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# THE VISCOSITY APPROXIMATION METHOD FOR THE IMPLICIT MIDPOINT RULE OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Dedicated to Professor Jong Kyu Kim on the occasion of his retirement

**Abstract.** The purpose of this paper is to introduce a viscosity approximation method for the implicit midpoint rule of nonexpansive mappings in Banach spaces. The strong convergence of this viscosity method is proved under certain assumptions imposed on the sequence of parameters. Applications to nonlinear variation inclusion problem and nonlinear Volterra integral equations are included. The results presented in the paper extend and improve some recent results announced in the current literature.

## 1. Introduction

The viscosity approximation method for nonexpansive mapping in Hilbert spaces was introduced by Moudafi [11], based on the ideas of Attouch [2]. Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [17].

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Let $X$ be a real Banach space, $T : X \to X$ a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$) and $f : X \to X$ a contraction mapping (i.e., $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in X$ and some $\alpha \in [0, 1)$).

The explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n)Tx_n, n \ge 0, \tag{1.1}$$

where $I$ is the identity of $X$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is well known [8,11,17] that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point of $T$.

The implicit midpoint rule is one of the powerful methods for solving ordinary differential equations, see [3,4,7,14-16] and the references therein. For instance, consider the initial value problem for the differential equation $y'(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where $f$ is a continuous function from $R^d$ to $R^d$. The implicit midpoint rule is that which generates a sequence $\{y_n\}$ via the relation

$$\frac{1}{h}(y_{n+1} - y_n) = f(\frac{y_{n+1} + y_n}{2}). \tag{1.2}$$

The implicit midpoint rule has been extended [1] to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the implicit procedure:

$$x_{n+1} = (1 - t_n)x_n + t_n T(\frac{x_{n+1} + x_n}{2}), n \ge 0, \tag{1.3}$$

Recently, Xu et al [18] in a Hilbert spaces introduced the following process:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_{n+1} + x_n}{2}), n \ge 0, \tag{1.4}$$

where $T$ is a nonexpansive mapping. They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.

Motivated and inspired by the research going on in this direction. The purpose of this paper is to introduce a viscosity approximation method for the implicit midpoint rule of nonexpansive mappings in the framework of Banach spaces. More precisely, we consider the following iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n T(\frac{x_{n+1} + x_n}{2}) + e_n, n \ge 1, \tag{1.5}$$

Under certain assumptions imposed on the sequence of parameters, the strong convergence of this viscosity method is proved. Applications to nonlinear variation inclusion problem and nonlinear Volterra integral equations are included.

## 2. Preliminaries

Throughout the paper, $X$ is a real Banach space with norm $\|\cdot\|$ and dual space $X^*$. Let $T$ be a nonlinear mapping. We denote the fixed point set of $T$ by $\mathrm{Fix}(T)$.

Let $\rho : [0, \infty) \to [0, \infty)$ be the *modulus of smoothness* of $X$ defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : x, y \in X, \|x\| = \|y\| = 1\}. \qquad (2.1)$$

A Banach space $X$ is said to be *uniformly smooth* if $\frac{\rho(t)}{t} \to 0$ as $t \to 0$. Let $q$ be a fixed real number with $q > 1$. Then a Banach space $E$ is said to be *$q$-uniformly smooth* if there exists a constant $b > 0$ such that $\rho(t) \leq bt^q$ for all $t > 0$. It is well known that every $q$-uniformly smooth Banach space is uniformly smooth.

Let $J_q(q > 1)$ denote the *generalized duality mapping* from $X$ into $2^{X^*}$ given by

$$J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\}, \forall x \in X, \qquad (2.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X$ and $X^*$. In particular, $J_2 := J$ is called the normalized duality mapping on $X$. It is also known (e.g., [[19], p.1128]) that

$$j_q(x) = \|x\|^{q-2}J(x), x \neq 0. \qquad (2.3)$$

We next provide some properties for the duality mapping.

