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A FURTHER JOURNEY IN THE "TERRA INCOGNITA" OF THE NEWTON-KANTOROVICH METHOD

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Abstract. In this paper, we prove an a-posteriori convergence theorem for Newton-Kantorovich approximations where the Fréchet derivative of the involved operator satisfies the Hölder continuity and center-Hölder continuity conditions.

1. Introduction

Let X and Y be Banach spaces, we denote with $B(x_0, R)$ the closed ball in X centered at x_0 and of radius R. We assume that the operator $f : B(x_0, R) \to Y$ is Fréchet differentiable at interior points of $B(x_0, R)$ with $f'(x_0)$ invertible. In previous papers [3], [4] and [5] we studied the convergence of Newton-Kantorovich approximations

$$x_n = x_{n-1} - f'(x_{n-1})^{-1} f(x_{n-1}), \quad (n \in \mathbb{N}) \tag{1.1}$$

under the Hölder continuity condition for f', i.e. we suppose that the constant

$$k := \sup_{x,y \in \overset{\circ}{B}(x_0,R), x \neq y} \frac{\|f'(x) - f'(y)\|}{\|x - y\|^\theta}, \quad (0 < \theta \leq 1), \tag{1.2}$$

is finite.

The first who considered the approximations (1.1) under the hypothesis (1.2) has been Vertgeim in [11] and [12]. He defined

$$b := ||f'(x_0)^{-1}||, \quad a := ||f'(x_0)^{-1} f(x_0)||, \quad \xi := a^\theta b k,$$

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and he showed that, if $\xi \leq \left(\frac{\theta}{1+\theta}\right)^{\theta}$, then the modified Newton-Kantorovich method defined by

$$x_n = x_{n-1} - f'(x_0)^{-1} f(x_{n-1}) \,, \quad (n \in \mathbb{N}) \tag{1.3}$$

converges to a solution of the equation $f(x) = 0$.

In [12], Vertgeim gived also some sufficient conditions for the convergence of the Newton-Kantorovich method, but there is an unknown area, called *terra incognita*, between the convergence regions of two methods. In [10], [6], [3], [7], [8], [4] and [5], there are further steps to recovering this area, but a gap between the two regions remains.

Galperin's approach to the problem in [7] and [8] is very original since he makes use of the theory of two-dimensional generators. Moreover, fixed $0 < \theta \leq 1$, the value of the Galperin's convergence curve is larger than the convergence's curve defined in [5], but we think that our results can be interesting since the values of Galperin's convergence curves are not directly computable.

In this paper, f' satisfies the Hölder continuity condition and therefore also the weaker *center-Hölder continuity condition*. We recall that an operator $g : B(x_0, R) \to Y$ satisfies the center-Hölder continuity condition if the constant

$$k_0 := \sup_{x \in \overset{\circ}{B}(x_0,R),\, x \neq x_0} \frac{\|g(x) - g(x_0)\|}{\|x - x_0\|^{\theta}} \,, \quad (0 < \theta \leq 1), \tag{1.4}$$

is finite.

Note that k_0 is, at least in some cases, strictly less than k (for examples when $\theta = 1$ and $\frac{k}{k_0}$ is arbitrarily large see [2]). The case $k_0 < k < +\infty$ here considered has been studied by Argyros in a significant paper [1]. He established an a-posteriori convergence theorem which, in some cases, improves the result in [3]. Moreover, he proved that, if $\xi_0 := a^{\theta} b k_0 \leq \left(\frac{\theta}{1+\theta}\right)^{\theta}$, then the approximations (1.3) converge.

Using techniques similar to the ones used in [4] and [5], we obtain an improvement of the results in [5] (Theorem 2.3) and in [1] (Theorem 3.1) which will be given in the following sections.

2. New estimates for a majorizing sequence

If $\xi_0 = a^\theta\, bk_0$ is strictly less than 1, we can define the sequence

$$r_0 = 0, \quad r_1 = a, \quad r_{n+1} = r_n + \frac{bk(r_n - r_{n-1})^{1+\theta}}{(1+\theta)(1-bk_0 r_n^\theta)}\,, \quad (n \in \mathbb{N})\,. \tag{2.1}$$

It is easy to show that the sequence $(r_n)_{n\in\mathbb{N}}$ is a majorizing sequence of (1.1), i.e.,

$$||x_n - x_{n-1}|| \le r_n - r_{n-1}\,, \quad (n \in \mathbb{N})\,. \tag{2.2}$$

In the sequel, we suppose $0 < \theta < 1$ since the case $\theta = 1$ is well known. In fact, in this case, the sequence $(r_n)_{n\in\mathbb{N}}$ converges, if and only if, $ab(k+k_0) \le \frac{1}{2}$.

