Nonlinear Functional Analysis and Applications
Vol. 23, No. 1 (2018), pp. 31-48
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

STABILITY OF STOCHASTIC SIRS MODEL WITH VARIABLE DIFFUSION RATES

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Abstract. In this paper, we extend the classical SIRS epidemic model from a deterministic framework to a stochastic one and formulate it as a stochastic differential equation. We prove the global existence of unique solution. Using the Lyapunov method, we find sufficient conditions for the stochastic asymptotic stability of equilibrium solutions of this model. Finally, establish the existence of a unique ergodic stationary distribution and illustrate our results.

1. Introduction

According to the World Health Organization (WHO), infectious diseases are responsible for a quarter to a third of all deaths worldwide. As of 2008, four of the top ten causes of death were due to infectious diseases and in low-income countries, five of the top killers were due to infectious diseases [14, 19]. The SIR epidemic model is one of the most important models in epidemiological patterns and disease control. Kermack and McKendrick [6] initially proposed and

[0]Received April 18, 2017. Revised September 30, 2017.
[0]2010 Mathematics Subject Classification: 60H10, 60H30, 92D30.
[0]Keywords: SIRS model, stochastic asymptotic stability, Lyapunov function.
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studied the classical deterministic SIR (susceptible-infected-recovered) model. From then on, many authors have developed and investigated the SIR epidemic model [2, 4, 5, 13]. Further, assume that the removed or recovered individuals lose immunity and return to the susceptible compartment, such a model is SIRS type. It is more reasonable than the SIR model.

The deterministic SIRS model can be expressed by the following ordinary differential equations:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \mu S + \gamma R, \\ \frac{dI}{dt} &= \beta SI - (k + \mu + \alpha)I, \\ \frac{dR}{dt} &= kI - (\mu + \gamma)R, \end{aligned} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ represent the number of susceptible, infective and recovered individuals at time t respectively, Λ is the recruitment rate of the population, μ is the natural death rate, α is the death rate due to disease, β is the infection coefficient, k is the recovery rate of the infective individuals, γ is the rate at which recovered individuals lose immunity and return to the susceptible class.

The model (1.1) can have at most two equilibrium solutions, namely an infection-free equilibrium solution $E_1 = (S_1, I_1, R_1)$, where

$$S_1 = \frac{\Lambda}{\mu}, \qquad I_1 = 0, \qquad R_1 = 0$$

and an endemic equilibrium solution $E_2 = (S_2, I_2, R_2)$, where

$$\begin{aligned} S_2 &= \frac{k + \alpha + \mu}{\beta} = \frac{S_1}{\mathcal{R}_0}, \\ I_2 &= \frac{\mu(\mu + \gamma)(k + \alpha + \mu)(\mathcal{R}_0 - 1)}{\beta(k\mu + (\mu + \gamma)(\alpha + \mu))}, \\ R_2 &= \frac{k\mu(k + \alpha + \mu)(\mathcal{R}_0 - 1)}{\beta(k\mu + (\mu + \gamma)(\alpha + \mu))}. \end{aligned}$$

The endemic equilibrium solution exists if the condition

$$\mathcal{R}_0 = \frac{\Lambda\beta}{\mu(k + \alpha + \mu)} > 1,$$

holds, where $\mathcal{R}_0$ is the basic reproduction number.

León [9] have investigated the global stability of infection-free equilibrium and endemic equilibrium solutions of the model (1.1) by using Lyapunav function.

In the real life, any system is inevitably affected by the environmental noise, it is an important component in an ecosystem. May [12] has revealed that due to environmental fluctuation, the birth rate, death rate, transmission coefficient and other parameters involved with the system exhibit random fluctuations to a greater or lesser extent. Mao *et al.* [11] found the presence of even a small amount of white noise can suppress a potential population explosion. Therefore, it is important to investigate the effect of random fluctuations in the environment on population dynamics.

So, we perturbed the deterministic system (1.1) by a white noise and obtained a stochastic counterpart by replacing the rates β by $\beta + F_1(S,I,R)\frac{dW_1}{dt}$ and k by $k+F_2(S,I,R)\frac{dW_2}{dt}$, where F_i, $i = 1,2$ are locally Lipschitz-continuous functions on $\mathbb{D}$ and W_i, $i = 1,2$ are *i.i.d.* Wiener processes defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where

$$\mathbb{D} = \left\{ (S,I,R) \in \mathbb{R}^3 : S > 0, \;\; I > 0, \;\; R > 0, \;\; S + I + R \leq \frac{\Lambda}{\mu} \right\}.$$

The stochastic SIRS model takes a form as,

$$\begin{aligned} dS &= (\Lambda - \beta SI - \mu S + \gamma R)\, dt - SIF_1(S,I,R)dW_1, \\ dI &= (\beta SI - (k+\mu+\alpha)I)\, dt + SIF_1(S,I,R)dW_1 - IF_2(S,I,R)dW_2, \\ dR &= (kI - (\mu+\gamma)R)\, dt + IF_2(S,I,R)dW_2. \end{aligned} \tag{1.2}$$

Since the diffusion coefficients F_i, $i = 1,2$ are arbitrary local Lipschitz-continuous functions, we have a family of stochastic SIRS model. Similar models are discussed in [8, 10, 15, 16, 17].

The rest of the paper is organized as follows. In Section 3, we discuss the existence of a unique global solution for the stochastic SIRS model (1.2). In Section 4, we discuss the stochastic asymptotic stability of infection free equilibrium and the endemic equilibrium with the help of Lyapunov functions. In Section 5, we show the existence of a unique ergodic stationary distribution. In Section 6, we visualize our result.

