Nonlinear Functional Analysis and Applications
Vol. 22, No. 5 (2017), pp. 1091-1105
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

A NEW SYSTEM OF GENERALIZED IMPLICIT SET-VALUED VARIATIONAL INCLUSIONS IN BANACH SPACES

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

Abstract. In this paper, we introduce and study a new system of generalized implicit set-valued variational inclusions involving generalized accretive operators in Banach spaces. By using the resolvent technique for the generalized accretive operator, we prove that this new system of generalized implicit set-valued variational inclusions is equivalent to a system of nonlinear operators equations. We give an algorithm for finding the solution of the new system of generalized implicit set-valued variational inclusions by employing the iterative method. Moreover, we show the existence of the solution of the new system of generalized implicit set-valued variational inclusions and the convergence of the iterative sequences generated by the algorithm under some suitable assumptions.

[0]Received October 30, 2017. Revised November 28, 2017.
[0]2010 Mathematics Subject Classification: 49J40, 47H10.
[0]Keywords: Generalized implicit set-valued variational inclusion, set-valued mapping, q-uniformly smooth Banach space, resolvent operator, iterative algorithm, convergence.
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1. Introduction

It is well known that the variational inequality theory plays an important role in many fields, such as mechanics, physics, differential equations, structural analysis, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. Because of its wide applications, variational inequality problems have been generalized in various directions for the past years. As a generalization of the variational inequality, the variational inclusion provides an important mathematical model to describe some problems arising in optimization and control, network and transportation, economics and finance. Various theoretical results, numerical algorithms and applications have been studied extensively for variational inequalities and variational inclusions in the literature (see, for example, [1, 2, 5, 27, 28] and the references therein).

On the other hand, monotonicity concerned with nonlinear mappings in Banach spaces has been studied by many authors due to its wide applications in variational inequalities, complementarity problems, fixed point theorems, mathematical programming, differential equations and variational inclusions. It is also well known that the resolvent operator technique involving monotonicity is an efficient and popular method for solving variational inequalities and variational inclusions. Recently, as generalizations of the classical maximal monotone operators, Huang and Fang [11] introduced and studied a class of η-maximal monotone operators and investigated the corresponding resolvent operators under some assumptions. Fang and Huang [6] further studied a class of generalized H-accretive operators, which provides a general framework of H-accretive operators, classical accretive operators, H-monotone operators and maximal monotone operators (see, for example, [7, 26, 28]). Moreover, Fang and Huang [6] considered a class of variational inclusions via the technique of resolvent of H-accretive operators. Zou and Huang [30] introduced and studied a class of system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces via corresponding resolvent operator technique. Luo and Huang [17] further presented a (H,ϕ)-η-monotone operator in Banach spaces and studied a class of variational inclusions via the proximal mapping method.

Recently, as a generalization of the work due to Chen [3], Zhou et al. [29] introduced and studied system of generalized implicit variational inclusions involving a class of (H,ϕ)-accretive operators in Banach spaces. Some related work, we refer the reader to [10, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23] and the references therein.

Inspired and motivated by the work mentioned above, in this paper, we introduce and study the following system of generalized implicit set-valued variational inclusions involving a class of (H, ϕ)-accretive operators

$$\begin{cases} 0 \in F(x) + T(x, y, z) + M(g(x)), \\ 0 \in G(y) + S(x, y, z) + N(h(y)), \\ 0 \in P(z) + A(x, y, z) + Q(f(z)). \end{cases}$$

By employing the resolvent operator technique, we show that to find the solution of the above system is equivalent to solve a system of nonlinear operator equations. Based on this equivalence, we give an iteration algorithm for approximate solutions of the system of generalized implicit set-valued variational inclusions. We prove the existence of solutions of the system of generalized implicit set-valued variational inclusions and the convergence of iterative sequences from the proposed algorithm under some suitable conditions. The results presented in this paper generalize the main results due to Chen [3] and Zhou et al. [29].

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space with its dual X^*. Let $\langle \cdot, \cdot \rangle$ denote the pair between X and X^*. Let 2^X and $CB(X)$ denote the family of all subsets of X and the family of all the nonempty closed and bounded subsets of X, respectively. Let $D(\cdot, \cdot)$ be the Hausdorff metric on $CB(X)$ defined by

$$D(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \ \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}, \quad \forall A, B \in CB(X).$$

It is well known that the generalized duality mapping J_q on X is defined by

$$J_q(x) = \{ f^* \in X^* \mid \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \}, \quad \forall x \in X,$$

where $q > 1$ is a given constant. When $q = 2$, J_2 is called the normalized duality mapping. We know that J_q is single-valued if and only if X^* is strictly convex.

