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ON EXISTENCE OF FIXED POINT FOR PATA TYPE 2-CONVEX CONTRACTION MAPPINGS

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Abstract. In this paper, existence of fixed point for Pata type 2-convex contraction mapping in complete metric space has been proved. This study is a natural continuation of Istraescu [3].

1. Introduction

In 1922, Banach [2] proved the existence of fixed point in a complete metric space (X, d). The mapping f has been considered to be a contraction and f takes points of X to itself. Later, several interpretations for the existence

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of fixed point with weaker conditions to contraction were given. One such classical and interesting is the following definition given by Istraescu [3].

Definition 1.1. A continuous mapping $f : X \to X$ is said to be convex contractive of order 2 if there exist two constants $a, b \in [0, 1)$ such that the following conditions hold:

(1) $a + b < 1$,
(2) $d(f^2(x), f^2(y)) \leq ad(f(x), f(y)) + bd(x, y)$ for all $x, y \in X$.

Throughout the paper, Θ denotes the class of all increasing functions $\Psi : [0, 1] \to [0, \infty)$ which vanishes with continuity at 0. In a recent paper, Pata [4] obtained the following refinement of the classical Banach contraction principle.

Let $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ be any constants. For all, $x, y \in X$

$$d(f(x), f(y)) \leq (1 - \epsilon)d(x, y) + \Lambda\epsilon^\alpha \Psi(\epsilon)\Big[1 + ||x|| + ||y||\Big]^\beta, \qquad (1.1)$$

for all $\epsilon \in [0, 1]$, where $||x|| = d(x, x_0)$ for arbitrary $x_0 \in X$ and $\Psi \in \Theta$.

Theorem 1.2. ([4]) *Let (X, d) be a complete metric space and let $f : X \to X$ be a Pata refinement of contraction Mapping. Then f has a unique fixed point in X.*

In this paper, we define the Pata type 2-convex contraction and prove the existence of fixed point in metric spaces which generalizes the result of [3, 4].

The following lemma is used to prove our main result.

Lemma 1.3. ([1]) *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\delta > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \delta$, $d(x_{m_k-1}, x_{n_k}) < \delta$ and*

(1) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \delta$;
(2) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \delta$;
(3) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \delta$.

Using above Lemma 1.3, we get

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \delta$$

and

$$\lim_{k\to\infty} d(x_{m_k-2}, x_{n_k-2}) = \delta.$$

2. Existence of fixed point

In this section, we prove the existence of unique fixed point for Pata type 2-convex contraction mapping. Let (X, d) be a metric space. In the sequel, we write $||x|| = d(x, x_0)$, where x_0 is arbitrary element in X.

Definition 2.1. Let (X, d) be a complete metric space. A continuous map $f : X \to X$ is said to be Pata type 2-convex contraction if for all $x, y \in X$, $\Psi \in \Theta$ and for every $\epsilon \in [0, 1]$, f satisfies the inequality:

$$\begin{aligned} d(f^2(x), f^2(y)) \leq (1-\epsilon) \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big]^{\beta}, \end{aligned} \tag{2.1}$$

where, $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ and $k \in [0, 1]$ are any constants.

Now, we show that all convex contraction of order 2 is an particular case of Pata type 2-convex contraction. Let $d = a + b$ in Definition 1.1 and consider the Bernoulli's inequality $(1 + rt) \leq (1 + t)^r$, for all $r \geq 1$ and $t \in [-1, \infty)$. Then

$$\begin{aligned} d(f^2(x), f^2(y)) &\leq ad(f(x), f(y)) + bd(x, y) \\ &\leq (a+b) \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ &= d \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ &\leq (1-\epsilon) \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ &\quad + (d + \epsilon - 1)\Big[||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big] \\ &\leq (1-\epsilon) \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ &\quad + d\bigg(1 + \frac{\epsilon - 1}{d}\bigg)\Big[1 + ||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big] \\ &\leq (1-\epsilon) \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ &\quad + d\epsilon^{\frac{1}{d}}\Big[1 + ||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big] \\ &\leq (1-\epsilon) \max\Big\{d(f(x), f(y)), d(x, y)\Big\} \\ &\quad + d\epsilon\epsilon^{\frac{1-d}{d}}\Big[1 + ||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big]. \end{aligned} \tag{2.2}$$

Comparing this with Pata type 2-convex contraction, we have that convex contraction of order 2 is actually a special case of Pata type 2-convex contraction with $\Lambda = d$, $\Psi(\epsilon) = \epsilon^{\frac{1-d}{d}}$ and $\alpha = \beta = 1$. It is also clear that mappings given by [3, 4] were also Pata type 2-convex contraction.