In the papers [4] and [5], we give some conditions involving the parameters a, b and k which assure the convergence of (1.1). In this paper the hypotheses for the convergence involve not only the parameters a, b, k but also k_0 and we improve previous results from [4] and [5].

In the last section we give a comparison with Theorem 5 of [1] and we show that our result is an improvement of Theorem 5. As in above papers [4] and [5], we use two parallel induction processes on the sequences $(r_n)_{n\in\mathbb{N}}$ and $\left(\frac{r_{n+1}}{r_n}\right)_{n\in\mathbb{N}}$. Similarly as in [5], in order to prove our main result, we need prove that the inequality

$$\frac{\left(1-\frac{1}{t}\right)^\theta}{\left[\left(k_0(1+\theta)\right)^{\frac{1}{1-\theta}} + \left(kt(t-1)^\theta\right)^{\frac{1}{1-\theta}}\right]^{1-\theta}} \le \frac{1}{k_0(1+\theta)+kt} \tag{2.3}$$

holds for every $t \ge 1$ and for every $0 < \theta < 1$.
If $t = 1$ the above inequality is obviously satisfied. For $t > 1$, we have yet observed in [5] that, rewriting the inequality (2.3) in the following way

$$\frac{\left(\frac{t-1}{t}\right)^{\frac{\theta}{1-\theta}}\left(1+\frac{1}{(t-1)^\theta}\,\frac{kt(t-1)^\theta}{k_0(1+\theta)}\right)^{\frac{1}{1-\theta}}}{1+\left(\frac{kt(t-1)^\theta}{k_0(1+\theta)}\right)^{\frac{1}{1-\theta}}} \le 1\,,$$

we can apply this elementary inequality

$$\frac{(1+\delta\, x)^\alpha}{1+x^\alpha} \le (1+\delta^{\frac{\alpha}{\alpha-1}})^{\alpha-1}\,, \quad x \ge 0,\ \alpha,\ \delta > 0$$

with

$$\delta := \frac{1}{(t-1)^\theta}\,, \quad x := \frac{kt(t-1)^\theta}{k_0(1+\theta)}\,, \quad \alpha := \frac{1}{1-\theta}\,.$$

Finally we have

$$\frac{\left(\frac{t-1}{t}\right)^{\frac{\theta}{1-\theta}}\left[1+\frac{1}{(t-1)^\theta}\,\frac{kt(t-1)^\theta}{k_0(1+\theta)}\right]^{\frac{1}{1-\theta}}}{1+\left(\frac{kt(t-1)^\theta}{k_0(1+\theta)}\right)^{\frac{1}{1-\theta}}}\le\left(\frac{t-1}{t}\right)^{\frac{\theta}{1-\theta}}\left(1+\frac{1}{t-1}\right)^{\frac{\theta}{1-\theta}}=1\,.$$

Let $h_0 : [1,+\infty[\to\mathbb{R}$ be defined by

$$h_0(t) := \left(1-\frac{1}{t}\right)^\theta \frac{1+\theta}{\left[\left(k_0(1+\theta)\right)^{\frac{1}{1-\theta}}+\left(kt(t-1)^\theta\right)^{\frac{1}{1-\theta}}\right]^{1-\theta}}\,. \tag{2.4}$$

Since $h_0(1) = \lim_{t\to+\infty} h_0(t) = 0$ and $h(t) > 1$ for $t > 1$, the function h_0 has a global maximum on $]1,+\infty[$ which we denote by c_0. As in [5], we can explicitly calculate c_0. In fact, $h_0'(t) = 0$ if and only if

$$\theta(\left(k_0(1+\theta)\right)^{\frac{1}{1-\theta}}+(kt(t-1)^\theta)^{\frac{1}{1-\theta}}) = ((1+\theta)t-1)(kt(t-1)^\theta)^{\frac{1}{1-\theta}}$$

from which it follows that

$$(t(t-1)^\theta)^{\frac{1}{1-\theta}} = \frac{\theta\, k_0^{\frac{1}{1-\theta}}\,(1+\theta)^{\frac{\theta}{1-\theta}}}{k^{\frac{1}{1-\theta}}\,(t-1)}\,.$$

So resolving for t we obtain

$$c_0 = \frac{k+\sqrt{k^2+4k_0\,k\,(1+\theta)^\theta\,\theta^{1-\theta}}}{2k}\,. \tag{2.5}$$

Now we can prove the following.