In this paper, we always suppose that X is a real Banach space with J_q being single-valued and $\mathcal{H}$ is a real Hilbert space. If $X = \mathcal{H}$, then J_2 is an identity mapping in $\mathcal{H}$. We say that the Banach space X is uniformly smooth if its smooth modulus ρ_X defined by

$$\rho_X(\tau) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\| - 1) : \|x\| \leq 1, \|y\| \leq \tau \right\}$$

satisfies $\lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0$.

We note that X is uniformly smooth if and only if J_q is single-valued and is uniformly continuous on any bounded subset of X (see [25]). For $1 < q \leq 2$, we say that X is $q-$uniformly smooth if

$$\rho_X(\tau) \leq d\tau^q$$

for all $\tau > 0$, where d is a positive constant. It is also well known that the smooth modulus for a Hilbert space is $(1+\tau^2)^{\frac{1}{2}} - 1$ and thus Hilbert space is 2-uniformly smooth (see [4]).

Lemma 2.1. ([24]) *Let X be a uniformly smooth real Banach space. Then X is q-uniformly smooth if and only if there is a constant c such that for any x and y in X one has*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q.$$

Now we list some definitions and preliminary results which will be used throughout the paper.

Definition 2.2. ([6]) Let T and H be two single-valued operators from X to X. We say that T is

(i) accretive, if

$$\langle Tx - Ty, J_q(x-y)\rangle \geq 0, \quad \forall x, y \in X;$$

(ii) strictly accretive, if

$$\langle Tx - Ty, J_q(x-y)\rangle \geq 0, \quad \forall x, y \in X$$

holds, and equality holds if and only if $x = y$;

(iii) r-strongly accretive, if there is a positive constant r such that

$$\langle Tx - Ty, J_q(x-y)\rangle \geq r\|x-y\|^q, \quad \forall x, y \in X;$$

(iv) r-strongly accretive relative to H, if there is a positive constant r such that

$$\langle Tx - Ty, J_q(Hx - Hy)\rangle \geq r\|x-y\|^q, \quad \forall x, y \in X;$$

(v) s-Lipschitz, if there is a positive constant s such that

$$\|Tx - Ty\| \leq s\|x-y\|, \quad \forall x, y \in X.$$

Definition 2.3. ([6]) A set-valued mapping $M\colon X \rightrightarrows X$ is said to be

(i) accretive, if

$$\langle u - v, J_q(x-y)\rangle \geq 0, \quad \forall x, y \in X, u \in Mx, v \in My;$$

(ii) m-accretive, if M is accretive, and for all $\lambda > 0$, $(I + \lambda M)X = X$, where I denotes the identity mapping on X.

Remark 2.4. ([29]) If X is a Hilbert space, then from Definitions 2.2 and 2.3, we can get the concepts of monotonicity, strict monotonicity, strong monotonicity, relatively strong monotonicity and maximal monotonicity, respectively.

Definition 2.5. ([6]) Let $H: X \to X$ be an operator and $M: X \to 2^X$ be a set-valued mapping. We say that M is H-accretive, if M is accretive, and for all $\lambda > 0$, $(H + \lambda M)X = X$.

Definition 2.6. ([29]) Let H and $\phi: X \to X$ be two mappings and $M: X \to 2^X$ be a set-valued mapping. We say that M is (H, ϕ)-accretive, if M is accretive, and $(H + \phi \circ M)X = X$.

Remark 2.7. ([29]) It is easy to see that (H, ϕ)-accretive mapping includes H-accretive mapping as a special case. Moreover, if $H = I$, then H-accretivity becomes m-accretivity. If $X = \mathcal{H}$ and $H = I$, then it reduces to maximally monotone operators.

Definition 2.8. ([29]) Let $H: X \to X$ be a single-valued mapping and $M: X \to 2^X$ be an (H, ϕ)-accretive mapping. Then the resolvent operator $R^H_{M,\phi}: X \to X$ with respect to H and M is defined by

$$R^H_{M,\phi}(u) = (H + \phi \circ M)^{-1}(u), \quad \forall u \in X.$$

Lemma 2.9. ([3]) *Let $H: X \to X$ be s-strongly accretive and $M: X \to 2^X$ be (H, ϕ)-accretive. Then the resolvent operator $R^H_{M,\phi}: X \to X$ with respect to H and M is $\frac{1}{s}$-Lipschtiz, i.e.,*

$$\|R^H_{M,\phi}(u) - R^H_{M,\phi}(v)\| \leq \frac{1}{s}\|u - v\|.$$

Lemma 2.10. ([18]) *Let $T : X \to CB(X)$ be a set-valued mapping. Then for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq (1 + \varepsilon)D(T(x), T(y)).$$