Now, we prove the main result of this paper.

Theorem 2.2. *Let (X, d) be a complete metric space and let $f : X \to X$ be a Pata type 2-convex contraction. Suppose, there exists an element $x_0 \in X$ such that the picard iterative sequence of x_0 satisfies that $d(f^n(x_0), f^{n-1}(x_0))$ is non-increasing. Then f has a unique fixed point in X. Moreover, for a fixed element $x_0 \in X$, the sequence generated as $x_{n+1} = f(x_n)$ converges to a point $x \in X$.*

Proof. Let $x_0 \in X$ be the element which satisfies our assumption. Let, $x_{n+1} = f(x_n)$ and $c_n = d(x_n, x_0)$.
Claim (1): We prove that $\{c_n\}$ is bounded. For $n \geq 3$, we get

$$\begin{aligned}
c_n &= d(x_n, x_0) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_2) + d(x_2, x_1) + d(x_1, x_0) \\
&\leq (1-\epsilon)\max\Big\{d(x_n, x_1), d(x_{n-1}, x_0)\Big\} + 3c_1 \\
&\quad + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x_n|| + ||x_1|| + ||x_{n-1}||\Big]^{\beta} \\
&\leq (1-\epsilon)\max\Big\{d(x_n, x_1), d(x_{n-1}, x_0)\Big\} + 3c_1 \\
&\quad + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x_n|| + ||x_1|| + d(x_{n-1}, x_n) + d(x_n, x_0)\Big]^{\beta} \\
&\leq (1-\epsilon)[c_n + c_1] + 3c_1 + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + 2c_n + 2c_1\Big]^{\alpha}.
\end{aligned}$$

Accordingly, since $\Big[1 + 2c_n + 2c_1\Big]^{\alpha} \leq 2^{\alpha}\ c_n^{\alpha}\ (1 + 2c_1)^{\alpha}$, it implies that

$$c_n \leq (1-\epsilon)c_n + a\epsilon^{\alpha}\Psi(\epsilon)c_n^{\alpha} + b,$$

for some $a, b > 0$ and hence,

$$\epsilon c_n \leq a\epsilon^{\alpha}\Psi(\epsilon)2c_n^{\alpha} + b.$$

If there is a subsequence $c_{n_i} \to \infty$, the choice $\epsilon = \epsilon_i = (1+b)/c_{n_i}$ leads to the contradiction that $1 \leq a(1+b)^{\alpha}\Psi(\epsilon_i) \to 0$. Hence, $\{c_n\}$ is bounded. Since, $d(x_n, x_{n-1})$ is non-increasing, let $\lim\limits_{n\to\infty} d(x_n, x_{n-1}) = d$. For $n \geq 2$,

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(f^2(x_{n-1}), f^2(x_{n-2})) \\
&\qquad \leq (1-\epsilon)\max\Big\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2})\Big\}
\end{aligned}$$

$$+ \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x_n|| + ||x_{n-1}|| + ||x_{n-1}|| + ||x_{n-2}||\Big]^{\beta}$$
$$\leq (1-\epsilon)\max\Big\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2})\Big\} + K\epsilon\Psi(\epsilon).$$

Now, as $n \to \infty$, we get $d \leq K\Psi(\epsilon)$ and hence $d = 0$.

