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THE MINIMAX PROBLEM OF TWO VECTOR-VALUED FUNCTIONS

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Abstract. In this paper, we study the minimax problem in a partial ordered Hausdorff topological vector space. We obtain a vector form Ky Fan minimax inequality theorem which involves two vector-valued functions. The proof of this theorem don't use separation theorems and nonlinear scalarization functions. Forthermore, we also use the fixed point theorem of set-valued mappings by Browder [1] to prove some results of minimax problem of two vector-valued functions, and the minimax theorem in [16] is extended to vector form.

1. Introduction

In 1928, von Neumann [15] obtained minimax theorem on finite dimensional simplex, it plays an important role in modern analysis, and has important applications in game theory and economics. This theorem is generalized by many scholars. In 1958, Sion [17] proved a minimax theorem under the conditions of compact convex, semicontinuous functions and quasi convexity by using Knaster, Kuratowski, Mazurkiewicz [10] and Helly's theorem. Park [16] proposed a simple method to prove Sion's minimax theorem in 2010. With the development of vector optimization, vector-valued minimax theorem has been widely studied in recent decades. In 1989, Ferro [7] proved a minimax theorem with general conditions of vector-valued function by using the separation theorem. In 1991, Ferro [8] gave another symmetric form of minimax theorem; In 1991, Tanaka [18] didn't use Ferro's inclusive assumptions, but obtained weakly minimax theorem. Tan *et al.* generalized the results of Tanaka in

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1996. In 2010, Li *et al.* [13] gave a class of minimax problems in lexicographic order.

On the other hand, Ky Fan established a minimax inequality theorem in 1972. This theorem has wide applications in variational inequality, game theory, and control theory, etc. Ky Fan's minimax inequality is generalized to the vector-valued form in recent years. In 1996, Chang *et al.* [2] proved minimax inequality for vector-valued functions on W-spaces. In 1998, Li and Wing [14] gave a type of minimax inequality for vector-valued functions. Ky Fan's minimax inequality for vector set-valued mappings are also studied by [11], [12].

Theorem 1.1. (Ky Fan Minimax Inequality) *Let E be a Hausdorff topological vector space and let X be a nonempty compact convex subset of E. Suppose that $f : X \times X \to R$ satisfies the following:*

(1) *For each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous;*
(2) *For each fixed $x \in X$, $f(x, \cdot)$ is quasiconcave.*

Then

$$\min_{x\in X}\sup_{y\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

In this paper, we establish a new type of Ky Fan minimax inequality for two vector-valued functions and we extend the result of [3] to the vector form. Finally, we adopt the method of [16] to prove a minimax theorem of two vector-valued functions.

2. Preliminaries

Let E be a Hausdorff topological vector space and let $C \subset E$ be a closed convex cone, *i.e.*, C is a closed set of E, $\lambda C \subseteq C$ for $\lambda \ge 0$ and $C + C \subseteq C$; C is said to be a pointed closed convex cone in E if $C \cap C = \{\theta\}$, where θ is the zero in E. Let C be a pointed closed convex cone C on E. Then there is a partial order "$\le$" on E defined by $x \le y$ if and only if $y - x \in C$. Let $\text{int}C \neq \emptyset$, where $\text{int}C$ denotes the interior of C, we have

$$x < y \Leftrightarrow y - x \in \text{int}C; \quad x \not< y \Leftrightarrow y - x \notin \text{int}C.$$

A subset $D \subset E$ is said to be upper bounded if there exists $v \in E$ such that $x \le v$ for all $x \in D$, in an analogous way, we define lower bounded of D. A subset $D \subset E$ is said to be bounded if it is both upper bounded and lower bounded.

Let $A \subset E$ be a nonempty set, a point $z \in A$ is said to be a minimal point of A if $A \bigcap (z - C) = \{z\}$. $\text{Min}A$ will be the set of all minimal points of A, similarly, $\text{Max}A$ will be the set of all maximal points of A. A point $z \in A$

is said to be a weakly minimal point of A if $A\bigcap(z-\text{intC})=\emptyset$, $\text{Min}_w A$ will denote the set of all weakly minimal points, in an analogous way, $\text{Max}_w A$ will denote the set of all weakly maximal points. It is easy to know that $\text{Min}A \subset \text{Min}_w A$ and $\text{Max}A \subset \text{Max}_w A$ provided $\text{int}C \neq \emptyset$, (see Ferro [7, 8]).