Definition 2.11. A set-valued mapping $A : X \to CB(X)$ is said to be k-D-Lipschitz continuous if there exists a constant $k > 0$ such that

$$D(A(x), A(y)) \leq k\|x - y\|, \quad \forall x, y \in X.$$

3. Main Results

Assume that $H\colon X \to X$ and $S, T, A : X \times X \times X \to X$ are single-valued mappings. Let $g, h, f\colon X \to X$ be strongly accretive operators and $F, G, P, M, N, Q\colon X \to 2^X$ be set-valued mappings. We consider the following system of generalized implicit set-valued variational inclusions:

$$\begin{cases} 0 \in F(x) + T(x, y, z) + M(g(x)), \\ 0 \in G(y) + S(x, y, z) + N(h(y)), \\ 0 \in P(z) + A(x, y, z) + Q(f(z)). \end{cases} \tag{3.1}$$

Some special cases of (3.1) are as follows.

(i) If $F = G = P = 0$, then problem (3.1) reduces to the following problem:

$$\begin{cases} 0 \in T(x, y, z) + M(g(x)), \\ 0 \in S(x, y, z) + N(h(y)), \\ 0 \in A(x, y, z) + Q(f(z)), \end{cases} \tag{3.2}$$

which was introduced and studied by Zhou et al. [29].

(ii) If $A = 0$, $Q = 0$ and $S, T : X \times X \to X$ are two single-valued mappings, then problem (3.2) reduces to the following problem:

$$\begin{cases} 0 \in T(x, y) + M(g(x)), \\ 0 \in S(x, y) + N(h(y)). \end{cases} \tag{3.3}$$

We would like to point out that problem (3.3) was studied by Chen [3].

(iii) If $X = \mathcal{H}$ and $g = h = I$, then system (3.3) reduces to the following problem:

$$\begin{cases} 0 \in T(x, y) + M(x), \\ 0 \in S(x, y) + N(y) \end{cases}$$

which is the variational inclusion system investigated in [8].

(iv) If $g = h = I$, $M = N$, and $T(x, y) = B(x) = S(y, x)$, then system (3.3) becomes

$$0 \in B(x) + M(x)$$

which is called the generalized variational inclusion and is studied by Fang and Huang [6, 7] in both Hilbert and Banach spaces.

(v) If $X = \mathcal{H}$, M is maximally monotone, and B is strongly monotone and Lipschitz continuous, then Case (iv) is studied by Huang [9].

(vi) If φ is a proper convex lower semicontinuous functional on a Hilbert space $\mathcal{H}$ and $M = \partial\varphi$, then Case (iv) is the problem of finding $x \in \mathcal{H}$ such that

$$\langle B(x), x - y\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in \mathcal{H}.$$

(vii) If φ is the indicator function of a nonempty closed convex set, then Case (vi) is to find $x \in \mathcal{H}$ such that

$$\langle B(x), x - y\rangle \geq 0, \quad \forall y \in \mathcal{H}.$$

Lemma 3.1. *Let ϕ, φ and ψ be three single-valued mappings from X to X such that*

$$\begin{cases} \phi(-x-u) = -\phi(x) - \phi(u), & \forall x, y \in X,\ \forall u \in F(y), \\ \varphi(-x-v) = -\varphi(x) - \varphi(v), & \forall x, y \in X,\ \forall v \in G(y), \\ \psi(-x-w) = -\psi(x) - \psi(w), & \forall x, y \in X,\ \forall w \in P(y). \end{cases} \tag{3.4}$$

Assume that $H : X \to X$ is strongly accretive. Let $M\colon X \to 2^X$ be an (H,ϕ)-accretive mapping, $N\colon X \to 2^X$ be an (H,φ)-accretive mapping, and $Q\colon X \to 2^X$ be an (H,ψ)-accretive mapping. Then $(x,y,z) \in X \times X \times X$ is a solution of system (3.1) *if and only if $(x,y,z) \in X \times X \times X$ satisfies the following system*

$$\begin{cases} g(x) = R^H_{M,\phi}\left[H(g(x)) - (\phi\circ T)(x,y,z) - \phi(u)\right], \\ h(y) = R^H_{N,\varphi}\left[H(h(y)) - (\varphi\circ S)(x,y,z) - \varphi(v)\right], \\ f(z) = R^H_{Q,\psi}\left[H(f(z)) - (\psi\circ A)(x,y,z) - \psi(w)\right], \end{cases}$$

where $u \in F(x)$, $v \in G(y)$, $w \in P(z)$ and

$$R^H_{M,\phi} = (H + \phi\circ M)^{-1}, \quad R^H_{N,\varphi} = (H + \varphi\circ N)^{-1}, \quad R^H_{Q,\psi} = (H + \psi\circ Q)^{-1}.$$