**Lemma 2.1.** (Cioranescu [6]) *Let $1 < q < \infty$. Then, we have the followings:*

- (i) *The Banach space $X$ is smooth if and only if the duality mapping $J_q$ is single-valued.*
- (ii) *The Banach space $X$ is uniformly smooth if and only if the duality mapping $J_q$ is single-valued and norm-to-norm uniformly continuous on bounded subsets of $X$.*

Using the concept of subdifferential, we know the following inequality:

**Lemma 2.2.** ([5]) *Let $q > 1$ and $X$ be a real normed space with the generalized duality mapping $J_q$ . Then, for any $x, y \in X$, we have*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle, \qquad (2.4)$$

*for all $j_q(x + y) \in J_q(x + y)$.*

**Lemma 2.3.** ([13]) *Let $C$ be a closed convex subset of a uniformly smooth Banach space $X$ and $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to a fixed point of $T$*

**Lemma 2.4.** ([9]) *Let $\{a_n\}$ and $\{\eta_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \tau_n + \eta_n, n \geq 1, \tag{2.5}$$

*where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\tau_n\}$ is a real sequence. Assume that $\sum\limits_{n=1}^{\infty} \eta_n < \infty$. Then the following results hold:*

(i) *If $\tau_n \leq \gamma_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$ and either $\limsup\limits_{n\to\infty} \frac{\tau_n}{\gamma_n} \leq 0$ or $\sum\limits_{n=1}^{\infty} |\tau_n| < \infty$, then $\lim\limits_{n\to\infty} a_n = 0$.*

**Lemma 2.5.** ([10]) *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q} b^{\frac{q}{q-1}}, \tag{2.6}$$

*for arbitrary positive real numbers $a$ and $b$.*

**Lemma 2.6.** ([12]) *Let $X$ be a real smooth Banach space with the generalized duality mapping $j_q$ for $q > 1$. Let $m \in \mathcal{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geq 0$ for all $i = 1, 2, ..., m$ with $\sum\limits_{i=1}^{m} t_i \leq 1$. Then we have*

$$\Big\| \sum_{i=1}^{m} t_i x_i \Big\|^q \leq \frac{\sum\limits_{i=1}^{m} t_i \|x_i\|^q}{q - (q - 1) \sum\limits_{i=1}^{m} t_i}. \tag{2.7}$$

## 3. Main Results

In this section, we first establish a crucial proposition and then prove our main theorem.

**Proposition 3.1.** *Let $X$ be a $q$-uniformly smooth Banach space, $T : X \to X$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, $f : X \to X$ a contraction with* *coefficient* $\alpha \in [0,1)$ *and* $\{e_n\}$ *a sequence in* $X$. *Let* $\{x_n\}$ *be generated by* $x_1 \in X$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n T(\frac{x_{n+1}+x_n}{2}) + e_n, n \geq 1, \tag{3.1}$$

*where* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in* $[0,1]$ *with* $\alpha_n + \lambda_n + \delta_n = 1$. *If* $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then* $\{x_n\}$ *is bounded.*

*Proof.* Let $\{y_n\}$ be defined by

$$y_{n+1} = \alpha_n f(y_n) + \lambda_n y_n + \delta_n T(\frac{y_{n+1}+y_n}{2}). \tag{3.2}$$

Then, we have

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|\alpha_n (f(x_n) - f(y_n)) + \lambda_n (x_n - y_n) \\ &\quad + \delta_n (T(\frac{x_{n+1}+x_n}{2}) - T(\frac{y_{n+1}+y_n}{2})) + e_n\|, \\ &\leq \alpha_n \|f(x_n) - f(y_n)\| + \lambda_n \|x_n - y_n\| \\ &\quad + \delta_n \|T(\frac{x_{n+1}+x_n}{2}) - T(\frac{y_{n+1}+y_n}{2})\| + \|e_n\| \\ &\leq \alpha_n \alpha \|x_n - y_n\| + \lambda_n \|x_n - y_n\| \\ &\quad + \frac{1}{2}\delta_n (\|x_n - y_n\| + \|x_{n+1} - y_{n+1}\|) + \|e_n\|. \end{aligned}$$