2. Preliminaries

Consider the d-dimensional stochastic differential equation of the form

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t), \quad X(t_0) = X_0, \tag{2.1}$$

with $t_0 \leq t \leq T < \infty$, where $f : \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^{d\times m}$ are Borel measurable, $W = \{W(t)\}_{t\geq t_0}$ is an $\mathbb{R}^m$-valued Wiener process, and X_0 is an $\mathbb{R}^d$-valued random variable defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

The infinitesimal generator $\mathcal{L}$ associated with the SDE (2.1) is given by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m} \left(g(x,t)g^T(x,t)\right)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Theorem 2.1. ([3], $\mathbb{D}$-invariance) *Let $\mathbb{D}$ and $\mathbb{D}_n$ be open sets in $\mathbb{R}^d$ with*

$$\mathbb{D}_n \subseteq \mathbb{D}_{n+1}, \qquad \overline{\mathbb{D}}_n \subseteq \mathbb{D} \qquad \text{and} \qquad \mathbb{D} = \bigcup_n \mathbb{D}_n$$

and suppose f and g, satisfy the existence and uniqueness conditions for solutions of (2.1), on each set $\{(t,x) : t > t_0, x \in \mathbb{D}_n\}$. Suppose there is a non-negative continuous function $V : \mathbb{D} \times [t_0, T] \to \mathbb{R}_+$ with continuous partial derivatives and satisfying $\mathcal{L}V \le cV$ for some positive constant c and $t > t_0$, $x \in \mathbb{D}$. If also,

$$\inf_{t > t_0,\ x \in \mathbb{D} \setminus \mathbb{D}_n} V(x,t) \to \infty \quad as \quad n \to \infty,$$

then, for any X_0 independent of W such that $\mathbb{P}(X_0 \in \mathbb{D}) = 1$, there is a unique Markovian, continuous time solution $X(t)$ of (2.1) with $X(0) = X_0$, and $X(t) \in \mathbb{D}$ for all $t > 0$ (a.s.).

Assume that f and g satisfy, in addition to the existence and uniqueness assumptions, $f(x^*,t) = 0$ and $g(x^*,t) = 0$, for equilibrium solution x^*, for $t \ge t_0$. Furthermore, let's assume that x_0 be a non-random constant with probability 1.

Definition 2.2. The equilibrium solution x^* of the SDE (2.1) is stochastically stable (stable in probability) if for every $\epsilon > 0$ and $s \ge t_0$

$$\lim_{x_0 \to x^*} \mathbb{P}\left(\sup_{s \le t < \infty} \|X_{s,x_0}(t) - x^*\| \ge \epsilon\right) = 0$$

where $X_{s,x_0}(t)$ denotes the solution of (2.1), satisfying $X(s) = x_0$, at time $t \ge s$.

Definition 2.3. The equilibrium solution x^* of the SDE (2.1), is said to be stochastically asymptotically stable if it is stochastically stable and

$$\lim_{x_0 \to x^*} \mathbb{P}\left(\lim_{t \to \infty} X_{s,x_0}(t) = x^*\right) = 1.$$

Definition 2.4. The equilibrium solution x^* of the SDE (2.1) is said to be globally stochastically asymptotically stable if it is stochastically stable and for every x_0 and every s

$$\mathbb{P}\left(\lim_{t \to \infty} X_{s,x_0}(t) = x^*\right) = 1.$$

The following theorem is a useful criterion for stochastic stability of equilibrium solutions in terms of Lyapunov function.

Theorem 2.5. ([1]) *Assume that f and g satisfy the existence and uniqueness assumptions and they have continuous coefficients with respect to t.*

(i) *Suppose that there exist a positive definite function $V \in C^{2,1}\left(U_h \times [t_0, \infty)\right)$, where $U_h = \{x \in \mathbb{R}^d : \|x - x^*\| < h\}$, for $h > 0$, such that for all $t \geq t_0$, $x \in U_h$: $\mathcal{L}V(x,t) \leq 0$. Then, the equilibrium solution x^* of* (2.1) *is stochastically stable.*

(ii) *If, in addition, V is decrescent (there exists a positive definite function V_1 such that $V(x,t) \leq V_1(x)$ for all $x \in U_h$) and $\mathcal{L}V(x,t)$ is negative definite, then the equilibrium solution x^* is stochastically asymptotically stable.*

(iii) *If the assumptions of part ii) hold for a radially unbounded function $V \in C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty)\right)$ defined everywhere, then the equilibrium solution x^* is globally stochastically asymptotically stable.*

Consider the d-dimensional stochastic differential equation

$$dX(t) = b(X)dt + \sum_{r=1}^{k} \sigma_r(X)dW_r(t), \tag{2.2}$$

and the diffusion matrix is defined as follows

$$A(x) = (a_{ij}(x)), \qquad a_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x).$$

The following lemma is a useful criterion for positive recurrence in terms of Lyapunov function [20].

Lemma 2.6. *The system* (2.2) *is positive recurrent if there is a bounded open subset H of $\mathbb{R}^d$ with a regular (i.e., smooth) boundary, and*

(i) *there exist some $i = 1, 2, \ldots, d$ and a positive constant k such that*

$$a_{ii}(x) \geq k \qquad \textit{for any} \qquad x \in H,$$

(ii) *there exists a non-negative function $V : H^c \to \mathbb{R}$ such that V is twice continuously differentiable and that for some $\theta > 0$,*

$$\mathcal{L}V(x) \leq -\theta, \qquad \textit{for any} \qquad x \in H^c.$$

Moreover, the positive recurrent process $X(t)$ has a unique stationary distribution $\mu(\cdot)$ with density in $\mathbb{R}^d$ such that for any Borel set $B \subset \mathbb{R}^d$

$$\lim_{t\to\infty} \mathbb{P}(t, x, B) = \mu(B),$$

and

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f(X(t))dt = \int\limits_{\mathbb{R}^d} f(x)\mu(dx)\right\} = 1$$

for all $x \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a function integrable with respect to the measure μ.