Proof. It is easy to see that (3.1) can be rewritten as follows:

$$\begin{cases} -T(x,y,z) - u \in M(g(x)), & u \in F(x), \\ -S(x,y,z) - v \in N(h(y)), & v \in G(y), \\ -A(x,y,z) - w \in Q(f(z)), & w \in P(z), \end{cases}$$

which is equivalent to

$$\begin{cases} -(\phi\circ T)(x,y,z) - \phi(u) \in (\phi\circ M)(g(x)), \\ -(\varphi\circ S)(x,y,z) - \varphi(v) \in (\varphi\circ N)(h(y)), \\ -(\psi\circ A)(x,y,z) - \psi(w) \in (\psi\circ Q)(f(z)), \end{cases}$$

that is,

$$\begin{cases} H(g(x)) - (\phi \circ T)(x,y,z) \in H(g(x)) + (\phi \circ M)(g(x)), \\ H(h(y)) - (\varphi \circ S)(x,y,z) \in H(h(y)) + (\varphi \circ N)(h(y)), \\ H(f(z)) - (\psi \circ A)(x,y,z) \in H(f(z)) + (\psi \circ Q)(f(z)). \end{cases}$$

Thus, it is easy to see that the conclusion follows directly from the definition of resolvent operator. □

Now we present the following algorithm for solving system (3.1).

Algorithm 3.2. *Assume that all the conditions of Lemma* 3.1 *hold. Moreover, assume that* $\phi \circ F : X \to CB(X)$, $\varphi \circ G : X \to CB(X)$, *and* $\psi \circ P : X \to CB(X)$. *For given* $x_0, y_0, z_0 \in X$, *choose* $\phi(u_0) \in (\phi \circ F)(x_0)$, $\varphi(v_0) \in (\varphi \circ G)(y_0)$, *and* $\psi(w_0) \in (\psi \circ P)(z_0)$. *Let*

$$\begin{cases} x_1 = x_0 - g(x_0) + R^H_{M,\phi}\left[H(g(x_0)) - (\phi \circ T)(x_0,y_0,z_0) - \phi(u_0)\right], \\ y_1 = y_0 - h(y_0) + R^H_{N,\varphi}\left[H(h(y_0)) - (\varphi \circ S)(x_0,y_0,z_0) - \varphi(v_0)\right], \\ z_1 = z_0 - f(z_0) + R^H_{Q,\psi}\left[H(f(y_0)) - (\psi \circ A)(x_0,y_0,z_0) - \psi(w_0)\right]. \end{cases}$$

Since $(\phi \circ F)(x_i) \in CB(X)$, $(\varphi \circ G)(y_i) \in CB(X)$, *and* $(\psi \circ P)(z_i) \in CB(X)$ *for* $i = 0, 1$, *by Lemma* 2.10, *we know that there exist* $\phi(u_1) \in (\phi \circ F)(x_1)$, $\varphi(v_1) \in (\varphi \circ G)(y_1)$, *and* $\psi(w_1) \in (\psi \circ P)(z_1)$ *such that*

$$\begin{cases} \|\phi(u_1) - \phi(u_0)\| \le \left(1 + \frac{1}{1+1}\right) D((\phi \circ F)(x_1), (\phi \circ F)(x_0)), \\ \|\varphi(v_1) - \varphi(v_0)\| \le \left(1 + \frac{1}{1+1}\right) D((\varphi \circ G)(y_1), (\varphi \circ G)(y_0)), \\ \|\psi(w_1) - \psi(w_0)\| \le \left(1 + \frac{1}{1+1}\right) D((\psi \circ P)(z_1), (\phi \circ P)(z_0)). \end{cases}$$