Definition 2.1. ([7]) Let E, Y be Hausdorff topological vector spaces, $X \subset E$ a nonempty convex subset, $C \subseteq Y$ a pointed closed convex cone, and $f : X \to Y$ a function.

(1) f is said to be properly quasi C-convex, if

$$\text{either } f(tx_1+(1-t)x_2) \leq f(x_1) \text{ or } f(tx_1+(1-t)x_2) \leq f(x_2)$$

for every $x_1, x_2 \in X$ and $t \in [0,1]$.

(2) f is said to be quasi C-convex if, for each $r \in X$, we have

$$\{x \in X : f(x) < r\}$$

is convex.

Obviously, properly quasi C-convex function implies that quasi C-convex, but the reverse is not true.

A function f is properly quasi C-concave for all $x \in X$ if and only if $-f$ is properly quasi C-convex; f is quasi C-concave for all $x \in X$ if and only if $-f$ is quasi C-convex.

Definition 2.2. ([9]) Let $D \subset E$ be a nonempty set and $C \subseteq E$ be a pointed closed convex cone.

(1) A point $z \in E$ is said to be the supremum of D and denote it by $\sup D$, if it satisfies the following:
(a) $x \leq z, \forall x \in D$; (b) $x \leq y, \forall x \in D \Rightarrow z \leq y$.
(2) A point $z \in E$ is said to be the infimum of D and denote it by $\inf D$, if it satisfies the following:
(a) $z \leq x, \forall x \in D$; (b) $y \leq x, \forall x \in D \Rightarrow z \geq y$.
(3) C is said to be a minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$.
(4) C is said to be strongly minihedral cone if any nonempty subset of D which is upper bounded has the supremum.

Remark 2.3. C is a minihedral cone implies that $\inf\{x, y\}$ exists for all $x, y \in E$. C is a strongly minihedral cone implies that any nonempty subset of D which is lower bounded has the infimum.

Definition 2.4. Let E, Y be Hausdorff topological vector spaces and $C \subset Y$ be a pointed closed convex cone.

(1) $f : E \to Y$ is quasi C-lower semicontinuous if $f^{-1}(y - C)$ is closed set for each $y \in Y$.
(2) $f : E \to Y$ is quasi C-upper semicontinuous if $f^{-1}(y + C)$ is closed set for each $y \in Y$.

In [5], we know that if E is compact and f is quasi C-upper semicontinuous, then there exists a maximal point for f. Similarly, f is quasi C-lower semicontinuous and E is compact imply that there exists a minimal point for f.

Using generalized semicontinuous introduced by [3], a function f is said to be C-lower semicontinuous from above at $x_0 \in X$ if for each sequence $\{x_n\} \subset X$, $x_n \to x_0$ such that $f(x_{n+1}) \leq f(x_n), \forall n \geq 1$ implies that $f(x_0) \leq f(x_n)$ for each $n \geq 1$. f is said to be C-upper semicontinuous from below at $x_0 \in X$ if for each sequence $\{x_n\} \subset X$, $x_n \to x_0$ such that $f(x_{n+1}) \geq f(x_n), \forall n \geq 1$ implies that $f(x_0) \geq f(x_n)$ for each $n \geq 1$. By Theorem 2.3 in [3], if E is compact and Y is a separable Huasdorff partial ordered topological vector space, then there exists the solution of Pareto's problem for f satisfying lower semicontinuous from above or upper semicontinuous from below condition.