3. Existence of a unique global solution

In this section, we discuss existence of a unique global solution of the model (1.2).

Theorem 3.1. *Let $(S(t_0), I(t_0), R(t_0)) = (S_0, I_0, R_0) \in \mathbb{D}$, and (S_0, I_0, R_0) is independent of W. Then the stochastic SIRS model (1.2) admits a unique continuous time, Markovian global solution $(S(t), I(t), R(t))$ on $t \geq t_0$ and this solution is invariant (a.s) with respect to $\mathbb{D}$.*

Proof. We use Theorem 2.1 and follow ideas of [18]. Since the coefficients of the system (1.2) are locally Lipschitz-continuous on $\mathbb{D}$, for any initial value $(S_0, I_0, R_0) \in \mathbb{D}$, there is a unique local solution on $t \in [t_0, \tau(\mathbb{D}))$, where $\tau(\mathbb{D})$ is the random time of first exit of stochastic process $(S(t), I(t), R(t))$ from the domain $\mathbb{D}$, started in $(S(s), I(s), R(s)) = (S_0, I_0, R_0) \in \mathbb{D}$ at the initial time $s \in [t_0, \infty)$. To make this solution global, we need to prove that

$$\mathbb{P}(\tau(\mathbb{D}) = \infty) = 1 \quad \text{a.s.}$$

Let

$$\mathbb{D}_n := \Big\{(S, I, R) : e^{-n} < S < \frac{\Lambda}{\mu} - e^{-n},\ \ e^{-n} < I < \frac{\Lambda}{\mu} - e^{-n},$$
$$e^{-n} < R < \frac{\Lambda}{\mu} - e^{-n},\ \ S + I + R \leq \frac{\Lambda}{\mu}\Big\},$$

for $n \in \mathbb{N}$. The system (1.2), has a unique solution up to stopping time $\tau(\mathbb{D}_n)$. Let

$$\begin{aligned} V(S, I, R) = S - \ln S + \left(\frac{\Lambda}{\mu} - S\right) - \ln\left(\frac{\Lambda}{\mu} - S\right) + I - \ln I \\ + \left(\frac{\Lambda}{\mu} - R\right) - \ln\left(\frac{\Lambda}{\mu} - R\right), \end{aligned} \tag{3.1}$$

defined on $\mathbb{D}$ and assume that $\mathbb{E}\left(V(S, I, R)\right) < \infty$. Note that $V(S, I, R) \geq 4$ for $(S, I, R) \in \mathbb{D}$. Let $W(S, I, R, t) = e^{-c(t-s)}V(S, I, R)$, defined on $\mathbb{D} \times [s, \infty)$, where

$$c = \frac{1}{4}\left(3\mu + \gamma + \alpha + 2k + \frac{\beta\Lambda}{\mu} + \frac{\beta\Lambda^2}{\mu^2} + (\mu+\gamma)\frac{\Lambda}{\mu}\right) + \frac{1}{4}\sup_{(S,I,R)\in\mathbb{D}}\left(\frac{3}{2}\frac{\Lambda^2}{\mu^2}F_1^2(S,I,R) + F_2^2(S,I,R)\right).$$

Apply the infinitesimal operator $\mathcal{L}$ on equation (3.1), we obtain

$$\begin{aligned}\mathcal{L}V(S,I,R) =& (\Lambda - \beta SI - \mu S + \gamma R)\left(\frac{1}{\frac{\Lambda}{\mu} - S} - \frac{1}{S}\right) \\ &+ (\beta SI - (k+\mu+\alpha)I)\left(1 - \frac{1}{I}\right) + (kI - (\mu+\gamma)R)\left(\frac{1}{\frac{\Lambda}{\mu} - R} - 1\right) \\ &+ \frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{1}{\left(\frac{\Lambda}{\mu} - S\right)^2} + \frac{1}{S^2} + \frac{1}{I^2}\right) \\ &+ \frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{1}{I^2} + \frac{1}{\left(\frac{\Lambda}{\mu} - R\right)^2}\right).\end{aligned}$$

After a little algebra, we have

$$\begin{aligned}\mathcal{L}V(S,I,R) =& \mu - \frac{\beta SI}{\left(\frac{\Lambda}{\mu} - S\right)} + \frac{\gamma R}{\left(\frac{\Lambda}{\mu} - S\right)} - \frac{\Lambda}{S} + \beta I + \mu - \frac{\gamma R}{S} + \beta SI - \beta S \\ &- (k+\mu+\alpha)I + k + \mu + \alpha + \frac{kI}{\left(\frac{\Lambda}{\mu} - R\right)} - kI - \frac{(\mu+\gamma)R}{\left(\frac{\Lambda}{\mu} - R\right)} \\ &+ (\mu+\gamma)R + \frac{1}{2}\frac{S^2I^2}{\left(\frac{\Lambda}{\mu} - S\right)^2}F_1^2(S,I,R) + \frac{1}{2}I^2F_1^2(S,I,R) \\ &+ \frac{1}{2}S^2F_1^2(S,I,R) + \frac{1}{2}F_2^2(S,I,R) + \frac{1}{2}\frac{I^2}{\left(\frac{\Lambda}{\mu} - R\right)^2}F_2^2(S,I,R).\end{aligned}$$