By induction, we can obtain the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *satisfying* $\phi(u_n) \in (\phi \circ F)(x_n)$, $\varphi(v_n) \in (\varphi \circ G)(y_n)$, $\psi(w_n) \in (\psi \circ P)(z_n)$, *and*

$$\begin{cases} x_{n+1} = x_n - g(x_n) + R^H_{M,\phi}\left[H(g(x_n)) - (\phi \circ T)(x_n,y_n,z_n) - \phi(u_n)\right], \\ \qquad \|\phi(u_{n+1}) - \phi(u_n)\| \le \left(1 + \frac{1}{n}\right) D((\phi \circ F)(x_{n+1}), (\phi \circ F)(x_n)), \\ y_{n+1} = y_n - h(y_n) + R^H_{N,\varphi}\left[H(h(y_n)) - (\varphi \circ S)(x_n,y_n,z_n) - \varphi(v_n)\right], \\ \qquad \|\varphi(v_{n+1}) - \varphi(v_n)\| \le \left(1 + \frac{1}{n}\right) D((\varphi \circ G)(y_{n+1}), (\varphi \circ G)(y_n)), \\ z_{n+1} = z_n - f(z_n) + R^H_{Q,\psi}\left[H(f(y_n)) - (\psi \circ A)(x_n,y_n,z_n) - \psi(w_n)\right], \\ \qquad \|\psi(w_{n+1}) - \psi(w_n)\| \le \left(1 + \frac{1}{n}\right) D((\psi \circ P)(z_{n+1}), (\phi \circ P)(z_n)) \end{cases} \tag{3.5}$$

for $n = 0, 1, 2, \cdots$.

Remark 3.3. If $F = G = P = 0$, then Algorithm 3.2 reduces to Algorithm 3.1 in Zhou et al. [29].

Next we give the existence of solutions of system (3.1) and the convergence of the sequences generated by Algorithm 3.2.

Theorem 3.4. *Assume that X is a q-uniformly smooth Banach space, $g\colon X \to X$ is σ-strongly accretive and ξ-Lipschitz continuous, $h\colon X \to X$ is α-strongly accretive and β-Lipschitz continuous, and $f\colon X \to X$ is κ-strongly accretive and θ-Lipschitz continuous. Let ϕ, φ and ψ be three single-valued mappings from X to X satisfying* (3.4). *Assume that $\phi \circ T\colon X \times X \to X$ is λ-strongly accretive relative to g with respect to its first variable and δ_1-Lipschitz continuous, δ_2-Lipschitz continuous, and δ_3-Lipschitz continuous with respect to its first, second and third variable, respectively. Let $\varphi \circ S\colon X \times X \to X$ be ν-strongly accretive relative to h with respect to its second variable and s_1-Lipschitz continuous, s_2-Lipschitz continuous and s_3-Lipschitz continuous with respect to its first, second and third variable, respectively. Let $\psi \circ A\colon X \times X \to X$ be μ-strongly accretive relative to f with respect to its third variable and η_1-Lipschitz continuous, η_2-Lipschitz continuous and η_3-Lipschitz continuous with respect to its first, second and third variable, respectively. Moreover, let $H\colon X \to X$ be r-strongly accretive and τ-Lipschitz continuous and M, N and $Q\colon X \to 2^X$ be (H,ϕ)-accretive, (H,φ)-accretive, and (H,ψ)-accretive, respectively, and let $\phi \circ F : X \to CB(X)$, $\varphi \circ G : X \to CB(X)$, and $\psi \circ P : X \to CB(X)$ be ξ_F-D-Lipschitz continuous, ξ_G-D-Lipschitz continuous, and ξ_P-D-Lipschitz continuous, respectively. If*

$$\begin{cases} 0 < (1 - q\sigma + c\xi^q)^{\frac{1}{q}} + \frac{\xi}{r}(1 - qr + c\tau^q)^{\frac{1}{q}} + \frac{1}{r}(\xi^q - q\lambda + c\delta_1^q)^{\frac{1}{q}} \\ \qquad + \frac{s_1+\eta_1+\xi_F}{r} < 1, \\ 0 < (1 - q\alpha + c\beta^q)^{\frac{1}{q}} + \frac{\beta}{r}(1 - qr + c\tau^q)^{\frac{1}{q}} + \frac{1}{r}(\beta^q - q\nu + cs_2^q)^{\frac{1}{q}} \\ \qquad + \frac{\delta_2+\eta_2+\xi_G}{r} < 1, \\ 0 < (1 - q\kappa + c\theta^q)^{\frac{1}{q}} + \frac{\theta}{r}(1 - qr + c\tau^q)^{\frac{1}{q}} + \frac{1}{r}(\theta^q - q\mu + c\eta_3^q)^{\frac{1}{q}} \\ \qquad + \frac{\delta_3+s_3+\xi_P}{r} < 1, \end{cases} \tag{3.6}$$

where c is the constant in Lemma 2.1*, then there is (x^*, y^*, z^*) which solves system* (3.1) *and the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm* 3.2 *converge strongly to x^*, y^* and z^*, respectively.*