It implies that

$$(1 - \frac{1}{2}\delta_n)\|x_{n+1} - y_{n+1}\| \leq (\alpha_n \alpha + \lambda_n + \frac{1}{2}\delta_n)\|x_n - y_n\| + \|e_n\|.$$

Therefore, we obtain that

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &\leq \frac{2\alpha_n \alpha + 2\lambda_n + \delta_n}{2 - \delta_n}\|x_n - y_n\| + \frac{2}{2-\delta_n}\|e_n\| \\ &= (1 - \frac{2\alpha_n(1-\alpha)}{2-\delta_n})\|x_n - y_n\| + \frac{2}{2-\delta_n}\|e_n\|. \end{aligned} \tag{3.3}$$

By the assumptions and Lemma 2.4 (ii), we conclude that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

We next show that $\{y_n\}$ is bounded. Indeed, for $p \in Fix(T)$, we have

$$\begin{aligned}
\|y_{n+1} - p\| &= \|\alpha_n(f(y_n) - p) + \lambda_n(y_n - p) + \delta_n(T(\frac{y_{n+1} + y_n}{2}) - p)\|, \\
&\leq \alpha_n\|f(y_n) - p\| + \lambda_n\|y_n - p\| + \delta_n\|T(\frac{y_{n+1} + y_n}{2}) - p\| \\
&\leq \alpha_n(\|f(y_n) - f(p)\| + \|f(p) - p\|) + \lambda_n\|y_n - p\| \\
&\quad + \frac{1}{2}\delta_n(\|y_n - p\| + \|y_{n+1} - p\|) \\
&\leq \alpha_n\alpha\|y_n - p\| + \alpha_n\|f(p) - p\| + \lambda_n\|y_n - p\| \\
&\quad + \frac{1}{2}\delta_n(\|y_n - p\| + \|y_{n+1} - p\|).
\end{aligned}$$

It implies that

$$(1 - \frac{1}{2}\delta_n)\|y_{n+1} - p\| \leq (\alpha_n\alpha + \lambda_n + \frac{1}{2}\delta_n)\|y_n - p\| + \alpha_n\|f(p) - p\|.$$

Hence, we have

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \frac{2\alpha_n\alpha + 2\lambda_n + \delta_n}{2 - \delta_n}\|y_n - p\| + \frac{2\alpha_n}{2 - \delta_n}\|f(p) - p\| \\
&= (1 - \frac{2\alpha_n(1 - \alpha)}{2 - \delta_n})\|y_n - p\| + \frac{2\alpha_n}{2 - \delta_n}\|f(p) - p\|.
\end{aligned} \tag{3.4}$$

This shows that $\{y_n\}$ is bounded from Lemma 2.4 (i) and hence $\{x_n\}$ is also bounded. □

**Lemma 3.2.** *Let $X$ be a uniformly convex and $q$-uniformly smooth Banach space, $T : X \to X$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : X \to X$ a contraction with coefficient $\alpha \in [0, 1)$. Let $\{x_n\}$ be generated by $y_1 \in X$ and*

$$y_{n+1} = \alpha_n f(y_n) + \lambda_n y_n + \delta_n T(\frac{y_{n+1} + y_n}{2}), n \geq 1, \tag{3.5}$$

*where $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

*Then we have the following statements:*

(1) $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.

(2) $\lim_{n\to\infty} \|Ty_n - y_n\| = 0.$

(3) $\limsup_{n\to\infty} \langle z - f(z), j_q(z - y_{n+1})\rangle \leq 0$, *for* $z \in Fix(T)$.