Since $S>0$, $I>0$, $R>0$ and $S+I+R\leq \dfrac{\Lambda}{\mu}$, we have,

$$\begin{aligned}
\mathcal{L}V(S,I,R) &\leq 3\mu+\gamma+\alpha+2k+\beta I+\beta SI+(\mu+\gamma)R\\
&\quad + S^2F_1^2(S,I,R)+\frac{1}{2}I^2F_1^2(S,I,R)+F_2^2(S,I,R),\\
\mathcal{L}V(S,I,R) &\leq 3\mu+\gamma+\alpha+2k+\frac{\beta\Lambda}{\mu}+\frac{\beta\Lambda^2}{\mu^2}+(\mu+\gamma)\frac{\Lambda}{\mu}\\
&\quad + \sup_{(S,I,R)\in\mathbb{D}}\left(\frac{3}{2}\frac{\Lambda^2}{\mu^2}F_1^2(S,I,R)+F_2^2(S,I,R)\right)\\
&= 4c.
\end{aligned}$$

So, $\mathcal{L}V(S,I,R)\leq cV(S,I,R)$, since $V(S,I,R)\geq 4$, for $(S,I,R)\in\mathbb{D}$. Hence

$$\mathcal{L}W(S,I,R,t)=e^{-c(t-s)}\left(-cV(S,I,R)+\mathcal{L}V(S,I,R)\right)\leq 0.$$

Note that, $\inf\limits_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n} V(S,I,R) > n+2$, for $n\in\mathbb{N}$. Now define $\tau_n := \min\{t,\tau(\mathbb{D}_n)\}$ and apply Dynkin's formula to get

$$\begin{aligned}
\mathbb{E}\left[W(S(\tau_n),I(\tau_n),R(\tau_n),\tau_n)\right] &= \mathbb{E}\left[W(S(s),I(s),R(s),s)\right]\\
&\quad + \mathbb{E}\left[\int_s^{\tau_n}\mathcal{L}W(S(u),I(u),R(u),u)du\right]\\
&\leq \mathbb{E}\left[W(S(s),I(s),R(s),s)\right]\\
&= \mathbb{E}\left[V(S(s),I(s),R(s))\right]=\mathbb{E}\left[V(S_0,I_0,R_0)\right].
\end{aligned}$$

Next, to show that $\mathbb{P}(\tau(\mathbb{D}_n)<t)=0$, we take the expected value of $e^{c(t-\tau_n)}V(S(\tau_n),I(\tau_n),R(\tau_n))$.

$$\begin{aligned}
\mathbb{E}\left[e^{c(t-\tau_n)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right] &= \mathbb{E}\left[e^{c(t-s)}e^{-c(\tau_n-s)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right]\\
&= \mathbb{E}\left[e^{c(t-s)}W(S(\tau_n),I(\tau_n),R(\tau_n),\tau_n)\right]\\
&\leq e^{c(t-s)}\mathbb{E}\left[V(S_0,I_0,R_0)\right],
\end{aligned}$$

and obtain

$$\begin{aligned}
0 &\le \mathbb{P}(\tau(\mathbb{D}) < t) \\
&\le \mathbb{P}(\tau(\mathbb{D}_n) < t) = \mathbb{P}(\tau_n < t) = \mathbb{E}(\mathbf{1}_{\tau_n<t}) \\
&\le \mathbb{E}\left(e^{c(t-\tau_n)} \frac{V(S(\tau(\mathbb{D}_n)), I(\tau(\mathbb{D}_n)), R(\tau(\mathbb{D}_n)))}{\inf\limits_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n} V(S,I,R)} \mathbf{1}_{\tau_n<t} \right) \\
&\le e^{c(t-s)} \frac{\mathbb{E}\left(V(S_0,I_0,R_0)\right)}{\inf\limits_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n} V(S,I,R)} \\
&\le e^{c(t-s)} \frac{\mathbb{E}\left(V(S_0,I_0,R_0)\right)}{n+2},
\end{aligned}$$

where $\mathbf{1}$ is the indicator function. Since $e^{c(t-s)} \dfrac{\mathbb{E}\left(V(S_0,I_0,R_0)\right)}{n+2} \to 0$, as $n \to \infty$ for all $(S_0, I_0, R_0) \in \mathbb{D}_n$ (for large n), and for all fixed $t \in [s,\infty)$. Thus $\mathbb{P}(\tau(\mathbb{D}) < t) = \mathbb{P}(\tau(\mathbb{D}_n) < t) = 0$, for $(S_0, I_0, R_0) \in \mathbb{D}$ and $t \ge t_0$, that is,

$$\mathbb{P}(\tau(\mathbb{D}) = \infty) = 1.$$

This proves the invariance property and the global existence of the solution $(S(t), I(t), R(t))$ on $\mathbb{D}$. Uniqueness and continuity of the solution is obtained by Theorem 2.1. □

4. Stochastic Asymptotic Stability of Infection-Free and Endemic Equilibrium States

In this section, we study the stochastic asymptotic stability of equilibrium solutions of (1.2). Generally, an epidemic model admits two types of equilibrium solution. The first one is the infection-free equilibrium solution E_1, whose global stability means biologically that the disease always dies out. The second one is the endemic equilibrium solution E_2. Epidemiologically, if E_2 is globally asymptotically stable, the disease will persist at the endemic equilibrium level if it is initially present.

The stochastic SIRS model (1.2) can have at most two equilibrium solutions, namely an infection-free equilibrium solution $E_1 = (S_1, I_1, R_1)$, and an endemic equilibrium solution $E_2 = (S_2, I_2, R_2)$. The endemic equilibrium solution exists if the condition

$$\mathcal{R}_0 > 1, \quad \text{and} \quad F_i(S_2, I_2, R_2) = 0, \qquad i = 1, 2,$$

holds.

Theorem 4.1. *The infection-free equilibrium solution* $E_1 = (S_1, I_1, R_1)$ *of* (1.2) *is globally stochastically asymptotically stable on* $\mathbb{D}$, *if* $\alpha\mu \geq \Lambda\beta$.