Definition 2.5. ([16]) Let E be a Hausdorff partial ordered topological vector space and $X \subset E$ be a nonempty convex subset. A set-valued map $T : X \to 2^X$ is said to be a Fan-Browder map if it satisfies the following conditions:

(1) $T(x)$ is nonempty convex set for all $x \in X$;
(2) There exists some finite set $\{y_1, y_2, \cdots y_n\} \subseteq X$ such that

$$X = \bigcup_{i=1}^{n} \mathrm{int} T^{-1} y_i.$$

Lemma 2.6. ([6]) *Let E be a Hausdorff topological vector space and $D \subset E$ be a nonempty subset. A set-valued map $G : D \to 2^E$ is closed valued, and for any finite set $\{x_1, x_2, \cdots, x_n\} \subset D$ such that co$\{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$. If $G(x)$ is compact for at least one $x \in D$, then $\bigcap_{x \in D} G(x) \neq \emptyset$.*

Lemma 2.7. ([1]) *Let $X \subset E$ be nonempty compact convex subset and $T : X \to 2^X$ be a map satisfying the following conditions:*

(1) *$T(x)$ is a nonempty convex set for all $x \in X$;*
(2) *$T^{-1}y = \{x \in X : y \in Tx\}$ is open set for all $y \in X$.*

Then T has a fixed point.

3. Main results

Theorem 3.1. *Let X, Y be Hausdorff topological vector spaces, $D \subseteq X$ be a compact convex subset, and $C \subset Y$ be a strongly minihedral cone with $intC \neq \emptyset$. Suppose that $f, g : D \times D \to Y$ are two upper bounded functions and satisfy the following conditions:*

(1) *$y \to g(x, y)$ is quasi C-lower semicontinuous for all $x \in D$;*
(2) *$x \to f(x, y)$ is properly quasi C-concave for all $y \in D$;*
(3) *$g(x, y) \leq f(x, y)$ for each $(x, y) \in D \times D$.*

Then, for every $z \in \mathrm{Min}_w \bigcup_{y\in D} \sup_{x\in D} g(x, y)$ such that

$$z \not> \sup_{x\in D} f(x, x).$$

Proof. Since C is a strongly minihedral cone and $f(x, y)$ has upper bounded for all $(x, y) \in D \times D$, we have $\sup_{x\in D} g(x, y) \neq \emptyset$ and $\sup_{x\in D} f(x, x) \neq \emptyset$. Since $g(x, y)$ is quasi C-lower semicontinuous for each fixed $x \in D$, we know that $\sup_{x\in D} g(x, y)$ is quasi C-lower semicontinous. Hence, $\mathrm{Min} \sup_{x\in D} g(x, y) \neq \emptyset$ for all $y \in D$, we get $\mathrm{Min}_w \bigcup_{y\in D} \sup_{x\in D} g(x, y) \neq \emptyset$.

Let $\alpha = \sup_{x\in D} f(x, x)$, define two maps $F : D \to 2^D$ and $G : D \to 2^D$ by

$$F(x) = \{y \in D : f(x, y) \leq \alpha\}, \quad G(x) = \{y \in D : g(x, y) \leq \alpha\},$$

for all $x \in D$. By the assumption (1), $G(x)$ is closed set for all $x \in D$. By the assumption (2), we get $\{x \in D : \alpha - f(x, y) \notin C\}$ is convex set for all $y \in D$. In fact, let $x_1, x_2 \in D$ such that

$$\alpha - f(x_1, y) \notin C, \quad \alpha - f(x_2, y) \notin C,$$

for all $y \in D$. Now, suppose that $f(x(t), y) \leq \alpha$ is true for all y, where $x(t) = tx_1 + (1 - t)x_2$ for every $t \in (0, 1)$. Fixed each $y \in D$, $f(x, y)$ is properly quasi C-concave for all $x \in D$, then there exists t_0 such that

$$\text{either } f(x(t_0), y) \geq f(x_1, y) \quad \text{or} \quad f(x(t_0), y) \geq f(x_2, y)$$

holds. So we have $x_1 \notin D$ or $x_2 \notin D$, which is a contradiction. Thus for any finite points $x_i \in D, i = 1, 2, \cdots n$, we have

$$co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i).$$

By the assumption (3), $F(x) \subseteq G(x)$ for all $x \in D$, for every finite set $\{x_1, x_2, \cdots, x_n\}$ of D such that

$$co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i).$$

By Lemma 2.6, we have $\bigcap_{x\in D} G(x) \neq \emptyset$, thus there exists $y_0 \in D$ such that

$$y_0 \in G(x),$$

for all $x \in D$, that is, $g(x, y_0) \leq \alpha$ implies that $\sup_{x\in D} g(x, y_0) \leq \alpha$ for all $x \in D$. Thus, for every $z \in \mathrm{Min}_w \bigcup_{y\in D} \sup_{x\in D} g(x, y)$, we have $z \not> \alpha$. □

Remark 3.2. Theorem 3.1 is Ky-Fan's minimax inequality when $Y = R$, $f = g$.