*Proof.* (1) To see this, we apply (3.2) to get

$$\begin{aligned}
\|y_{n+1} - y_n\| =& \|\alpha_n f(y_n) + \lambda_n y_n + \delta_n T(\frac{y_{n+1}+y_n}{2}) \\
&- (\alpha_{n-1} f(y_{n-1}) + \lambda_{n-1} y_{n-1} + \delta_{n-1} T(\frac{y_n + y_{n-1}}{2}))\| \\
\leq& \|\delta_n (T(\frac{y_{n+1}+y_n}{2}) - T(\frac{y_n+y_{n-1}}{2})) \\
&+ (\delta_n - \delta_{n-1}) T(\frac{y_n+y_{n-1}}{2}) \\
&+ \alpha_n (f(y_n) - f(y_{n-1})) + (\alpha_n - \alpha_{n-1}) f(y_{n-1}) \\
&+ \lambda_n (y_n - y_{n-1}) + (\lambda_n - \lambda_{n-1}) y_{n-1}\| \\
\leq& \frac{1}{2}\delta_n (\|y_{n+1} - y_n\| + \|y_n - y_{n-1}\|) \\
&+ |\delta_n - \delta_{n-1}| \|T(\frac{y_n+y_{n-1}}{2})\| \\
&+ \alpha_n \alpha \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| \\
&+ \lambda_n \|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|y_{n-1}\| \\
\leq& \frac{1}{2}\delta_n \|y_{n+1} - y_n\| + (\frac{1}{2}\delta_n + \alpha_n\alpha + \lambda_n)\|y_n - y_{n-1}\| \\
&+ |\delta_n - \delta_{n-1}| \|T(\frac{y_n+y_{n-1}}{2})\| + |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| \\
&+ |\lambda_n - \lambda_{n-1}| \|y_{n-1}\|.
\end{aligned}$$

Since $\{y_n\}$ is bounded, so are $\{f(y_n)\}$ and $\{T(\frac{y_n+y_{n+1}}{2})\}$. Let

$$M \geq \sup_{n\geq 1}\{\|y_{n-1}\|, \|f(y_{n-1})\|, \|T(\frac{y_n+y_{n-1}}{2})\|\}.$$

It implies that

$$\begin{aligned}
(1 - \frac{1}{2}\delta_n)\|y_{n+1} - y_n\| \leq& (\frac{1}{2}\delta_n + \alpha_n\alpha + \lambda_n)\|y_n - y_{n-1}\| \\
&+ (|\delta_n - \delta_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}|)M \\
\leq& (\frac{1}{2}\delta_n + \alpha_n\alpha + \lambda_n)\|y_n - y_{n-1}\| \\
&+ 2M(|\delta_n - \delta_{n-1}| + |\alpha_n - \alpha_{n-1}|).
\end{aligned}$$

Hence, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \frac{(\frac{1}{2}\delta_n+\alpha_n\alpha+\lambda_n)}{1-\frac{1}{2}\delta_n}\|y_n-y_{n-1}\| \\ &\quad + \frac{2M}{1-\frac{1}{2}\delta_n}(|\delta_n-\delta_{n-1}|+|\alpha_n-\alpha_{n-1}|) \\ &\le (1-\frac{2\alpha_n(1-\alpha)}{2-\delta_n})\|y_n-y_{n-1}\| \\ &\quad + \frac{4M}{2-\delta_n}(|\delta_n-\delta_{n-1}|+|\alpha_n-\alpha_{n-1}|).\end{aligned} \tag{3.6}$$

By virtue of the conditions (i) and (ii), it follows from Lemma 2.4 that

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.7}$$

(2) Since,

$$\begin{aligned}\|Ty_n-y_n\| &\le \|Ty_n-y_{n+1}\|+\|y_{n+1}-y_n\| \\ &= \|\alpha_n f(y_n)+\lambda_n y_n+\delta_n T(\frac{y_{n+1}+y_n}{2})-Ty_n\|+\|y_{n+1}-y_n\| \\ &\le \alpha_n\|f(y_n)-Ty_n\|+\lambda_n\|y_n-Ty_n\| \\ &\quad +\delta_n\|T(\frac{y_{n+1}+y_n}{2})-Ty_n\|+\|y_{n+1}-y_n\| \\ &\le \alpha_n\|f(y_n)-Ty_n\|+\lambda_n\|y_n-Ty_n\| \\ &\quad +\frac{1}{2}\delta_n\|y_{n+1}-y_n\|+\|y_{n+1}-y_n\|.\end{aligned}$$