Proof. We use Theorem 2.5 and define a Lyapunov function

$$V_1(S,I,R) = \frac{1}{2}\left(S - \frac{\Lambda}{\mu} + I + R\right)^2 + \frac{\Lambda}{\mu}I + \frac{\Lambda}{\mu}R. \tag{4.1}$$

The infinitesimal generator $\mathcal{L}$ acting on the Lyapunov function can be written as:

$$\begin{aligned}
\mathcal{L}V_1(S,I,R) = &(\Lambda - \beta SI - \mu S + \gamma R)\left(S - \frac{\Lambda}{\mu} + I + R\right)\\
&+ (\beta SI - (k+\mu+\alpha)I)\left(S - \frac{\Lambda}{\mu} + I + R + \frac{\Lambda}{\mu}\right)\\
&+ (kI - (\mu+\gamma)R)\left(S - \frac{\Lambda}{\mu} + I + R + \frac{\Lambda}{\mu}\right)\\
&+ \frac{1}{2}S^2 I^2 F_1^2(S,I,R)(1-1-1+1)\\
&+ \frac{1}{2}I^2 F_2^2(S,I,R)(1-1-1+1).\\
\mathcal{L}V_1(S,I,R) = &\left(S - \frac{\Lambda}{\mu} + I + R\right)[\Lambda - \beta SI - \mu S + \gamma R\\
&+\beta SI - (k+\mu+\alpha)I + kI - (\mu+\gamma)R]\\
&+ \frac{\Lambda}{\mu}\beta SI - \frac{\Lambda}{\mu}(k+\mu+\alpha)I + \frac{\Lambda}{\mu}kI - \frac{\Lambda}{\mu}(\mu+\gamma)R.\\
\mathcal{L}V_1(S,I,R) = &-\mu\left(S - \frac{\Lambda}{\mu} + I + R\right)^2 - \alpha I(S - \frac{\Lambda}{\mu} + I + R)\\
&+ \frac{\Lambda}{\mu}\beta SI - \frac{\Lambda}{\mu}(\mu+\alpha)I - \frac{\Lambda}{\mu}(\mu+\gamma)R.\\
\mathcal{L}V_1(S,I,R) = &-\mu\left(S - \frac{\Lambda}{\mu} + I + R\right)^2 - \alpha I^2 - \left(\alpha - \frac{\Lambda}{\mu}\beta\right)SI\\
&- \alpha IR - \Lambda I - \frac{\Lambda}{\mu}(\mu+\gamma)R.
\end{aligned}$$

If $\alpha\mu \geq \Lambda\beta$, then $\mathcal{L}V_1(S,I,R)$ becomes negative definite on $\mathbb{D}$. By Theorem 2.5, the infection-free equilibrium solution E_1 of the stochastic SIRS model (1.2) is globally stochastically asymptotically stable on $\mathbb{D}$. □

The above theorem concludes that if $\alpha - \dfrac{\Lambda\beta}{\mu} \geq 0$, then the disease will die out. From this we conclude that $\mathcal{R}_0 < 1$ because the stability condition

$\alpha - \dfrac{\Lambda\beta}{\mu} \geq 0$ can be written in terms of the basic reproduction number as follows

$$\frac{\Lambda\beta}{\mu} \leq \alpha < \alpha + \mu + k \quad \Rightarrow \quad \frac{\Lambda\beta}{\mu(k+\alpha+\mu)} < 1.$$

Theorem 4.2. *The endemic equilibrium solution,* $E_2 = (S_2, I_2, R_2)$ *of the system* (1.2) *is stochastically asymptotically stable on* $\mathbb{D}$ *if* $\mathcal{R}_0 > 1$ *and satisfies* $\eta(S, I, R) \leq 0$, *where*

$$\begin{aligned}\eta(S,I,R) = &-(\mu + 2a\mu)(S - S_2)^2 - (\alpha + \mu)(I - I_2)^2 \\ &- (\mu + 2c(\mu+\gamma))(R - R_2)^2 \\ &+ \frac{1}{2} S^2 F_1^2(S,I,R)\left(2a\left(\frac{\Lambda}{\mu}\right)^2 + bI_2\right) \\ &+ \frac{1}{2} F_2^2(S,I,R)\left(bI_2 + 2c\left(\frac{\Lambda}{\mu}\right)^2\right),\end{aligned}$$

for $a = \dfrac{\mu}{\gamma}$, $b = \dfrac{\alpha + 2\mu}{\beta} + 2aS_2$ *and* $c = \dfrac{\alpha+2\mu}{2k}$.

Proof. Note that the conditions $\mathcal{R}_0 > 1$ and $F_i(S_2, I_2, R_2) = 0$, are needed for the existence of the endemic equilibrium solution. Define a Lyapunaov function.

$$\begin{aligned}V_2(S,I,R) =&\frac{1}{2}(S - S_2 + I - I_2 + R - R_2)^2 + a(S - S_2)^2 \\ &+ b\left(I - I_2 - I_2 \ln\left(\frac{I}{I_2}\right)\right) + c(R - R_2)^2,\end{aligned} \tag{4.2}$$

where a, b and c are positive constants and are chosen later.