Theorem 2.1. *Let h_0 be defined by (2.4) and c_0 by (2.5). Setting*

$$r(c_0) = \frac{(1+\theta)^{\frac{1}{\theta}}}{b^{\frac{1}{\theta}}\left(\left(k_0(1+\theta)\right)^{\frac{1}{1-\theta}}+(kc_0(c_0-1)^\theta)^{\frac{1}{1-\theta}}\right)^{\frac{1-\theta}{\theta}}}\;, \tag{2.6}$$

suppose that

$$a^\theta b \le h_0(c_0)\,. \tag{2.7}$$

Then the estimates

$$r_n \le r(c_0)\left(1-\frac{1}{c_0^n}\right),\quad \frac{r_{n+1}}{r_n}\le\frac{1-\frac{1}{c_0^{n+1}}}{1-\frac{1}{c_0^n}} \tag{2.8}$$

hold for all $n\in\mathbb{N}$. Consequently, the sequence $(r_n)_{n\in\mathbb{N}}$, being increasing and bounded, is converging to some $r_*\le r(c_0)<1$.

Proof. We prove the estimate (2.8) by induction on n. We proceed with two parallel induction processes on the two inequalities in (2.8).
If $n = 1$, we have from (2.7)

$$\begin{aligned} r_1 = a &\le \left(1 - \frac{1}{c_0}\right) \frac{(1+\theta)^{\frac{1}{\theta}}}{b^{\frac{1}{\theta}} \left(\left(k_0(1+\theta)\right)^{\frac{1}{1-\theta}} + (kc_0(c_0-1)^{\theta})^{\frac{1}{1-\theta}} \right)^{\frac{1-\theta}{\theta}}} \\ &= r(c_0) \left(1 - \frac{1}{c_0}\right) \end{aligned}$$

and

$$\frac{r_2}{r_1} = 1 + \frac{bka^{\theta}}{(1+\theta)(1 - bk_0a^{\theta})} .$$

From (2.7) and (2.3), it follows that

$$\begin{aligned} \frac{r_2}{r_1} &\le 1 + \frac{bkh_0(c_0)}{(1+\theta)(1 - bk_0h_0(c_0))} \\ &\le 1 + \frac{k}{(k_0(1+\theta) + kc_0)\left(1 - \frac{k_0(1+\theta)}{k_0(1+\theta)+kc_0}\right)} \\ &= 1 + \frac{1}{c_0} = \frac{1 - \frac{1}{c_0^2}}{1 - \frac{1}{c_0}} , \end{aligned}$$

where the first equality follows from an easy computation.
Suppose now

$$r_{n-1} \le r(c_0)\left(1 - \frac{1}{c_0^{n-1}}\right), \quad \frac{r_n}{r_{n-1}} \le \frac{1 - \frac{1}{c_0^n}}{1 - \frac{1}{c_0^{n-1}}} .$$

We have

$$r_n \le \frac{1 - \frac{1}{c_0^n}}{1 - \frac{1}{c_0^{n-1}}} r_{n-1} \le r(c_0)\left(1 - \frac{1}{c_0^n}\right)$$

and

$$1 - \frac{r_{n-1}}{r_n} \le \frac{\frac{1}{c_0^{n-1}} - \frac{1}{c_0^n}}{1 - \frac{1}{c_0^n}} .$$

Then

$$\begin{aligned}
\frac{r_{n+1}}{r_n} &= 1+\frac{bkr_n^\theta\left(1-\frac{r_{n-1}}{r_n}\right)^{1+\theta}}{(1+\theta)(1-bk_0r_n^\theta)} \\
&\le 1+\frac{bkr^\theta(c_0)\left(1-\frac{1}{c_0^n}\right)^\theta\left(\frac{1}{c_0^{n-1}}-\frac{1}{c_0^n}\right)^{1+\theta}}{(1+\theta)\left(1-\frac{1}{c_0^n}\right)^{1+\theta}\left(1-bk_0r^\theta(c_0)\left(1-\frac{1}{c_0^n}\right)^\theta\right)} \\
&= 1+\frac{bkr^\theta(c_0)(c_0-1)^{1+\theta}}{(1+\theta)(c_0^n-1)\big(c_0^{\theta n}-bk_0r^\theta(c_0)\,(c_0^n-1)^\theta\big)}\,.
\end{aligned}$$