Claim (2): The sequence $\{x_n\}$ is Cauchy. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 1.3, there exist subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k> m_k>k$ such that

$$\begin{aligned}\delta &\leq d(x_{m_k}, x_{n_k})\\ &= d(f^2(x_{m_k-2}), f^2(x_{n_k-2}))\\ &\leq (1-\epsilon)\max\Big\{d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-2}, x_{n_k-2})\Big\} + K\epsilon\Psi(\epsilon).\end{aligned}$$

Now, as $k \to \infty$, we get $\delta \leq K\Psi(\epsilon)$ which is a contradiction. Therefore, $\{x_n\}$ is Cauchy. Since X is complete, there exists $x \in X$ such that $x_n \to x$. Since f is continuous, if follows that $f(x_n) \to f(x)$ and hence x is a fixed point of f. For the uniqueness of fixed point, suppose that x and y are fixed points of f. Then

$$d(f^2(x), f^2(y)) \leq (1-\epsilon)\max\Big\{d(f(x), f(y)), d(x,y)\Big\} + K\epsilon\Psi(\epsilon).$$

Therefore, we get $d(x,y) \leq K\Psi(\epsilon)$ and hence $x = y$. This completes the proof. □

Corollary 2.3. ([4]) *Let (X,d) be a complete metric space and let $f : X \to X$ be a Pata refinement of contraction mapping. Then f has a unique fixed point in X.*

Proof. Let x and y be two elements of X. For $\epsilon = 0$, f satisfies nonexpansive condition and hence, for $x_0 \in X$, the picard iterative sequence satisfies $d(f^n(x_0), f^{n-1}(x_0))$ is non-increasing. Since f is a Pata type contraction mapping, for all $\epsilon \in [0,1]$, $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$, we get

$$\begin{aligned}d(f^2(x), f^2(y)) &= d\Big(f(f(x)), f(f(y))\Big)\\ &\leq (1-\epsilon)d(f(x),(y)) + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x|| + ||y||\Big]^{\beta}\\ &\leq (1-\epsilon)\max\{d(f(x), f(y)), d(x,y)\}\\ &\quad + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big]^{\beta}.\end{aligned} \tag{2.3}$$

Therefore by Theorem 2.2, f has a unique fixed point in X. □

In some situations a function is not a contraction but the iterate of it is a contraction. This turns out to suffice to get the conclusion of the contraction mapping theorem for the original function.

Theorem 2.4. *Let X be a complete metric space and $h : X \to X$ be a mapping such that some iterate $h^N : X \to X$ is a contraction. Then h has a unique fixed point.*

Let f be a mapping from a metric space (X, d) to itself and the second iterate of f, that is, f^2 be a contraction on X. Now, we generalize these kind of mappings in the sense of Pata type mapping and give sufficient condition for the existence of fixed point for the generalization.

Definition 2.5. Let (X, d) be a complete metric space. A mapping $f : X \to X$ is said to be Pata type 2-contraction if for all $x, y \in X$, $\Psi \in \Theta$ and for every $\epsilon \in [0, 1]$, f satisfies the inequality:

$$d(f^2(x), f^2(y)) \leq (1-\epsilon)d(x,y) + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x|| + ||y||\Big]^{\beta}, \tag{2.4}$$

where, $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ and $k \in [0, 1]$ are any constants.

Corollary 2.6. *Let (X, d) be a complete metric space and let $f : X \to X$ be a nonexpansive and Pata type 2-contraction. Then f has a unique fixed point in X.*

Proof. Since f is a nonexpansive, it follows that for $x_0 \in X$, $d(f^n(x_0), f^{n-1}(x_0))$ is non-increasing. Also,

$$\begin{aligned} d(f^2(x), f^2(y)) &\leq (1-\epsilon)d(x,y) + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x|| + ||y||\Big]^{\beta} \\ &\leq (1-\epsilon)\max\{d(f(x), f(y)), d(x,y)\} \\ &\quad + \Lambda\epsilon^{\alpha}\Psi(\epsilon)\Big[1 + ||x|| + ||y|| + ||f(x)|| + ||f(y)||\Big]^{\beta}. \end{aligned} \tag{2.5}$$

Therefore by Theorem 2.2, f has a unique fixed point in X. □

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