The infinitesimal generator $\mathcal{L}$ acting on the Lyapunov function V_2 can be written as:

$$\begin{aligned}&\mathcal{L}V_2(S,I,R) \\ &= (S - S_2 + I - I_2 + R - R_2)\left(\Lambda - \mu S - (\mu+\alpha)I - \mu R\right) \\ &\quad + 2a(S - S_2)(\Lambda - \beta SI - \mu S + \gamma R) \\ &\quad + b\,(I - I_2)\left(\beta S - (k + \mu + \alpha)\right) + 2c(R - R_2)\left(kI - (\mu+\gamma)R\right) \\ &\quad + \frac{1}{2} S^2 F_1^2(S,I,R)\left(2aI^2 + bI_2\right) + \frac{1}{2} F_2^2(S,I,R)\left(bI_2 + 2cI^2\right).\end{aligned} \tag{4.3}$$

The following identities help to simplify $\mathcal{L}V_2(S, I, R)$

(i) $\Lambda - \mu S - (\mu+\alpha)I - \mu R = -\mu(S - S_2 + I - I_2 + R - R_2) - \alpha(I - I_2)$,
(ii) $\Lambda - \beta SI - \mu S + \gamma R = -\beta(S - S_2)I - \beta S_2(I - I_2) - \mu(S - S_2) + \gamma(R - R_2)$,

(iii) $\beta S - (k + \mu + \alpha) = \beta(S - S_2)$,
(iv) $kI - (\mu + \gamma)R = k(I - I_2) - (\mu + \gamma)(R - R_2)$.

Substituting the above identities into (4.3), we get

$$\begin{aligned}
\mathcal{L}V_2(S, I, R) = &- \mu \left(S - S_2 + I - I_2 + R - R_2\right)^2 - \alpha(S - S_2)(I - I_2) \\
&- \alpha(I - I_2)(R - R_2) - 2a\beta(S - S_2)^2 I - 2a\mu(S - S_2)^2 \\
&- 2a\beta S_2(S - S_2)(I - I_2) + 2a\gamma(S - S_2)(R - R_2) \\
&- \alpha(I - I_2)^2 + b\beta(S - S_2)(I - I_2) \\
&+ 2ck(I - I_2)(R - R_2) - 2c(\mu + \gamma)(R - R_2)^2 \\
&+ \frac{1}{2}S^2 F_1^2(S, I, R)\left(2aI^2 + bI_2\right) + \frac{1}{2}F_2^2(S, I, R)\left(bI_2 + 2cI^2\right).
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_2(S, I, R) = &- (\mu + 2a\beta I + 2a\mu)(S - S_2)^2 - (\alpha + \mu)(I - I_2)^2 \\
&- (\mu + 2c(\mu + \gamma))(R - R_2)^2 - (2\mu - 2a\gamma)(S - S_2)(R - R_2) \\
&- (2\mu + \alpha + 2a\beta S_2 - b\beta)(S - S_2)(I - I_2) \\
&- (\alpha + 2\mu - 2ck)(I - I_2)(R - R_2) \\
&+ \frac{1}{2}S^2 F_1^2(S, I, R)\left(2aI^2 + bI_2\right) + \frac{1}{2}F_2^2(S, I, R)\left(bI_2 + 2cI^2\right).
\end{aligned}$$

Choose $a = \dfrac{\mu}{\gamma}$, $b = \dfrac{\alpha + 2\mu}{\beta} + 2aS_2$ and $c = \dfrac{\alpha + 2\mu}{2k}$, we have

$$\begin{aligned}
\mathcal{L}V_2(S, I, R) = &- (\mu + 2a\beta I + 2a\mu)(S - S_2)^2 - (\alpha + \mu)(I - I_2)^2 \\
&- (\mu + 2c(\mu + \gamma))(R - R_2)^2 + \frac{1}{2}S^2 F_1^2(S, I, R)\left(2aI^2 + bI_2\right) \\
&+ \frac{1}{2}F_2^2(S, I, R)\left(bI_2 + 2cI^2\right).
\end{aligned}$$

Hence $\mathcal{L}V_2(S, I, R) = 0$ only at (S_2, I_2, R_2) and by the choice of suitable functions $F_i(S, I, R)$, one can obtain $\mathcal{L}V_2(S, I, R) < 0$ on $\mathbb{D} \setminus (S_2, I_2, R_2)$. Hence $\mathcal{L}V_2(S, I, R)$ is negative definite on $\mathbb{D}$ for some suitable $F_i(S, I, R)$, $i = 1, 2$. Therefore, by Theorem 2.5, the endemic equilibrium is stochastically asymptotically stable on $\mathbb{D}$ if $\mathcal{R}_0 > 1$ and for some suitable $F_i(S, I, R)$, $i = 1, 2$ such that $F_i(S_2, I_2, R_2) = 0$, $i = 1, 2$ and satisfies the condition $\eta(S, I, R) \leq 0$ on $\mathbb{D}$. □

5. Stationary Distribution and Positive Recurrence

In this section, we discuss positive recurrence and stationary distribution of the model (1.2).

Theorem 5.1. *The solution* $(S(t), I(t), R(t))$ *of system* (1.2) *with any positive initial value* $(S(0), I(0), R(0)) \in \mathbb{D}$, *where* $F_1(S,I,R) = f_1(S - S_2) \not\equiv 0$ *and* $F_2(S,I,R) = f_2(I - I_2) \not\equiv 0$ *is positive recurrent and admits a unique ergodic stationary distribution in* $\mathbb{D}$ *if* $\mathcal{R}_0 > 1$ *and*

$$\begin{aligned}\eta_1(S,I,R) =&(\mu + 2a_1\mu)(S-S_2)^2 + (\alpha+\mu)(I-I_2)^2\\ &-\frac{1}{2}\left(\frac{\Lambda}{\mu}\right)^2 F_1^2(S,I,R)\left(2a_1\left(\frac{\Lambda}{\mu}\right)^2 + b_1 I_2\right)\\ &-\frac{1}{2}F_2^2(S,I,R)\left(b_1 I_2 + 2c_1\left(\frac{\Lambda}{\mu}\right)^2\right)\end{aligned}$$

is positive on H^c, *for* $a_1 = \dfrac{\mu}{\gamma}$, $b_1 = \dfrac{\alpha+2\mu}{\beta} + 2a_1 S_2$ *and* $c_1 = \dfrac{\alpha+2\mu}{2k}$.