It then follows that

$$(1-\lambda_n)\|Ty_n-y_n\| \le \alpha_n\|f(y_n)-Ty_n\|+(1+\frac{1}{2}\delta_n)\|y_{n+1}-y_n\|,$$

and we have

$$\begin{aligned}\|Ty_n-y_n\| &\le \frac{\alpha_n}{1-\lambda_n}\|f(y_n)-Ty_n\|+\frac{1+\frac{1}{2}\delta_n}{1-\lambda_n}\|y_{n+1}-y_n\| \\ &= \frac{\alpha_n}{\alpha_n+\delta_n}\|f(y_n)-Ty_n\|+\frac{1+\frac{1}{2}\delta_n}{\alpha_n+\delta_n}\|y_{n+1}-y_n\|.\end{aligned}$$

By conditions (i), (iii) and (3.7), we obtain that

$$\lim_{n\to\infty}\|Ty_n-y_n\|=0. \tag{3.8}$$

(3) Let $z_t = tf(z_t)+(1-t)Tz_t$, for $t\in(0,1)$. Then it follows from Lemma 2.3 that $z_t\to z\in Fix(T)$ as $t\to 0$.

On the other hand, from Lemma 2.2 we have

$$\begin{aligned}\|z_t - y_n\|^q &= \|t(f(z_t) - y_n) + (1-t)(Tz_t - y_n)\|^q \\ &\le (1-t)^q \|Tz_t - y_n\|^q + qt\langle f(z_t) - y_n, j_q(z_t - y_n)\rangle \\ &\le (1-t)^q (\|Tz_t - Ty_n\| + \|Ty_n - y_n\|)^q \\ &\quad + qt\langle f(z_t) - z_t, j_q(z_t - y_n)\rangle + qt\langle z_t - y_n, j_q(z_t - y_n)\rangle \\ &\le (1-t)^q (\|z_t - y_n\| + \|Ty_n - y_n\|)^q \\ &\quad + qt\langle f(z_t) - z_t, j_q(z_t - y_n)\rangle + qt\|z_t - y_n\|^q.\end{aligned}$$

This shows that

$$\langle z_t - f(z_t), j_q(z_t - y_n)\rangle \le \frac{(1-t)^q}{qt}(\|z_t - y_n\| + \|Ty_n - y_n\|)^q + \frac{qt-1}{qt}\|z_t - y_n\|^q. \tag{3.9}$$

From (3.8), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle z_t - f(z_t), j_q(z_t - y_n)\rangle &\le \frac{(1-t)^q}{qt}M^q + \frac{qt-1}{qt}M^q \\ &= \frac{(1-t)^q + qt - 1}{qt}M^q,\end{aligned} \tag{3.10}$$

where $M = \limsup_{n\to\infty}\|z_t - y_n\|$, for $t \in (0,1)$. Note that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Lemma 2.1 (ii), we know that $j_q$ is norm-to-norm uniformly continuous on bounded subsets of $X$. Since $z_t \to z$ as $t \to 0$, we have

$$\|j_q(z_t - y_n) - j_q(z - y_n)\| \to 0$$

as $t \to 0$. Observe that

$$\begin{aligned}&|\langle z_t - f(z_t), j_q(z_t - y_n)\rangle - \langle z - f(z), j_q(z - y_n)\rangle| \\ &\le |\langle z_t - z + z - f(z) + f(z) - f(z_t), j_q(z_t - y_n)\rangle - \langle z - f(z), j_q(z - y_n)\rangle| \\ &\le |\langle z_t - z, j_q(z_t - y_n)\rangle| + |\langle z - f(z), j_q(z_t - y_n) - j_q(z - y_n)\rangle| \\ &\quad + |\langle f(z) - f(z_t), j_q(z_t - y_n)\rangle| \\ &\le (1+\alpha)\|z_t - z\|\|z_t - y_n\|^{q-1} + \|z - f(z)\|\|j_q(z_t - y_n) - j_q(z - y_n)\|^{q-1}.\end{aligned}$$