Proof. By Lemma 2.9, we know that $R^H_{M,\phi}$, $R^H_{N,\varphi}$ and $R^H_{Q,\psi}$ are $\frac{1}{r}$-Lipschitz. Thus, by (3.5), one has

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \big\|x_n - g(x_n) + R^H_{M,\phi}\left[H(g(x_n)) - (\phi \circ T)(x_n, y_n, z_n) - \phi(u_n)\right] - x_{n-1} \\ &\quad + g(x_{n-1}) - R^H_{M,\phi}\left[H(g(x_{n-1})) - (\phi \circ T)(x_{n-1}, y_{n-1}, z_{n-1}) - \phi(u_{n-1})\right]\big\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - g(x_n) - x_{n-1} + g(x_{n-1})\| \\
&\quad + \|R^H_{M,\phi}\left[H(g(x_n)) - \phi \circ T(x_n, y_n, z_n) - \phi(u_n)\right] \\
&\quad - R^H_{M,\phi}\left[H(g(x_{n-1})) - \phi \circ T(x_{n-1}, y_{n-1}, z_{n-1}) - \phi(u_{n-1})\right]\| \\
&\leq \|x_n - g(x_n) - x_{n-1} + g(x_{n-1})\| + \frac{1}{r}\|H(g(x_n)) - \phi \circ T(x_n, y_n, z_n) \\
&\quad - H(g(x_{n-1})) + \phi \circ T(x_{n-1}, y_{n-1}, z_{n-1})\| + \frac{1}{r}\|\phi(u_n) - \phi(u_{n-1})\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq \|x_n - g(x_n) - x_{n-1} + g(x_{n-1})\| \\
&\quad + \frac{1}{r}\|H(g(x_n)) - H(g(x_{n-1})) - g(x_n) + g(x_{n-1})\| \\
&\quad + \frac{1}{r}\|g(x_n) - g(x_{n-1}) - \phi \circ T(x_n, y_n, z_n) + \phi \circ T(x_{n-1}, y_{n-1}, z_{n-1})\| \\
&\quad + \frac{1}{r}\|\phi(u_n) - \phi(u_{n-1})\| \\
&\leq \|x_n - g(x_n) - x_{n-1} + g(x_{n-1})\| \\
&\quad + \frac{1}{r}\|H(g(x_n)) - H(g(x_{n-1})) - g(x_n) + g(x_{n-1})\| \\
&\quad + \frac{1}{r}\|g(x_n) - g(x_{n-1}) - \phi \circ T(x_n, y_n, z_n) + \phi \circ T(x_{n-1}, y_n, z_n)\| \\
&\quad + \frac{1}{r}\|\phi \circ T(x_{n-1}, y_n, z_n) - \phi \circ T(x_{n-1}, y_{n-1}, z_n)\| \\
&\quad + \frac{1}{r}\|\phi \circ T(x_{n-1}, y_{n-1}, z_n) - \phi \circ T(x_{n-1}, y_{n-1}, z_{n-1})\| \\
&\quad + \frac{1}{r}\|\phi(u_n) - \phi(u_{n-1})\|. \qquad (3.7)
\end{aligned}
$$

Since g is σ-strongly accretive and ξ-Lipschitz continuous, it follows from Lemma 2.1 that

$$
\begin{aligned}
&\|x_n - g(x_n) - x_{n-1} + g(x_{n-1})\|^q \\
&\leq \|x_n - x_{n-1}\|^q + c\|g(x_n) - g(x_{n-1})\|^q \\
&\quad - q\langle g(x_n) - g(x_{n-1}), J_q(x_n - x_{n-1})\rangle \\
&\leq (1 - q\sigma + c\xi^q)\|x_n - x_{n-1}\|^q. \qquad (3.8)
\end{aligned}
$$

By the r-strong accretivity and τ-Lipschitz continuity of H, one has

$$\begin{aligned}
&\|H(g(x_n)) - H(g(x_{n-1})) - g(x_n) + g(x_{n-1})\|^q \\
&\le \|g(x_n) - g(x_{n-1})\|^q + c\|H(g(x_n)) - H(g(x_{n-1}))\|^q \\
&\quad - q\langle H(g(x_n)) - H(g(x_{n-1})), J_q(g(x_n) - g(x_{n-1}))\rangle \\
&\le (1 - qr + c\tau^q)\|g(x_n) - g(x_{n-1})\|^q \\
&\le \xi^q(1 - qr + c\tau^q)\|x_n - x_{n-1}\|^q. \qquad (3.9)
\end{aligned}$$