By using the generalized semicontinuous introduced by [3], we may get the following result.

Theorem 3.3. *Let X be a Hausdorff topological vector space, Y be a separable Banach space, $D \subseteq X$ be a compact convex subset, and $C \subset Y$ be a strongly minihedral cone with $intC \neq \emptyset$. Suppose that $f, g : D \times D \to Y$ are two upper bounded functions and satisfy the following conditions:*

(1) *$\sup_{x\in D} g(x, y)$ is C-lower semicontinuous from above on D;*
(2) *$\{y \in D : g(x, y) \leq \sup_{x\in D} f(x, x)\}$ is closed for all $x \in D$;*
(3) *$x \to f(x, y)$ is properly quasi C-concave for all $y \in D$;*
(4) *$g(x, y) \leq f(x, y)$ for each $(x, y) \in D \times D$.*

Then, for every $z \in \mathrm{Min}_w \bigcup_{y\in D} \sup_{x\in D} g(x, y)$ such that

$$z \not> \sup_{x\in D} f(x, x).$$

Theorem 3.4. *Let X and Y be nonempty convex subsets of two Hausdorff topological vector spaces, respectively, F be a Hausdorff topological vector space, and $C \subset F$ be a strongly minihedral cone with $intC \neq \emptyset$. Suppose that $f, s, t, g : X \times Y \to F$ are functions such that $f(x, y)$ is lower bounded for all $(x, y) \in X \times Y$, and $g(x, y)$ is upper bounded for all $(x, y) \in X \times Y$, and satisfy the following assumptions:*

(1) *$f(x, y) \leq s(x, y) \leq t(x, y) \leq g(x, y)$ for each $(x, y) \in X \times Y$;*
(2) *For each $y \in Y$, $s(x, y)$ is quasi C-concave for all $x \in X$ and for each $x \in Y$, $t(x, y)$ is quasi C-convex for all $y \in X$;*
(3) *For each $r \in F$, there exists $x_i \in X, i = 1, 2, \cdots, m$ such that $Y = \bigcup_{i=1}^{m} int\{y \in Y : f(x_i, y) > r\}$;*
(4) *For each $r \in F$, there exists $y_j \in Y, j = 1, 2, \cdots, n$ such that $X = \bigcup_{j=1}^{n} int\{x \in X : g(x, y_j) < r\}$.*

Then, $\sup_{x\in X} \inf_{y\in Y} g(x, y) \not< \inf_{y\in Y} \sup_{x\in X} f(x, y)$.

Proof. Since C is a strongly minihedral cone, $f(x, y)$ has lower bounded and $g(x, y)$ has upper bounded for all $(x, y) \in D \times D$, by the assumption (1), we know that both $\sup_{x\in X}\inf_{y\in Y} g(x,y)$ and $\inf_{y\in Y}\sup_{x\in X} f(x,y)$ exist. Now, suppose that

$$\sup_{x\in X}\inf_{y\in Y} g(x,y) \not< \inf_{y\in Y}\sup_{x\in X} f(x,y)$$

is not true. Then there exists $r \in F$ such that

$$\sup_{x\in X}\inf_{y\in Y} g(x,y) < r < \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Let a function $T : X \times Y \to 2^{X\times Y}$ defined by $T(x,y) = \{\overline{x} \in X : s(\overline{x},y) > r\} \times \{\overline{y} \in Y : t(x,\overline{y}) < r\}$. Then, by the assumption (2), we get $T(x,y)$ is convex for each $(x,y) \in X \times Y$. For each $(\overline{x},\overline{y}) \in X \times Y$, we have

$$\begin{aligned} T^{-1}(\overline{x},\overline{y}) &= \{x \in X : t(x,\overline{y}) < r\} \times \{y \in Y : s(\overline{x},y) > r\} \\ &\supseteq \{x \in X : g(x,\overline{y}) < r\} \times \{y \in Y : f(\overline{x},y) > r\} \\ &\supseteq \mathrm{int}\{x \in X : g(x,\overline{y}) < r\} \times \mathrm{int}\{y \in Y : f(\overline{x},y) > r\}. \end{aligned}$$