So, as $t \to 0$, we get

$$\langle z_t - f(z_t), j_q(z_t - y_n)\rangle \to \langle z - f(z), j_q(z - y_n)\rangle. \tag{3.11}$$

From (3.10), as $t \to 0$, it follows that

$$\limsup_{n\to\infty}\langle z - f(z), j_q(z - y_n)\rangle \le 0. \tag{3.12}$$

Combining (3.7) and (3.12), we get that

$$\limsup_{n\to\infty}\langle z - f(z), j_q(z - y_{n+1})\rangle \le 0. \tag{3.13}$$

this ompletes the proof. □

**Theorem 3.3.** *Let $X$ be a uniformly convex and $q$-uniformly smooth Banach space, $T : X \to X$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, $f : X \to X$ a contraction with coefficient $\alpha \in [0, 1)$ and $\{e_n\}$ a sequence in $X$. Let $\{x_n\}$ be generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n T(\frac{x_{n+1} + x_n}{2}) + e_n, n \geq 1, \tag{3.14}$$

*where $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

*Then the sequence $\{x_n\}$ defined by (3.14) is strongly convergent to a fixed point $z$ of $T$.*

*Proof.* From Lemma 2.2, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
\|y_{n+1} - z\| &= \|\alpha_n(f(y_n) - z) + \lambda_n(y_n - z) \\
&\quad + \delta_n(T(\frac{y_{n+1} + y_n}{2}) - z)\|^q, \\
&\leq \|\lambda_n(y_n - z) + \delta_n(T(\frac{y_{n+1} + y_n}{2}) - z)\|^q \\
&\quad + q\alpha_n\langle (f(y_n) - z), j_q(y_{n+1} - z)\rangle \\
&\leq \frac{1}{1 - (q-1)(1-\alpha_n)}(\lambda_n\|y_n - z\|^q \\
&\quad + \delta_n\|T(\frac{y_{n+1} + y_n}{2}) - z)\|^q \\
&\quad + q\alpha_n\langle f(y_n) - f(z), j_q(y_{n+1} - z)\rangle \\
&\quad + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\lambda_n}{\alpha_n q + 1 - \alpha_n}\|y_n - z\|^q \\
&\quad + \frac{\delta_n}{\alpha_n q + 1 - \alpha_n}\|\frac{y_{n+1} + y_n}{2} - z\|^q \\
&\quad + q\alpha_n\alpha\|y_n - z\|\|y_{n+1} - z\|^{q-1} \\
&\quad + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle \\
&\leq \frac{\lambda_n}{\alpha_n q + 1 - \alpha_n}\|y_n - z\|^q \\
&\quad + \frac{\delta_n}{\alpha_n q + 1 - \alpha_n}(\frac{1}{2}\|y_n - z\|^q + \frac{1}{2}\|y_{n+1} - z\|^q) \\
&\quad + q\alpha_n\alpha(\frac{1}{q}\|y_n - z\|^q + \frac{q-1}{q}\|y_{n+1} - z\|^q) \\
&\quad + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle \\
&\leq \frac{\lambda_n + \frac{1}{2}\delta_n + \alpha_n\alpha(\alpha_n q + 1 - \alpha_n)}{\alpha_n q + 1 - \alpha_n}\|y_n - z\|^q \\
&\quad + \frac{\frac{1}{2}\delta_n + (q-1)\alpha_n\alpha(\alpha_n q + 1 - \alpha_n)}{\alpha_n q + 1 - \alpha_n}\|y_{n+1} - z\|^q \\
&\quad + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\frac{(1 - q\alpha_n\alpha + \alpha_n\alpha)(\alpha_n q + 1 - \alpha_n) - \frac{1}{2}\delta_n}{\alpha_n q + 1 - \alpha_n}\|y_{n+1} - z\|^q \\
&\leq \frac{\lambda_n + \frac{1}{2}\delta_n + \alpha_n\alpha(\alpha_n q + 1 - \alpha_n)}{\alpha_n q + 1 - \alpha_n}\|y_n - z\|^q \\
&\quad + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|y_{n+1} - z\|^q \\
&\leq \frac{\lambda_n + \frac{1}{2}\delta_n + \alpha_n\alpha(\alpha_n q + 1 - \alpha_n)}{(1 - q\alpha_n\alpha + \alpha_n\alpha)(\alpha_n q + 1 - \alpha_n) - \frac{1}{2}\delta_n}\|y_n - z\|^q \\
&\quad + \frac{q\alpha_n(\alpha_n q + 1 - \alpha_n)}{(1 - q\alpha_n\alpha + \alpha_n\alpha)(\alpha_n q + 1 - \alpha_n) - \frac{1}{2}\delta_n}\langle f(z) - z, j_q(y_{n+1} - z)\rangle \\
&\leq (1 - \frac{\alpha_n q(1 - \alpha - \alpha\alpha_n(q-1))}{(1 - q\alpha_n\alpha + \alpha_n\alpha)(\alpha_n q + 1 - \alpha_n) - \frac{1}{2}\delta_n})\|y_n - z\|^q \\
&\quad + \frac{q\alpha_n(\alpha_n q + 1 - \alpha_n)}{(1 - q\alpha_n\alpha + \alpha_n\alpha)(\alpha_n q + 1 - \alpha_n) - \frac{1}{2}\delta_n}\langle f(z) - z, j_q(y_{n+1} - z)\rangle.
\end{aligned}
\tag{3.15}
$$

