Nonlinear Functional Analysis and Applications
Vol. 22, No. 5 (2017), pp. 1081-1089
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

INTERMEDIATE SPACES RELATED TO ANALYTIC SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. In this paper, we deal with the theory of interpolation spaces between initial Banach functional spaces and the domain of an elliptic differential operator, which is to be the infinitesimal generator of an analytic semigroup. Based on the theory of basic properties of Besob spaces and characterized by the domain of the infinitesimal generator, we can be applied to L^1-valued control problems or the maximal regularity for the abstract initial value problems in the negative space.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order in $L^1(\Omega)$:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x), \tag{1.1}$$

[0]Received June 18, 2017. Revised November 28, 2017.
[0]2010 Mathematics Subject Classification: 47D03, 46B70.
[0]Keywords: Intermediate space, Besob space, infinitesimal generator, elliptic differential operator, analytic semigroup.
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where $(a_{i,j}(x) : i, j = 1, \cdots, n)$ is a positive definite symmetric matrix for each $x \in \Omega$, $a_{i,j} \in C^1(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$. The operator

$$\mathcal{A}'(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\cdot) + c(x) \tag{1.2}$$

is the formal adjoint of $\mathcal{A}$.

For $1 < p < \infty$, we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$
$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

For $p' = p/(p-1)$, we can also define the realization $\mathcal{A}'$ in $L^{p'}(\Omega)$ under Dirichlet boundary condition by $A'_{p'}$:

$$D(A'_{p'}) = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega),$$
$$A'_{p'}u = \mathcal{A}'u \quad \text{for} \quad u \in D(A'_{p'}).$$

It is known that $-A_p$ and $-A'_{p'}$ generate analytic semigroups in $L^p(\Omega)$ and $L^{p'}(\Omega)$, respectively, and $A_p^* = A'_{p'}$. For brevity, we assume that $0 \in \rho(A_p)$. From the result of Seeley [11] (see also Triebel [13, p. 321]) we obtain that

$$[D(A_p), L^p(\Omega)]_{\frac{1}{2}} = W_0^{1,p}(\Omega),$$

and hence, may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A'_{p'})^*.$$

Let $(A'_{p'})'$ be the adjoint of $A'_{p'}$ considered as a bounded linear operator from $D(A'_{p'})$ to $L^{p'}(\Omega)$. Let A be the restriction of $(A'_{p'})'$ to $W_0^{1,p}(\Omega)$. Then by the interpolation theory, the operator A is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction A' of $(A_p)'$ belonging to $B(L^{p'}(\Omega), D(A_p)^*)$ to $W_0^{1,p'}(\Omega)$ is an isomorphism from $W_0^{1,p'}(\Omega)$ to $W^{-1,p'}(\Omega)$. For $q \in (1, \infty)$, we set

$$H_{p,q} = (W_0^{1,p}, W^{-1,p})_{1/q,q}.$$

As seen in proposition 3.1 in Jeong [7], the operators $-A$ and $-A'$ generate an analytic semigroup in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively. Furthermore, $-A$ also generates an analytic semigroup in $H_{p,q}$. The space $H_{p,q}$ is ζ-convex(as for the definition and fundamental results of a ζ-convex space, see [2, 1]), and the inequality

$$||(A)^{is}||_{B(W^{-1,p}(\Omega))} \le Ce^{\gamma|s|}, \quad -\infty < s < \infty$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$.

Let us consider

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = Bw(t), & t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{1.3}$$

where the controller B is a bounded linear operator from some Banach space U to $L^1(\Omega)$, and $w \in L^q(0,T;U)$ for $1 < q < \infty$. Noting that if $1 < p < n/(n-1)$ we may consider $L^1(\Omega) \subset W^{-1,p}(\Omega)$, and so, we cannot express $u(t)$ using the solution semigroup since B is a mapping into $W^{-1,p}(\Omega)$ not into $H_{p,q}$. Therefore, based on the theory of the definition and basic properties of Besob spaces, we will show that if $\frac{1}{p'} < 1/n(1-2/q')$ then