By (3) and (4), we obtain that $X \times Y$ is covered by $\{\mathrm{int}T^{-1}(x_i,y_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$, and then T is a Fan-Browder map. By Lemma 2.7, we get a point $(x_0,y_0) \in X \times Y$ such that $(x_0,y_0) \in T(x_0,y_0)$. By the definition of T, we have

$$t(x_0,y_0) < r < s(x_0,y_0),$$

which is a contradiction. This completes the proof. □

Remark 3.5. For $F = R$ in Theorem 3.4, we obtain the results of [16] in scalar case.

Theorem 3.6. *Let X, Y be nonempty compact convex subsets of two Hausdorff topological vector spaces, respectively. Let F be a Hausdorff topological vector space and $C \subset F$ be a strongly minihedral cone with $intC \neq \emptyset$. Suppose that $f, s, t, g : X \times Y \to F$ are functions such that $f(x,y)$ is lower bounded and $g(x,y)$ is upper bounded for all $(x,y) \in X \times Y$, and satisfy the following assumptions:*

(1) *$f(x,y) \leq s(x,y) \leq t(x,y) \leq g(x,y)$ for each $(x,y) \in X \times Y$;*
(2) *$y \to f(x,y)$ is quasi C-lower semicontinuous for all $x \in X$ and for each $x \in Y$, $t(x,y)$ is quasi C-convex for all $y \in X$;*
(3) *$x \to g(x,y)$ is quasi C-upper semicontinuous for all $y \in X$ and for each $y \in Y$, $s(x,y)$ is quasi C-concave for all $x \in X$.*

Then, for every $\alpha \in \mathrm{Min}_w \bigcup_{y\in Y}\sup_{x\in X} f(x,y)$, there exists

$$\beta \in \mathrm{Max}_w \bigcup_{x\in X}\inf_{y\in Y} g(x,y)$$

such that

$$\beta \not< \alpha.$$

Proof. Since C is a strongly minihedral cone, $f(x,y)$ has lower bounded and $g(x,y)$ has upper bounded for all $(x,y) \in D \times D$, and by assumption (1), we get $\sup_{x\in X} f(x,y) \neq \emptyset, \inf_{y\in Y} g(x,y) \neq \emptyset$. Since $f(x,\cdot)$ is quasi C-lower semicontinuous, $\sup_{x\in X} f(x,\cdot)$ is also quasi C-lower semicontinuous. Similarly, we get $\inf_{y\in Y} g(\cdot,y)$ is quasi C-upper semicontinuous. Since X,Y are compact, we get both $\mathrm{Min}_w \bigcup_{y\in Y} \sup_{x\in X} f(x,y)$ and $\mathrm{Max}_w \bigcup_{x\in X} \inf_{y\in Y} g(x,y)$ exist. Suppose that there exists $\alpha \in \mathrm{Min}_w \bigcup_{y\in Y} \sup_{x\in X} f(x,y)$ and each $\beta \in \mathrm{Max}_w \bigcup_{x\in X} \inf_{y\in Y} g(x,y)$ such that $\beta < \alpha$. Then, there exists $r \in F$ such that $\beta < r < \alpha$. By the similar argument of Theorem 3.4, we can obtain a contradiction. □

Remark 3.7. If $F = R$, then $\min_{y\in X} \sup_{x\in X} f(x,y) \leq \max_{x\in X} \inf_{y\in Y} g(x,y)$. If $f = s = t = g$, then $\min_{y\in X} \max_{x\in X} f(x,y) = \max_{x\in X} \min_{y\in Y} f(x,y)$.