If we consider the real function $w_0(x) := c_0^{\theta x} - bk_0 r^\theta(c_0)(c_0^x-1)^\theta$, $x \in [1,+\infty[$, it is easily seen that w_0 has an unique global minimum point x_0^*. In fact

$$w_0'(x) = \theta \log c_0\, c_0^{\theta x}\left(1 - bk_0 r^\theta(c_0)\left(\frac{c_0^x}{c_0^x-1}\right)^{1-\theta}\right) = 0$$

is equivalent to $c_0^x = (1-(bk_0r^\theta(c_0))^{\frac{1}{1-\theta}})^{-1}$ which has a unique root x_0^*. Moreover, from $w_0'(x_0^*)=0$ it follows

$$bk_0r^\theta(c_0) = \left(\frac{c_0^{x_0^*}-1}{c^{x_0^*}}\right)^{1-\theta}$$

and therefore

$$w_0(x) \ge w_0(x_0^*) = c_0^{\theta x_0^*} - (c_0^{x_0^*}-1)\,c_0^{(\theta-1)x_0^*} = c_0^{(\theta-1)x_0^*} = \big(1-(bk_0r^\theta(c_0))^{\frac{1}{1-\theta}}\big)^{1-\theta}.$$

Consequently

$$\frac{r_{n+1}}{r_n} \le 1 + \frac{bkr^\theta(c_0)(c_0-1)^{1+\theta}}{(1+\theta)(c_0^n-1)\big(1-(bk_0r^\theta(c_0))^{\frac{1}{1-\theta}}\big)^{1-\theta}}\,.$$

and from $r^{\theta}(c_0) := \dfrac{1+\theta}{b\Big(\big(k_0(1+\theta)\big)^{\frac{1}{1-\theta}} + (kc_0(c_0-1)^{\theta})^{\frac{1}{1-\theta}}\Big)^{1-\theta}}$ we have

$$\frac{r_{n+1}}{r_n} \le 1+$$

$$\frac{k(c_0-1)^{1+\theta}}{(c_0^n-1)\Big(\big(k_0(1+\theta)\big)^{\frac{1}{1-\theta}}+(kc_0(c_0-1)^{\theta})^{\frac{1}{1-\theta}}\Big)^{1-\theta}\left(1-\dfrac{\big(k_0(1+\theta)\big)^{\frac{1}{1-\theta}}}{\big(k_0(1+\theta)\big)^{\frac{1}{1-\theta}}+(kc_0(c_0-1)^{\theta})^{\frac{1}{1-\theta}}}\right)^{1-\theta}}$$

$$= 1 + \frac{c_0-1}{c_0(c_0^n-1)} = \frac{1-\frac{1}{c_0^{n+1}}}{1-\frac{1}{c_0^n}}\,.$$

□

As direct consequence of Theorem 2.1 we have the following theorem on the convergence of the sequence (1.1).

Theorem 2.2. *Suppose that (2.7) holds, with c_0 given by (2.5), and that $r_* := (bk)^{-\frac{1}{\theta}} t_* \le R$. Then the Newton-Kantorovich approximations* (1.1) *are well defined for all n, belong to $B(x_0, r_*)$ and converge to the unique solution x_* of the equation $f(x) = 0$. Moreover, the following estimates hold*

$$||x_n - x_{n-1}|| \le r_n - r_{n-1}\,, \quad (n \in \mathbb{N})\,,$$

$$||x_n - x_*|| \le r_* - r_n\,, \quad (n \in \mathbb{N})\,.$$

Theorem 2.2 is a real improvement of Theorem 2.3 of [5] in case $k_0 < k$. In the sequel we give an elementary example in which $k_0 < k$.