$$H_{p',q'} \subset C_0(\overline{\Omega}) \subset L^\infty(\Omega).$$

Thus, we may consider

$$H_{p,q} = H^*_{p'.q'} \supset C_0(\overline{\Omega})^* \supset L^1(\Omega)$$

and B is bounded mapping from U to $H_{p,q}$. Hence, it is possible to investigate the control problem for (1.3) in $H_{p,q}$. Consequently, in view of the maximal regularity result by Dore and Venni [4], the initial value problem (1.3) has a unique solution $u \in L^q\big(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\big) \bigcap W^{1,q}(0,T;H_{p,q})$ for any $u_0 \in W_0^{1,p}(\Omega)$.

2. Notations

Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ and closure $\overline{\Omega}$. $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω.

$C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω.

$W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p,\Omega} = (\sum_{\alpha \le m} ||D^\alpha f||^p_{p,\Omega})^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$||f||_{m,\infty,\Omega} = \max_{\alpha \le m} ||D^\alpha u||_{\infty,\Omega},$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $||\cdot||_{p,\Omega}$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p = 2$, we denote $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{2,p}(\Omega) = H_0^m(\Omega)$.

Let $p' = p/(p-1)$, $1 < p < \infty$. $W^{-1,p}(\Omega)$ stands for the dual space $W_0^{1,p'}(\Omega)^*$ of $W_0^{1,p'}(\Omega)$ whose norm is denoted by $||\cdot||_{-1,p}$.

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable.

$C^m([0,T];X)$ will denote the set of all m-times continuously differentiable functions from $[0,T]$ into X.

If X and Y are two Banach spaces, $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$.

For an interpolation couple of Banach spaces X_0 and X_1, $(X_0,X_1)_{\theta,p}$ for any $\theta \in (0,1)$ and $1 \le p \le \infty$ and $[X_0,X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively(see [13]).

3. Relationship of $H_{p,q} \subset L^1(\Omega)$ as a Besov space

Let A be the operator mentioned in Section 1. Then it was shown that the operators $-A$ generates an analytic semigroup in $W^{-1,p}(\Omega)$ (see [7]).

Lemma 3.1. *There exists a positive constant C such that for any $t > 0$*

$$||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))} \le Ct^{-\frac{1}{2}} \tag{3.1}$$

and

$$||(t+A)^{-1}||_{B(L^p(\Omega),W_0^{1,p}(\Omega))} \le Ct^{-\frac{1}{2}}. \tag{3.2}$$

Proof. Let A_p be the realization of (1.1) in $L^p(\Omega)$ in the distribution sense under the Dirichlet boundary condition. Then $-A_p$ generates an analytic semigroup in $L^p(\Omega)$, and A_p is the restriction of A to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Hence, (3.2) follows from the moment inequality

$$||u||_{W^{1,p}(\Omega)} \le C||u||^{\frac{1}{2}}_{W^{2,p}(\Omega)}||u||^{\frac{1}{2}}_{L^p(\Omega)}$$

and the estimate

$$||(t+A)^{-1}||_{B(L^p(\Omega))} \le Ct^{-1}$$

proved in [7, Eq(3.5)]. Replacing p by p' we get

$$||(t+A')^{-1}||_{B(L^{p'}(\Omega),W_0^{1,p'}(\Omega))} \le Ct^{-\frac{1}{2}},$$

where A' is the realization of (1.2) in $W^{-1,p'}(\Omega)$ under the Dirichlet boundary condition. Taking the adjoint we obtain (3.1). □

Let Y_0 and Y_1 be two Banach spaces contained in a locally convex linear topological space $\mathcal{Y}$ such that the identity mapping of Y_i $(i = 0,\ 1)$ into $\mathcal{Y}$ is continuous, and their norms will be denoted by $||\cdot||_i$. The algebraic sum $Y_0 + Y_1$ of Y_0 and Y_1 is the space of all elements $a \in \mathcal{Y}$ of the form $a = a_0 + a_1$, $a_0 \in Y_0$ and $a_1 \in Y_1$. The intersection $Y_0 \cap Y_1$ and the sum $Y_0 + Y_1$ are Banach spaces with the norms

$$||a||_{Y_0 \cap Y_1} = \max\ \{||a||_0, ||a||_1\}$$

and

$$||a||_{Y_0+Y_1} = \inf_a\ \{||a_0||_0 + ||a_1||_1\}, \quad a = a_0 + a_1,\ a_i \in Y_i,$$

respectively.

