty, \frac{t}{h}), \\ &\qquad M(Ax,\, Ty, \frac{2t}{h}) * M(Ay, Sx, \frac{2t}{h})\} \\ &= \min\{M(x, y, \frac{t}{h}), M(x, x, \frac{t}{h}), M(y, y, \frac{t}{h}) M(x, y, \frac{2t}{h}) * M(y, x, \frac{2t}{h})\} \\ &= M(x, y, \frac{t}{h}) \\ &\quad \cdots \\ &> M(x, y, \frac{t}{h^n}) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Therefore $x = y$. Hence A, S, B, T have a unique common fixed point in X. □

Example 2.2. Let $X = [1, 10]$ and M be the usual fuzzy metric on $(X, M, *)$. Define mappings $A, B, S, T : X \to X$ by

$$A_1 = 1, Ax = 2 \text{ if } x > 1$$
$$S_1 = 1, Sx = 3 \text{ if } x > 1$$
$$Bx = 1 \text{ if } x = 1 \text{ or } > 5, Bx = 3 \text{ if } 1 < x \leq 5$$
$$T_1 = 1, Tx = 6 \text{ if } 1 < x \leq 5, Tx = x - 4 \text{ if } x > 5.$$

Also define $M(Ax, By, t) = \frac{t}{[t+d(Ax,By)]}$ for all x, y in X and $t > 0$. Then A, B, S and T satisfy all the conditions of the above theorem with $h \in (0, 1)$ and have a unique fixed point $x = 1$. It may be noted in the example that A and S are reciprocally continuous compatible maps. But neither A nor S is continuous, not even at the common point $x = 1$. The mappings B and T are non compatible but pointwise R-weakly commuting, since they commute at their coincidence points. To see that B and T are not compatible, let us consider the sequence $\{x_n\}$ defined by

$$x_n = 5 + \frac{1}{n}, n \geq 1.$$

Then $Tx_n \to 1, Bx_n = 1, TBx_n = 1$ and $BTx_n = 3$. Hence B and T are non-compatible. Putting $S = T = I$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. Let A and B be two self mappings of a complete fuzzy metric space $(X, M, *)$ with $r * r \geq r$ such that

$$M(Ax, By, t) > \min\{M(x, y, \frac{t}{h}), M(Ax, x, \frac{t}{h}), M(By, y, \frac{t}{h}),$$
$$M(Ax, y, \frac{2t}{h}) * M(By, x, \frac{2t}{h})\}$$

for all x, y in X and $0 < h < 1$. Then A and B have a unique common fixed point.

Putting $S = T = I$ and $A = B$ in Theorem 2.1, we get the following corollary.

Corollary 2.4. Let A be a self mapping of a complete fuzzy metric space $(X, M, *)$ with $r * r \geq r$ such that

$$M(Ax, Ay, t) > \min\{M(x, y, \frac{t}{h}), M(Ax, x, \frac{t}{h}), M(Ay, y, \frac{t}{h}),$$
$$M(Ax, y, \frac{2t}{h}) * M(Ay, x, \frac{2t}{h})\}$$

Then A has unique fixed point.

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