Corollary 3.8. *Let X and Y be nonempty compact convex subsets of two Hausdorff topological vector spaces, repectively, F be a separable Hausdorff topological vector space and $C \subset F$ be a strongly minihedral cone with $intC \neq \emptyset$. Suppose that $f : X \times Y \to F$ is bounded for all $(x,y) \in X \times Y$, and satisfies the following assumptions:*

(1) *$y \to f(x,y)$ is quasi C-lower semicontinuous from above and quasi C-convex for each fixed $x \in X$;*
(2) *$x \to f(x,y)$ is quasi C-upper semicontinuous from below and quasi C-concave for each fixed $y \in Y$;*
(3) *For each $r \in F$, there exists $x_i \in X, i = 1,2,\cdots,m$ such that $Y = \bigcup_{i=1}^m \{y \in Y : f(x_i,y) > r\}$, where $\{y \in Y : f(x_i,y) > r\}$ is open set;*
(4) *For each $r \in F$, there exists $y_j \in Y, j = 1,2,\cdots,n$ such that $X = \bigcup_{j=1}^n \{x \in X : f(x,y_j) < r\}$, where $\{x \in X : f(x,y_j) < r\}$ is open set.*

Then, $\sup \bigcup_{x\in X} \mathrm{Min} f(x,Y) \not< \inf \bigcup_{y\in Y} \mathrm{Max} f(X,y)$.

Proof. Since X,Y are compact, F is separable, and $f(x,y)$ is bounded, by the assumptions (1), (2) and Theorem 2.3 in [3], we have

$$\sup \bigcup_{x\in X} \mathrm{Min} f(x,Y) \neq \emptyset, \quad \inf \bigcup_{y\in Y} \mathrm{Max} f(X,y) \neq \emptyset.$$

Suppose that $\sup \bigcup_{x\in X} \mathrm{Min} f(x,Y) \not< \inf \bigcup_{y\in Y} \mathrm{Max} f(X,y)$ is not true, there exists $r \in F$ such that

$$\sup \bigcup_{x\in X} \mathrm{Min} f(x,Y) < r < \inf \bigcup_{y\in Y} \mathrm{Max} f(X,y).$$

Obviously, which is a contradiction. □

Corollary 3.9. *Let X and Y be nonempty compact convex subsets of two Hausdorff topological vector spaces, respectively, F be a Hausdorff topological vector space, and $C \subset F$ be a pointed closed convex cone with $intC \neq \emptyset$. Suppose that $f, s, t, g : X \times Y \to F$ satisfy the following assumptions:*

(1) *$f(x,y) \leq s(x,y) \leq t(x,y) \leq g(x,y)$ for each $(x,y) \in X \times Y$;*
(2) *$f(x,y)$ is continuous for all $(x,y) \in X \times Y$ and for each $x \in X$,$t(x,y)$ is quasi C-convex for all $y \in X$;*
(3) *$g(x,y)$ is continuous for all $(x,y) \in X \times Y$ and for each $y \in Y$,$s(x,y)$ is quasi C-concave for all $x \in X$.*

Then $\mathrm{Min}\bigcup_{y\in Y}\mathrm{Max}_w f(X,y) \subset \mathrm{Max}\bigcup_{x\in X}\mathrm{Min}_w g(x,Y) + F\backslash intC$.

Proof. Since f, g are continuous and X, Y are compact. In [7], we get

$$\mathrm{Min}\bigcup_{y\in Y}\mathrm{Max}_w f(X,y) \neq \emptyset, \quad \mathrm{Max}\bigcup_{x\in X}\mathrm{Min}_w g(x,Y) \neq \emptyset.$$

Suppose that the conclusion is not true. Then there exists $r \in F$, for $\alpha \in \mathrm{Min}\bigcup_{y\in Y}\mathrm{Max}_w f(X,y)$ and for each $\beta \in \mathrm{Max}\bigcup_{x\in X}\mathrm{Min}_w g(x,Y)$ such that

$$\beta < r < \alpha,$$

which is a contradiction. □

Remark 3.10. If $f = s = t = g$, then we get

$$\mathrm{Min}\bigcup_{y\in Y}\mathrm{Max}_w f(X,y) \subset \mathrm{Max}\bigcup_{x\in X}\mathrm{Min}_w f(x,Y) + F\backslash \mathrm{int}C.$$

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