Proof. Define the following bounded open subset H of $\mathbb{R}^3$

$$H = \left\{(S,I,R) \in \mathbb{D}\middle| N < S, I < \frac{\Lambda}{\mu} - N, M < R < \frac{\Lambda}{\mu} - M\right\},$$

where N and M are positive constants to be chosen in the following, S_2 or $I_2 \notin \overline{H}$ and $R_2 \in H$. The diffusion matrix associated with the system (1.2) is given by

$$A(S,I,R) = \begin{pmatrix} S^2I^2F_1^2(S,I,R) & -S^2I^2F_1^2(S,I,R) & 0\\ -S^2I^2F_1^2(S,I,R) & S^2I^2F_1^2(S,I,R) + I^2F_2^2(S,I,R) & -I^2F_2^2(S,I,R)\\ 0 & -I^2F_2^2(S,I,R) & I^2F_2^2(S,I,R)\end{pmatrix}.$$

Since $\overline{H} \subset \mathbb{R}^3_+$, then

$$\begin{aligned}a_{22}(S,I,R) &= S^2I^2F_1^2(S,I,R) + I^2F_2^2(S,I,R)\\ &\geq \min_{(S,I,R)\in\overline{H}} S^2I^2F_1^2(S,I,R) + I^2F_2^2(S,I,R)\\ &\geq k_1,\end{aligned}$$

where k_1 is positive constant. This implies that, the condition (i) in Lemma 2.6 is satisfied. It remains for us to verify the condition (ii) in Lemma 2.6. We define the following non-negative function

$$\begin{aligned}V_2(S,I,R) =&\frac{1}{2}(S - S_2 + I - I_2 + R - R_2)^2 + a_1(S-S_2)^2\\ &+ b_1\left(I - I_2 - I_2\ln\left(\frac{I}{I_2}\right)\right) + c_1(R-R_2)^2,\end{aligned}$$

where $a_1 = \dfrac{\mu}{\gamma}$, $b_1 = \dfrac{\alpha+2\mu}{\beta} + 2a_1 S_2$ and $c_1 = \dfrac{\alpha+2\mu}{2k}$. Apply $\mathcal{L}$ on $V_2(S, I, R)$, we have

$$\begin{aligned}\mathcal{L}V_2(S,I,R) = &-(\mu + 2a_1\beta I + 2a_1\mu)(S-S_2)^2 - (\alpha+\mu)(I-I_2)^2 \\ &-(\mu + 2c_1(\mu+\gamma))(R-R_2)^2 + \frac{1}{2}S^2F_1^2(S,I,R)\left(2a_1I^2 + b_1I_2\right) \\ &+ \frac{1}{2}F_2^2(S,I,R)\left(b_1I_2 + 2c_1I^2\right).\end{aligned}$$

Since $S + I + R \le \dfrac{\Lambda}{\mu}$, we get

$$\mathcal{L}V_2(S,I,R) \le -\left(\eta_1(S,I,R) + (\mu + 2c_1(\mu+\gamma))(R-R_2)^2\right).$$

Since $\eta_1(S, I, R) > 0$ on H^c, we have

$$\begin{aligned}&\eta_1(S,I,R) + (\mu + 2c_1(\mu+\gamma))(R-R_2)^2 \\ &\ge \inf_{(S,I,R)\in H^c} (\mu + 2c_1(\mu+\gamma))(R-R_2)^2 = \theta > 0.\end{aligned}$$

From this, we have

$$\mathcal{L}V_2(S,I,R) \le -\theta \qquad \text{for all} \qquad (S,I,R) \in H^c.$$

So the condition (ii) of Lemma 2.6 is met. This completes the proof. □

6. Numerical Simulation

In this section we visualize our results. Consider the stochastic SIRS model.

$$\begin{aligned} dS &= (\Lambda - \beta SI - \mu S + \gamma R)\,dt - \left(\frac{\mu}{\Lambda}\right)^3 SI(S-S_2)dW_1, \\ dI &= (\beta SI - (k+\mu+\alpha)I)\,dt + \left(\frac{\mu}{\Lambda}\right)^3 SI(S-S_2)dW_1 \\ &\quad - \left(\frac{\mu}{\Lambda}\right)^2 I(I-I_2)dW_2, \\ dR &= (kI - (\mu+\gamma)R)\,dt + \left(\frac{\mu}{\Lambda}\right)^2 I(I-I_2)dW_2, \end{aligned} \tag{6.1}$$

where Λ, μ, β, k, α and γ are positive constants and an endemic equilibrium $E_2 = (S_2, I_2, R_2)$.

Global existence of a unique solution of the system (6.1) in

$$\mathbb{D} = \left\{(S,I,R) \in \mathbb{R}^3 : S > 0,\ \ I > 0,\ \ R > 0,\ \ S + I + R \le \frac{\Lambda}{\mu}\right\}$$

is proved by Theorem 3.1.

In Figure 1(A), 1(B), 1(C) and 2(A), 2(B), 2(C), dynamics of expected values of Susceptible, Infected and Recovered versus time are plotted. They show that Susceptible, Infected and Recovered populations, in average, settle around the equilibrium.

Figure 1(D), 1(E), 1(F) and 2(D), 2(E), 2(F) display the evaluation of the variances of Susceptible, Infected and Recovered versus time. As it is seen, variances rapidly go to zero. Hence the equilibrium solutions are approached.

Figure 1 verifies Theorem 4.1 which states, if $\alpha\mu - \Lambda\beta = 0.0475 \geq 0$ then the disease free equilibrium solution $E_1 = (18.1818, 0, 0)$ of the system (6.1) is globally stochastically asymptotically stable on $\mathbb{D}$.