Moreover, by assumptions, we have

$$\begin{aligned}
&\|g(x_n) - g(x_{n-1}) - \phi \circ T(x_n, y_n, z_n) + \phi \circ T(x_{n-1}, y_n, z_n)\|^q \\
&\le \|g(x_n) - g(x_{n-1})\|^q + c\|\phi \circ T(x_n, y_n, z_n) - \phi \circ T(x_{n-1}, y_n, z_n)\|^q \\
&\quad - q\langle \phi \circ T(x_n, y_n, z_n) - \phi \circ T(x_{n-1}, y_n, z_n), J_q(g(x_n) - g(x_{n-1}))\rangle \\
&\le (\xi^q - q\lambda + c\delta_1^q)\|x_n - x_{n-1}\|^q \qquad (3.10)
\end{aligned}$$

and

$$\|\phi \circ T(x_{n-1}, y_n, z_n) - \phi \circ T(x_{n-1}, y_{n-1}, z_n)\| \le \delta_2\|y_n - y_{n-1}\| \qquad (3.11)$$

with

$$\|\phi \circ T(x_{n-1}, y_{n-1}, z_n) - \phi \circ T(x_{n-1}, y_{n-1}, z_{n-1})\| \le \delta_3\|z_n - z_{n-1}\|. \qquad (3.12)$$

It follows from (3.5) and the ξ_F-D-Lipschitz continuity of $\phi \circ F$ that

$$\begin{aligned}
\|\phi(u_n) - \phi(u_{n-1})\| &\le \left(1 + \frac{1}{n-1}\right) D((\phi \circ F)(x_n), (\phi \circ F)(x_{n-1})) \\
&\le \left(1 + \frac{1}{n-1}\right) \xi_F\|x_n - x_{n-1}\|. \qquad (3.13)
\end{aligned}$$

Now combining (3.7)-(3.13), we have

$$\begin{aligned}
&\|x_n - x_n\| \\
&\le \left[(1 - q\sigma + c\xi^q)^{\frac{1}{q}} + \frac{\xi}{r}(1 - qr + c\tau^q)^{\frac{1}{q}} + \frac{1}{r}(\xi^q - q\lambda + c\delta_1^q)^{\frac{1}{q}}\right]\|x_n - x_{n-1}\| \\
&\quad + \left(1 + \frac{1}{n-1}\right)\frac{\xi_F}{r}\|x_n - x_{n-1}\| + \frac{\delta_2}{r}\|y_n - y_{n-1}\| + \frac{\delta_3}{r}\|z_n - z_{n-1}\|. \qquad (3.14)
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\le \left[(1-q\alpha+c\beta^q)^{\frac{1}{q}}+\frac{\beta}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\beta^q-q\nu+cs_2^q)^{\frac{1}{q}}\right]\|y_n-y_{n-1}\| \\
&\quad+\left(1+\frac{1}{n-1}\right)\frac{\xi_G}{r}\|y_n-y_{n-1}\|+\frac{s_1}{r}\|x_n-x_{n-1}\|+\frac{s_3}{r}\|z_n-z_{n-1}\| \quad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&\le \left[(1-q\kappa+c\theta^q)^{\frac{1}{q}}+\frac{\theta}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\theta^q-q\mu+c\eta_3^q)^{\frac{1}{q}}\right]\|z_n-z_{n-1}\| \\
&\quad+\left(1+\frac{1}{n-1}\right)\frac{\xi_P}{r}\|z_n-z_{n-1}\|+\frac{\eta_1}{r}\|x_n-x_{n-1}\|+\frac{\eta_2}{r}\|y_n-y_{n-1}\|. \quad (3.16)
\end{aligned}$$

It follows from (3.14)-(3.16) that

$$\begin{aligned}
&\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\|+\|z_{n+1}-z_n\| \\
&\le \varrho_n(\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|+\|z_n-z_{n-1}\|), \quad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
\varrho_n=\max\Bigg\{&(1-q\sigma+c\xi^q)^{\frac{1}{q}}+\frac{\xi}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\xi^q-q\lambda+c\delta_1^q)^{\frac{1}{q}} \\
&+\frac{s_1+\eta_1+(1+\frac{1}{n-1})\xi_F}{r}, \\
&(1-q\alpha+c\beta^q)^{\frac{1}{q}}+\frac{\beta}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\beta^q-q\nu+cs_2^q)^{\frac{1}{q}} \\
&+\frac{\delta_2+\eta_2+(1+\frac{1}{n-1})\xi_G}{r}, \\
&(1-q\kappa+c\theta^q)^{\frac{1}{q}}+\frac{\theta}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\theta^q-q\mu+c\eta_3^q)^{\frac{1}{q}} \\
&+\frac{\delta_3+s_3+(1+\frac{1}{n-1})\xi_P}{r}\Bigg\}.
\end{aligned}$$

