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HARMONIC INVEX SETS AND GENERALIZED HARMONIC VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we have defined various type of generalized harmonic variational inequality problems in harmonic invex set and studied their existence theorems under certain conditions.

1. Introduction

In 2014, Insan [5] has introduced the concept of harmonically convex set and harmonically convex functions and has studied the Hermite-Hadamard type inequalities for harmonically convex functions as an extension work of Hermite-Hadamard inequalities. We recall the concepts of harmonic sets and harmonic convex function introduced by Insan [5].

Definition 1.1. ([5]) Let $K \subset \mathbb{R} \setminus \{0\}$ be any set and $f : K \to \mathbb{R}$ be any map.

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(1) K is said to be *harmonically convex set* if

$$\frac{xy}{tx+(1-t)y} \in K$$

for all $x, y \in K$ and $t \in [0,1]$,

(2) f is said to be *harmonically convex function* on the harmonically convex set K if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in K$ and $t \in [0,1]$.

2. Generalized harmonic variational inequality problems

Let X be a topological vector space in a separable Banach space. Let $K \subset X \setminus \{0\}$ be any set. For $y = (y_1, y_2, \cdots) \in K$ and $v = (v_1, v_2, \cdots) \in X$, let

$$K(v) = \{y + tv : \; x, y \in K, \;\; t \in [0,1]\}$$

be a nonempty set in X. Let

$$\begin{aligned}
\Pi^2(K) &= K \circ K \\
&= \{xy = (x_1y_1, x_2y_2, \cdots) : x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots) \in K\}, \\
\frac{1}{K} &= \left\{\frac{1}{x} : \frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots\right), x = (x_1, x_2, \cdots) \in K\right\}, \\
\Pi^2(K; W) &= \frac{K \circ K}{W} = \left\{\frac{xy}{w} : x, y \in K, \;\; w \in K, \;\; t \in [0,1]\right\}
\end{aligned}$$

be the subspaces of X. For $x, y \in K$, let $I[y,x]$ be a path joining y and x contained in K and the map $\gamma_{xy} : [0,1] \to I[y,x]$ be continuous. The set K has the invariant harmonic convex (IHC) combination property in a given direction $v \in X$ if the following are satisfied:

(P1) for all $t \in [0,1]$, $v \in X$ and $y \in K$,

$$y + tv = (y_1 + tv_1, y_2 + tv_2, \cdots) \in K,$$

(P2) for all $x, y \in K$ and $v = v(x,y)$, we have

$$\frac{xy}{y+tv} = \left(\frac{x_1y_1}{y_1+tv_1}, \frac{x_2y_2}{y_2+tv_2}, \cdots\right) \in I[y,x] = \Pi^2(K; K(v)) \subset K$$

if and only if

$$\frac{y+tv}{xy} = \left(\frac{y_1+tv_1}{x_1y_1}, \frac{y_2+tv_2}{x_2y_2}, \cdots\right) \in I\left[\frac{1}{x}, \frac{1}{y}\right] \subset \frac{1}{K}.$$

For our need, we make the concept of harmonically η-Lipschitz continuous functions.

Definition 2.1. Let $F : K \to \mathbb{R} \setminus \{0\}$ be any mapping. For each $y \in K$, the set of harmonically η-Lipschitz continuous functions $LC_F(y)$ and $LC_F(1/y)$ are defined by

$$\begin{aligned} LC_F(y) &= \{x \in K \setminus \{0\} : |F(x) - F(y)| \\ &\qquad \le \mathrm{L}_1 \max\left\{ \|\eta(x,y)\|, \left\| \frac{xy}{\eta(x,y)} \right\| \right\} \right\} \end{aligned}$$

and

$$\begin{aligned} LC_F(1/y) &= \left\{ x \in K \setminus \{0\} : \left| F\left(\frac{1}{y}\right) - F\left(\frac{1}{x}\right) \right| \right. \\ &\qquad \left. \le \mathrm{L}_2 \max\left\{ \|\eta(x,y)\|, \left\| \frac{xy}{\eta(x,y)} \right\| \right\} \right\} \end{aligned}$$

respectively where L_1 and L_2 are the Lipschitz constants.