Example 2.3. We consider the function $f(x) = x^3$ in the interval $[-2, 2]$ and we choose $x_0 = 1$ as initial point. Fixed $0 \le \theta \le 1$, $\theta \ne \frac{2}{3}$, it is a simple computation to verify that

$$k_0 = \sup_{-2\le x\le 2} \frac{|f'(x) - f'(1)|}{|x-1|^{\theta}} = 3 \sup_{-2\le x\le 2} |x+1|\,|x-1|^{1-\theta} = 9$$

and

$$\begin{aligned} k &= \sup_{-2\le x,y\le 2\,,x\ne y} \frac{|f'(x)-f'(y)|}{|x-y|^{\theta}} \\ &= 3 \sup_{-2\le x,y\le 2} |x+y|\,|x-y|^{1-\theta} = 3\left(\frac{4}{2-\theta}\right)^{2-\theta} (1-\theta)^{1-\theta}\,. \end{aligned}$$

In order to prove that $k_0 < k$, we need show that

$$\left(\frac{4}{2-\theta}\right)^{2-\theta} (1-\theta)^{1-\theta} > 3 \,.$$

Setting

$$s(\theta) := \left(\frac{4}{2-\theta}\right)^{2-\theta} (1-\theta)^{1-\theta} \,, \quad 0 \le \theta < 1 \,,$$

we have $s(0) = \lim_{\theta \to 1} s(\theta) = 4$; moreover

$$s'(\theta) = \left(\frac{4}{2-\theta}\right)^{2-\theta} (1-\theta)^{1-\theta} \log\left(\frac{2-\theta}{4(1-\theta)}\right) \,,$$

and $s'(\theta) \ge 0$ if, and only if, $\theta \ge \frac{2}{3}$.

So s attains its minimum value only in $\frac{2}{3}$ and

$$s(\theta) > s\left(\frac{2}{3}\right) = 3 \text{ for all } 0 \le \theta \le 1 \,, \ \theta \ne \frac{2}{3}.$$

3. A Comparison with previous results

In this section, we prove that Theorem 2.2 improves Theorem 5 of [1]. We begin by recalling Lemma 1 of [1] of which we make use in the sequel.

Lemma 3.1. *([1]) Assume there exist parameters $k, k_0, a \ge 0$, $\theta \in]0,1[$ and $q \in [0,1[$ such that*

$$\left(k + \frac{(1+\theta)k_0}{(1-q)^\theta}\right) a^\theta b \le (1+\theta) q \,. \tag{3.1}$$

Then the sequence (2.1) is increasing and converges to some t_* such that $0 \le t_* \le \frac{a}{1-q}$.

Putting $t = \frac{1}{q}$ $(t > 1)$, condition (3.1) becomes:

$$\left(k + \frac{k_0\,(1+\theta)}{(1-\frac{1}{t})^\theta}\right) a^\theta b \le (1+\theta) \frac{1}{t}$$

i.e.,

$$a^\theta b \le \left(1 - \frac{1}{t}\right)^\theta \frac{1+\theta}{\left[k_0(1+\theta)t + k t^{1-\theta}\,(t-1)^\theta\right]} \,.$$

Define $g : [1, +\infty[\to \mathbb{R}$ by

$$g(t) := \left(1 - \frac{1}{t}\right)^{\theta} \frac{1+\theta}{\left[k_0(1+\theta)t + kt^{1-\theta}\,(t-1)^{\theta}\right]},$$

to show that our result is an improvement of Theorem 5 [1], we will prove in Theorem 3.3 below that $g(t) \le h_0(c_0)$ for all $t \ge 1$.

In order to simplify the proof of this theorem, we prove before the following.

Lemma 3.2. *Set*

$$\alpha(t) := \left(1 - \frac{1}{t}\right)^{\frac{1}{1-\theta}} + [t^{\theta}(1+\theta-t]^{\frac{1}{1-\theta}},$$

we have $\max\limits_{1\le t\le 1+\theta} \alpha(t) = \alpha(1) = \theta^{\frac{1}{1-\theta}}$ *and, in particular,* $\alpha(t) < \theta^{\frac{1}{1-\theta}}$ *for all* $1 < t \le 1+\theta$.

Proof. Since $\alpha(1) = \theta^{\frac{1}{1-\theta}}$ and $\alpha(1+\theta) = \left(\dfrac{\theta}{1+\theta}\right)^{\frac{1}{1-\theta}}$, it is sufficient to prove that α has only a critical point which is a minimum point. We have

$$\alpha'(t) = \frac{1}{(1-\theta)\,t^{\frac{1-2\theta}{(1-\theta)}}} \left[\frac{(t-1)^{\frac{\theta}{1-\theta}}}{t^{\frac{1+\theta}{1-\theta}}} - (1+\theta)(t-\theta)(1+\theta-t)^{\frac{\theta}{1-\theta}}\right] \tag{3.2}$$

and therefore $\alpha'(t) = 0$ if and only if

$$\frac{(t-1)^{\theta}}{t^{1+\theta}\,(t-\theta)^{1-\theta}\,(1+\theta-t)^{\theta}} = (1+\theta)^{1-\theta}. \tag{3.3}$$