Definition 3.2. ([8]) We say that an intermediate space Y of Y_0 and Y_1 belongs to

(i) the class $\underline{K}_\theta(Y_0, Y_1),\ 0 < \theta < 1$, if for any $a \in Y_0 \cap Y_1$,

$$||a||_Y \le c||a||_0^{1-\theta}||a||_1^\theta$$

where c is a constant;

(ii) the class $\overline{K}_\theta(Y_0, Y_1),\ 0 < \theta < 1$, if for any $a \in Y$ and $t > 0$ there exist $a_i \in Y_i$ $(i = 1,\ 2)$ such that $a = a_0 + a_1$ and

$$||a_0||_0 \le ct^{-\theta}||a||_Y, \quad ||a_1||_1 \le ct^{1-\theta}||a||_Y$$

where c is a constant;

(iii) the class $K_\theta(Y_0, Y_1)$, $0 < \theta < 1$, if the space Y belongs to both $\underline{K}_\theta(Y_0, Y_1)$ and $\overline{K}_\theta(Y_0, Y_1)$.

Here, we note that by replacing t with t^{-1} the condition in (ii) is rewritten as follows:

$$||a_0||_0 \le ct^{\theta}||a||_Y, \quad ||a_1||_1 \le ct^{\theta-1}||a||_Y.$$

The following result is due to Lions-Peetre [8, Theorem 2.3].

Proposition 3.3. *For $0 < \theta_0 < \theta < \theta_1 < 1$, if the spaces X_0 and X_1 belong to the class $K_{\theta_0}(Y_0, Y_1)$ and the class $K_{\theta_1}(Y_0, Y_1)$, respectively, then*

$$(X_0, X_1)_{\frac{\theta-\theta_0}{\theta_1-\theta_0},p} = (Y_0, Y_1)_{\theta,p}.$$

The following corollary is verified following the proof of Proposition 3.1.

Corollary 3.4. *If the space X_1 is of the class $K_{\theta_1}(Y_0, Y_1)$ and $0 < \theta < \theta_1 < 1$, then*

$$(Y_0, X_1)_{\frac{\theta}{\theta_1},p} = (Y_0, Y_1)_{\theta,p}.$$

If the space X_0 is of the class $K_{\theta_0}(Y_0, Y_1)$ and $0 < \theta_0 < \theta < 1$, then

$$(X_0, Y_1)_{\frac{\theta-\theta_0}{1-\theta_0},p} = (Y_0, Y_1)_{\theta,p}.$$

Proposition 3.5. *For $1 < p < \infty$, $L^p(\Omega)$ is of the class $K_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$.*

Proof. For any $u \in W_0^{1,p}(\Omega)$ and $t > 0$, from Lemma 3.1 and

$$u = A(t+A)^{-1}u + t(t+A)^{-1}u = (t+A)^{-1}Au + t(t+A)^{-1}u,$$

it follows

$$\begin{aligned}||u||_{p,\Omega} \leq & ||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))}||Au||_{-1,p,\Omega} \\ & + t||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))}||u||_{-1,p,\Omega} \\ \leq & Ct^{-\frac{1}{2}}||u||_{1,p,\Omega} + Ct^{\frac{1}{2}}||u||_{-1,p,\Omega}.\end{aligned}$$

By choosing $t > 0$ such that $t^{-1/2}||u||_{1,p,\Omega} = t^{1/2}||u||_{B(W_0^{1,p'}(\Omega))^*} = t^{1/2}||u||_{-1,p,\Omega}$, we obtain

$$||u||_{p,\Omega} \leq C||u||_{1,p,\Omega}^{\frac{1}{2}}||u||_{-1,p,\Omega}^{\frac{1}{2}}.$$