Now, let

$$\gamma_n = \frac{\alpha_n q(1-\alpha-\alpha\alpha_n(q-1))}{(1-q\alpha_n\alpha+\alpha_n\alpha)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}$$

and

$$\tau_n = \frac{q\alpha_n(\alpha_n q+1-\alpha_n)}{(1-q\alpha_n\alpha+\alpha_n\alpha)(\alpha_n q+1-\alpha_n)-\frac{1}{2}\delta_n}\langle f(z)-z, j_q(y_{n+1}-z)\rangle.$$

Then it follows from conditions (i) and (3.13) that $\gamma_n \subset (0,1)$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and

$$\limsup_{n\to\infty}\frac{\tau_n}{\gamma_n} = \limsup_{n\to\infty}\frac{\alpha_n q+1-\alpha_n}{1-\alpha-\alpha\alpha_n(q-1)}\langle f(z)-z, j_q(y_{n+1}-z)\rangle \le 0.$$

From Lemma 2.4, we have $\lim_{n\to\infty} y_n = z \in Fix(T)$, by Proposition 3.1, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, so $\lim_{n\to\infty} x_n = z \in Fix(T)$. This completes the proof. □

For the case $\lambda_n = 0$ for all $n \ge 1$, then we obtain the following result:

**Corollary 3.4.** *Let $X$ be a uniformly convex and $q$-uniformly smooth Banach space, $T : X \to X$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, $f : X \to X$ a contraction with coefficient $\alpha \in [0,1)$ and $\{e_n\}$ a sequence in $X$. Let $\{x_n\}$ be generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_{n+1}+x_n}{2}) + e_n, n \ge 1, \tag{3.16}$$

*where $\{\alpha_n\}$ is a sequences in $[0,1]$. Assume that*

(i) $\sum_{n=1}^{\infty}\alpha_n = \infty, \lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 0$;

(iii) $\sum_{n=1}^{\infty}\|e_n\| < \infty$ *or* $\lim_{n\to\infty}\|e_n\|/\alpha_n = 0$.