Figure 2 agrees to Theorem 4.2 which proves stochastic asymptotic stability of the endemic equilibrium solution $E_2 = (9.0000, 7.6190, 4.5714)$ to the system (6.1) on $\mathbb{D}$ under the assumption $\mathcal{R}_0 = 2.7778 > 1$ and η is negative definite, which requires non-negative of the constant

$$\phi = \mu + 2a\mu - \frac{1}{2}\left(\frac{\mu}{\Lambda}\right)^4 \left(2a\left(\frac{\Lambda}{\mu}\right)^2 + bI_2\right) = 3.5928$$

and

$$\xi = \alpha + \mu - \frac{1}{2}\left(\frac{\mu}{\Lambda}\right)^4 \left(bI_2 + 2c\left(\frac{\Lambda}{\mu}\right)^2\right) = 0.5965,$$

where $\eta(S, I, R) = -\phi(S - S_2)^2 - \xi(I - I_2)^2 - (\mu + 2c(\mu + \gamma))(R - R_2)^2$, for $a = \frac{\mu}{\gamma}$, $b = \frac{\alpha + 2\mu}{\beta} + 2aS_2$ and $c = \frac{\alpha + 2\mu}{2k}$. If we choose the parameter $\Lambda = 10, \mu = 0.4, \alpha = 0.2, \beta = 0.1, k = 0.3$ and $\gamma = 0.1$ then the conditions of Theorem 5.1 are holds, hence the system (6.1) is positive recurrent, moreover, the positive recurrent has unique stationary distribution in $\mathbb{D}$.

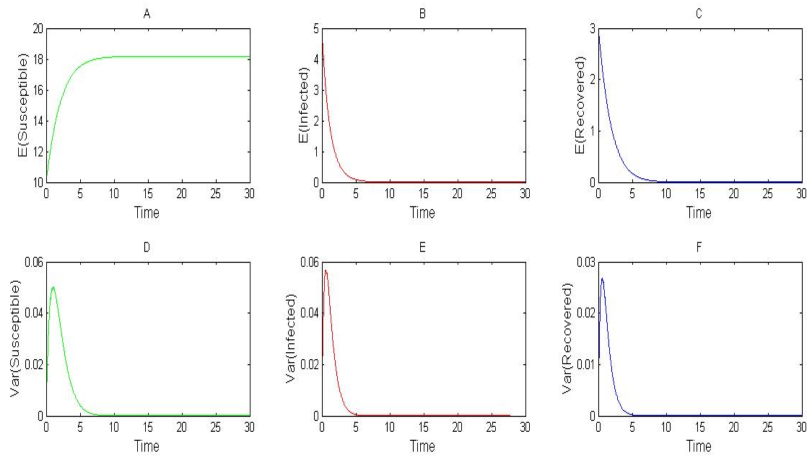

Figure 1. The disease free equilibrium $(S_1, I_1, R_1) = (18.1818, 0, 0)$ is globally asymptotically stochastically stable for the parameters: $\Lambda = 10, \mu = 0.55, \alpha = 0.45, \beta = 0.02, k = 0.1$ and $\gamma = 0.1$ ($R_0 = 0.3306 < 1$).

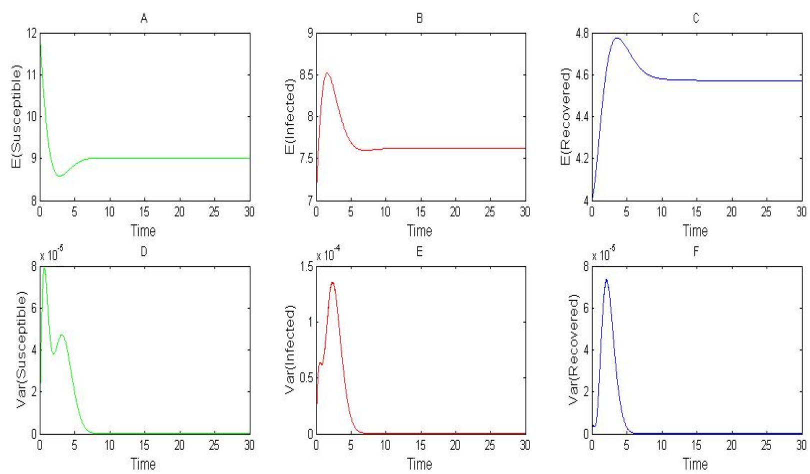

Figure 2. The endemic equilibrium $E_2 = (S_2, I_2, R_2) = (9.0000, 7.6190, 4.5714)$ is asymptotically stochastically stable for the parameters: $\Lambda = 10, \mu = 0.4, \alpha = 0.2, \beta = 0.1, k = 0.3$ and $\gamma = 0.1$ ($R_0 = 2.7778 > 1$).

7. Summary

In this paper, we consider general diffusion term. Hence, we have a family of stochastic SIRS model. We established a positive unique solution for the model (1.2), which is essential for any population dynamics models. We discussed stochastic asymptotic stability of infection-free and endemic equilibria with the help of invariance principle and Lyapunov's second method. Commonly, stochastic asymptotic stability of equilibria is connected to the basic reproduction number $\mathcal{R}_0$. A sufficient condition for stochastic asymptotic stability is found in terms of parameters and functional dependence on the variable. A remarkable fact of the criteria $\eta(S, I, R) \leq 0$ on $\mathbb{D}$ is that a sufficient condition for stability can be found even for general local Lipschitz continuous F_i's. Our results reveal that a certain type of stochastic perturbation may help stabilize the system and we also proved that the solution of the system (1.2) is positive recurrent and admits a unique ergodic stationary distribution in $\mathbb{D}$. Furthermore, we visualized our results numerically.

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