Therefore, $L^p(\Omega)$ belongs to the class $\underline{K}_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$. Put $u_0 = t(t+A)^{-1}u$ and $u_1 = A(t+A)^{-1}u$ for any $u \in L^p(\Omega)$. Then $u = u_0 + u_1$, and we obtain that

$$||u_0||_{1,p,\Omega} \leq t||(t+A)^{-1}u||_{B(L^p(\Omega),W_0^{1,p}(\Omega))}||u||_{p,\Omega} \leq Ct^{\frac{1}{2}}||u||_{p,\Omega}$$

$$||u_1||_{-1,p,\Omega} \leq C||(t+A)^{-1}u||_{1,p,\Omega} \leq Ct^{-\frac{1}{2}}||u||_{p,\Omega}.$$

Therefore, $L^p(\Omega)$ belongs to the class $\overline{K}_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$, and hence, it is of the class $K_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$. □

Theorem 3.6. *Let $1 < p < \infty$, $1 < q < \infty$ and $0 < \theta < 1$. If $1 - 2\theta - 1/p \neq 0$ and $2\theta - 2 + 1/p \neq 0$ then*

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} \mathring{B}_{p,q}^{1-2\theta}(\Omega), & \textit{for} \quad \theta < \frac{1}{2}(1-\frac{1}{p}), \\ B_{p,q}^{1-2\theta}(\Omega), & \textit{for} \quad \theta > \frac{1}{2}(1-\frac{1}{p}), \end{cases}$$

where $\mathring{B}_{p,q}^{1-2\theta}(\Omega) = \{u \in B_{p,q}^{1-2\theta}(\Omega) : u|_{\partial\Omega} = 0\}$. In particular, we obtain that

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q} = B_{p,q}^0(\Omega).$$

Proof. Let $0 < \theta < 1/2$. Then from Corollary 3.1, we obtain that

$$\begin{aligned}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} &= (W_0^{1,p}(\Omega), L^p(\Omega))_{2\theta,q} \\ &= (L^p(\Omega), W_0^{1,p}(\Omega))_{1-2\theta,q}.\end{aligned}$$

Therefore, in view of the result of Grisvard [6] (see also Triebel [13][6; p. 321]),

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} \mathring{B}^{1-2\theta}_{p,q}(\Omega), & for \quad 1-2\theta > \frac{1}{p}, \\ B^{1-2\theta}_{p,q}(\Omega), & for \quad 1-2\theta < \frac{1}{p}. \end{cases}$$

Let $1/2 < \theta < 1$. Then from Corollary 2.1, it follows

$$\begin{aligned}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} &= (L^p(\Omega), W^{-1,p}(\Omega))_{2\theta-1,q} \\ &= ((L^{p'}(\Omega), W_0^{-1,p'}(\Omega))_{2\theta-1,q'})^*.\end{aligned}$$

In view of Grisvard's theorem, if $2\theta - 1 - 1/p' \neq 0$ then

$$(L^{p'}(\Omega), W_0^{1,p'}(\Omega))_{2\theta-1,q'} = \begin{cases} \mathring{B}^{2\theta-1}_{p',q'}(\Omega), & for \quad 2\theta-1 > \frac{1}{p'}, \\ B^{2\theta-1}_{p',q'}(\Omega), & for \quad 2\theta-1 < \frac{1}{p'}. \end{cases}$$

From Theorem 4.8.2 in Triebel [13, p. 332], we obtain that

$$(\mathring{B}^{2\theta-1}_{p',q'}(\Omega))^* = B^{1-2\theta}_{p,q}(\Omega) \text{ and } (B^{2\theta-1}_{p',q'}(\Omega))^* = B^{1-2\theta}_{p,q}(\Omega)$$

according as $2\theta - 1 - 1/p' \gtrless 0$. Since $2\theta - 1 - 1/p' \neq 0$ if $1/2 < \theta < 1$ and $2\theta - 2 + 1/p \neq 0$, we get