*Then $\{x_n\}$ strongly converges to some $z \in Fix(T)$.*

**Remark 3.5.** Corollary 3.4 is the Banach space version of the Xu's result [18] with error term.

## 4. Applications

### 4.1 Application to nonlinear variational inclusion problem

Let $X$ be a real Banach space and $M : X \to 2^X$ an m-accretive operator. Then, the resolvent mapping $J_r^M : X \to X$ associated with M is defined by

$$J_r^M(x) = (I + rM)^{-1}(x), r > 0, \tag{4.1}$$

where $I$ is the identity operator on $X$. It is known that the $m$-accretiveness of $M$ implies that $J_r^M$ is a nonexpansive mapping.

The so-called monotone variational inclusion problem (in short, MVIP) is to find $x^* \in X$ such that

$$0 \in M(x^*). \tag{4.2}$$

From the definition of mapping $J_r^M$, it is easy to see that (MVIP) (4.2) is equivalent to find $x^* \in X$ such that

$$x^* \in Fix(J_r^M) \text{ for some } r > 0. \tag{4.3}$$

For any given starting point $x_1 \in X$, we define a sequence by

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_r^M(\frac{x_{n+1} + x_n}{2}) + e_n, n \geq 1, \tag{4.4}$$

where $f : X \to X$ is a mapping.

From Theorem 3.3, we have the following:

**Theorem 4.1.** *Let $X$ be a uniformly convex and $q$-uniformly smooth Banach space and $J_r^M : X \to X$ be the resolvent mapping associated with an m-accretive operator $M$ such that $Fix(J_r^M) \neq \emptyset$. Let $f : X \to X$ be a contraction with coefficient $\alpha \in [0, 1)$ and $\{e_n\}$ be a sequence in $X$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

*Then the sequence $\{x_n\}$ defined by* (4.4) *is strongly convergent to the solution of monotone variational inclusion problem* (4.2).

### 4.2 Application to nonlinear Volterra integral equations

Let us consider the following nonlinear Volterra integral equation:

$$x(t) = g(t) + \int_0^t F(t, s, x(s))\, \mathrm{d}s, t \in [0, 1], \tag{4.5}$$

where $g$ is a continuous function on [0, 1] and $F : [0, 1] \times [0, 1] \times R \to R$ is continuous and satises the following condition.

$$|F(t, s, x) - F(t, s, y)| \leq |x - y|, \quad t, s \in [0, 1],\ x, y \in R.$$

Define a mapping $T : L^2[0, 1] \to L^2[0, 1]$ by

$$T(x(t)) = g(t) + \int_0^t F(t, s, x(s))\, \mathrm{d}s, t \in [0, 1]. \tag{4.6}$$

It is easy to see that $T$ is a nonexpansive mapping. This means that to find the solution of integral equation (4.6) is reduced to find a fixed point of the nonexpansive mapping $T$ in $L^2[0, 1]$.

For any given function $x_1 \in L^2[0, 1]$, define a sequence of functions $\{x_n\}$ in $L^2[0, 1]$ by

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n T(\frac{x_{n+1} + x_n}{2}) + e_n, n \geq 1, \tag{4.7}$$

where $f$ is a mapping on $L^2[0, 1]$.

From Theorem 3.2 we have the following.

**Theorem 4.2.** *Let $F, g, T$ be the same mappings as above. Let $f$ be a contraction on $L^2[0, 1]$ with coefficient $\alpha \in [0, 1)$ and $\{e_n\}$ be a sequence in $L^2[0, 1]$. Let $Fix(T) \neq \emptyset$ and $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0;$

(ii) $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty;$

(iii) $\liminf_{n\to\infty} \delta_n > 0;$

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0.$

*Then the sequence $\{x_n\}$ defined by (4.7) is strongly convergent in $L^2[0, 1]$ to the solution of integral equation (4.5).*

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