We prove that (3.3) has a unique solution in $t \in]1, 1+\theta[$. Set

$$q(t) := \frac{(t-1)^{\theta}}{t^{1+\theta}\,(t-\theta)^{1-\theta}\,(1+\theta-t)^{\theta}},$$

from $q([1, 1+\theta[) = [0, +\infty[$ it follows that (3.3) admits at least a solution. In order to prove that this solution is unique, it is sufficient to show that $q'(t) > 0$ for every $t \in [1, 1+\theta[$. Since

$$q'(t) = t^{\theta}\,(t-1)^{\theta-1}\,(t-\theta)^{-\theta}\,(1+\theta-t)^{\theta-1}\,[2t^3 - (4+3\theta)\,t^2 + (3\theta^2+4\theta+2)\,t - \theta(1+\theta)^2],$$

we have $q'(t) > 0$ if and only if

$$u(t) := 2t^3 - (4+3\theta)\,t^2 + (3\theta^2+4\theta+2)\,t - \theta(1+\theta)^2 > 0.$$

If $\theta \geq \frac{2}{3}$, then u is strictly increasing and $u(t) > u(1) = \theta^2(1-\theta) > 0$.
If $0 < \theta < \frac{2}{3}$, then u admits only a minimum point $\frac{4+3\theta+\sqrt{4-9\theta^2}}{6}$ and

$$u(t) \geq u\left(\frac{4+3\theta+\sqrt{4-9\theta^2}}{6}\right) = \frac{8-\sqrt{(4-9\theta^2)^3}}{54} > 0\,.$$

Finally, from (3.2) it follows that $\alpha'(1) < 0$ and $\alpha'(1+\theta) > 0$, i.e. the only critical point of α is a minimum point.

□

Finally we obtain the following theorem.

Theorem 3.3. *If* $0 \leq k_0 \leq k$, h_0 *and* c_0 *are defined in Section* 2 *by* (2.4) *and* (2.5) *respectively, we have*

$$\max_{t\geq 1} g(t) < \max_{t\geq 1} h_0(t) = h_0(c_0)\,.$$

Proof. We can easily verify that $g'(t) \geq 0$ is equivalent to

$$k_0(1+\theta)(1+\theta-t) - kt^{-\theta}(t-1)^{1+\theta} \geq 0\,.$$

Then the function g is strictly decreasing on $[1+\theta, +\infty[$ and achieves its maximum in a point $s \in]1, 1+\theta[$ which satisfies the equality

$$k_0(1+\theta)(1+\theta-s) = ks^{-\theta}(s-1)^{1+\theta}\,.$$

Moreover

$$\max_{t\geq 1} g(t) = g(s) = \left(1-\frac{1}{s}\right)^{\theta} \frac{1+\theta}{k_0(1+\theta)s + \frac{k_0(1+\theta)s(1+\theta-s)}{s-1}}$$

$$= \left(1-\frac{1}{s}\right)^{\theta} \frac{s-1}{k_0\theta s}\,.$$

In order to prove that $g(s) < h_0(s) \leq h_0(c_0)$, we note that

$$h_0(s) = \left(1-\frac{1}{s}\right)^{\theta} \frac{1+\theta}{\left[\big(k_0(1+\theta)\big)^{\frac{1}{1-\theta}} + \left(\frac{k_0(1+\theta)(1+\theta-s)s^{1+\theta}}{s-1}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta}}$$

$$= \left(1-\frac{1}{s}\right)^{\theta} \frac{s-1}{k_0\left[(s-1)^{\frac{1}{1-\theta}} + [s^{1+\theta}(1+\theta-s)]^{\frac{1}{1-\theta}}\right]^{1-\theta}}\,.$$

Thus $g(s) < h_0(s)$ is equivalent to

$$\left(1 - \frac{1}{s}\right)^{\frac{1}{1-\theta}} + [s^\theta(1 + \theta - s)]^{\frac{1}{1-\theta}} < \theta^{\frac{1}{1-\theta}} ,$$

i.e.,

$$\alpha(s) < \theta^{\frac{1}{1-\theta}} . \tag{3.4}$$

Since s belongs to open interval $]1, 1 + \theta[$, (3.4) follows from Lemma 2.

□

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