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = B^{1-2\theta}_{p,q}(\Omega).$$

Consequently, we obtain that

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q} = B^{\theta}_{p,q}(\Omega), \text{ if } \quad 0 < \theta < \frac{1}{p}$$

and

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q} = B^{-\theta}_{p,q}(\Omega), \text{ if } \quad 0 < \theta < 1 - \frac{1}{p}.$$

Hence, if $0 < \theta < \min\{1/p, 1-1/p\}$, then

$$\begin{aligned}&(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q} \\ &= ((W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q}, (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q})_{\frac{1}{2},q} \\ &= (B^{\theta}_{p,q}(\Omega), B^{-\theta}_{p,q}(\Omega))_{\frac{1}{2},q} = B^0_{p,q}(\Omega).\end{aligned}$$

The last equality is obtained from Theorem 1 of section 4.3.1 in Treibel [13]. Hence the proof is complete. □

Theorem 3.7. *Let* $1 < p,\ q < \infty$*.*

(i) *If* $2/q - 2 + 1/p \neq 0$ *then*

$$H_{p,q} = \begin{cases} \overset{\circ}{B}{}^{1-\frac{2}{q}}_{p,q}(\Omega), & \text{if} \quad \frac{1}{q} < \frac{1}{2}(1 - \frac{1}{p}), \\ B^{1-\frac{2}{q}}_{p,q}(\Omega), & \text{if} \quad \frac{1}{q} > \frac{1}{2}(1 - \frac{1}{p}). \end{cases}$$

(ii) *If* $n/p^{'} < 1 - 2/q^{'}$ *then*

$$H_{p^{'},q^{'}} \subset C_0(\overline{\Omega}) \subset L^{\infty}(\Omega).$$

Proof. The relation (i) follows directly from Theorem 2.1. Let $1/p^{'} < 1/n(1 - 2/q^{'})$ which implies $2/q^{'} - 2 + 1/p^{'} < -1 - (n-1)/p^{'} < 0$ and $1/q^{'} < 1/2(1 - n/p^{'}) < 1/2(1 - 1/p^{'})$. Then from (i) and the imbedding theorem ([2; Theorem 4.6.1 in p. 327-328]), we obtain

$$H_{p^{'},q^{'}} = \overset{\circ}{B}{}^{1-\frac{2}{q^{'}}}_{p^{'},q^{'}}(\Omega) \subset C_0(\overline{\Omega})$$

Hence, the first inclusion in (ii) follows. □

Example 3.8. Let U be a Banach space, and let $w \in L^q(0,T;U)$ for $1 < q < \infty$. Consider the followingcontrol problem:

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = Bw(t), & t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{3.3}$$

where the controller B is a bounded linear operator from U to $L^1(\Omega)$. Here, A is an elliptic differential operator of second order in $L^1(\Omega)$ as seen in Section 1. By virtue of Theorem 3.2, we may consider

$$H_{p,q} = H^*_{p^{'}.q^{'}} \supset C_0(\overline{\Omega})^* \supset L^1(\Omega),$$

where $\frac{1}{p^{'}} < 1/n(1 - 2/q^{'})$. Since B is a bounded mapping into $H_{p,q}$, we are able to express $u(t)$ using the solution semigroup $S(t) = e^{At}$. Furthermore, it is possible to investigate the control problem for (3.3) in $H_{p,q}$. Consequently, in view of the maximal regularity result by Dore and Venni [4], the initial value problem (1.3) has a unique solution $u \in L^q\big(0,T;W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\big) \bigcap W^{1,q}(0,T;H_{p,q})$ for any $u_0 \in W^{1,p}_0(\Omega)$. As for the maximal regularity problem of (3.3) in Hilbert spaces, We refer to [3, 5]. Moreover, The observability of (3.3) is defined as

$$B^*S^*(t)f \equiv 0 \quad \text{implies} \quad f = 0$$

in a usual sense of [11-14] .

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