Let

$$\varrho=\max\left\{\bar{A},\bar{B},\bar{C}\right\},$$

where

$$\begin{cases}
\bar{A}=(1-q\sigma+c\xi^q)^{\frac{1}{q}}+\frac{\xi}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\xi^q-q\lambda+c\delta_1^q)^{\frac{1}{q}}+\frac{s_1+\eta_1+\xi_F}{r}, \\
\bar{B}=(1-q\alpha+c\beta^q)^{\frac{1}{q}}+\frac{\beta}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\beta^q-q\nu+cs_2^q)^{\frac{1}{q}}+\frac{\delta_2+\eta_2+\xi_G}{r}, \\
\bar{C}=(1-q\kappa+c\theta^q)^{\frac{1}{q}}+\frac{\theta}{r}(1-qr+c\tau^q)^{\frac{1}{q}}+\frac{1}{r}(\theta^q-q\mu+c\eta_3^q)^{\frac{1}{q}}+\frac{\delta_3+s_3+\xi_P}{r},
\end{cases}$$

Then it is easy to see that $\varrho_n \to \varrho$ and $0 < \varrho < 1$ by assumption condition (3.6). Thus, it follows from (3.17) that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X and so there exist x^*, y^* and z^* in X such that $x_n \to x^*$, $y_n \to y^*$ and $z_n \to z^*$ as $n \to \infty$.

Next, it follows from (3.13) that $\{\phi(u_n)\}$ is also a Cauchy sequence in X and so there exists $\bar{u} \in X$ such that $\phi(u_n) \to \bar{u}$. We now show that $\bar{u} \in (\phi \circ F)(x^*)$. In fact,

$$\begin{aligned} d(\bar{u}, (\phi \circ F)(x^*)) &= \inf_{b \in (\phi \circ F)(x^*)} \|\bar{u} - b\| \\ &\leq \|\phi(u_n) - \bar{u}\| + d(\phi(u_n), (\phi \circ F)(x^*)) \\ &\leq \|\phi(u_n) - \bar{u}\| + D((\phi \circ F)(x_n), (\phi \circ F)(x^*)) \\ &\leq \|\phi(u_n) - \bar{u}\| + \xi_F \|x_n - x^*\|. \end{aligned}$$

This implies that $d(\bar{u}, (\phi \circ F)(x^*)) = 0$ and so $\bar{u} \in (\phi \circ F)(x^*)$. Thus, there exists $u^* \in X$ such that $\bar{u} = \phi(u^*)$ and so $\phi(u_n) \to \phi(u^*)$. Similarly, we can find $v^*, w^* \in X$ such that $\varphi(v_n) \to \varphi(v^*)$ and $\psi(w_n) \to \psi(w^*)$. Therefore, it follows from the continuity of mappings $g, h, f, H, S, T, A, R^H_{M,\phi}$, $R^H_{N,\varphi}$, and $R^H_{Q,\psi}$ that

$$\begin{cases} g(x^*) = R^H_{M,\phi}\left[H(g(x^*)) - \phi \circ T(x^*, y^*, z^*) - \phi(u^*)\right], \\ h(y^*) = R^H_{N,\varphi}\left[H(h(y^*)) - \varphi \circ S(x^*, y^*, z^*) - \varphi(v^*)\right], \\ f(z^*) = R^H_{N,\psi}\left[H(f(z^*)) - \psi \circ A(x^*, y^*, z^*) - \psi(w^*)\right]. \end{cases}$$

Then it follows from Lemma 3.1 that the point (x^*, y^*, z^*) solves system (3.1). This completes the proof. □

Remark 3.5. We note that, if $F = G = P = 0$, then reduces Theorem 3.4 to Theorem 3.1 in Zhou et al. [29]. Therefore, the main results presented in this paper improve and generalize the main results due to Chen [3] and Zhou et al. [29].

Acknowledgments: This work was supported by the National Natural Science Foundation of China (11471230, 11671282, 11771067), the Applied Basic Project of Sichuan Province (2016JY0170), the Open Foundation of State Key Laboratory of Electronic Thin Films and Integrated Devices (KFJJ201611), and the Fundamental Research Funds for the Central Universities (2015NZYQ N70, ZYGX2015J098).

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