Let $\eta : K \times K \to X$ be any map. Let $F : K \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ be harmonically η-Lipschitz continuous near each $y \in K \setminus \{0\}$. For any nonlinear map $T : K \to X^*$, the pairing $\langle T(y), z \rangle$ is defined by

$$\langle T(y), z \rangle = \sum_i \langle T(y_i), z_i \rangle$$

for all $z \in X$, $y \in K$.

The generalized harmonic variational inequalities are defined as follows:

(a) The generalized harmonic variational inequality problem (GHVIP) is to find: $y \in K$ such that

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \ge 0 \quad \text{for all } x \in K. \tag{GHVIP}$$

(b) The generalized dual harmonic variational inequality problem (GDHVIP) is to find: $y \in K$ such that

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle \le 0 \quad \text{for all } x \in K. \tag{GDHVIP}$$

(c) The generalized harmonic complementarity problem (GHCP) is to find: $y \in K$ such that

$$\left\langle T(x), \frac{yy}{\eta(y,x)} \right\rangle = 0 \quad \text{for all } x \in K. \tag{GHCP}$$

In this section, we have studied the existence theorems of the above problems under certain conditions. For our need, we make the definition of invariant harmonically convex function in the invariant harmonic convex set K.

Definition 2.2. Let $K \subset \mathbb{R} \setminus \{0\}$ be any set and $f : K \to \mathbb{R}$ be any map.

(1) K is said to be *an IHC set* given in the direction $v \in \mathbb{R} \setminus \{0\}$ if K has the IHC combination properties P_1 and P_2.

(2) f is said to be *an invariant harmonically convex* (IHC) *function* on the IHC set K if

$$f\left(\frac{xy}{y+tv}\right) \leq tf(y) + (1-t)f(x) \tag{2.1}$$

for all $x, y \in K$, $t \in [0,1]$.

Definition 2.3. Let $\eta : K \times K \to X$ be any map.

(1) The set K is said to be *η-invex* if $y + t\eta(x,y) \in K$ for all $x, y \in K$ and $t \in [0,1]$, *i.e.*,

$$K(\eta) = \{y + t\eta(x,y) \in K : x, y \in K, t \in [0,1]\}$$

η-invex set,

(2) The *kernel* of K is defined by

$$Ker(K(\eta)) = \{y \in K : y + t\eta(x,y) \in K(\eta), x \in K, t \in [0,1]\}.$$

For our need, we define the concept of harmonic η-invex set and orthonormal η-invex set as follows:

Definition 2.4. Let $\eta : K \times K \to X$ be any map. The set K is said to be

(1) *a harmonic η-invex set* if for all $x, y \in K$ and $t \in [0,1]$,

$$\begin{aligned} y + t\eta(x,y) \in K \iff \frac{xy}{y+t\eta(x,y)} &= \left(\frac{y+t\eta(x,y)}{xy}\right)^{-1} \\ &= \left(\frac{1}{x} + t\frac{\eta(x,y)}{xy}\right)^{-1} \in K, \end{aligned}$$

(2) *an orthonormal harmonic η-invex set* if K is harmonic η-invex set and there exists an orthonormal basis $B \subset K$ such that each $z \in K$ can be written as $z = \sum_{b \in B} < z, b > b$ with norm $\|z\|^2 = \sum_{b \in B} |< z, b >|^2$ and for each $y \in K$ and $t \in [0,1]$,

$$y + t\eta(x,y) \in K \iff \frac{1}{y+t\eta(x,y)} \in K$$

for all $x \in A(y) = \{v \in K : vy = 1\}$.

As an extension, we introduce the concept of harmonically pre η-invex (incave) function associated with $\lambda(t)$ on harmonic η-invex set where the mapping $\lambda : [0,1] \to \mathbb{R}_+$ satisfies the condition

$$\lim_{t\to 0} \lambda(t) = 0 \quad \text{and} \qquad \lim_{t\to 0} \frac{\lambda(t)}{t} = 1.$$

Definition 2.5. Let $F : K \to \mathbb{R} \setminus \{0\}$ be any map. The mapping F is said to be

(1) *harmonically pre η-invex (incave)* on harmonic η-invex set K associated with $\lambda(t)$ (defined above) if

$$F\left(\frac{xy}{y + t\eta(x,y)}\right) \geq (\leq) \lambda(t) F(y) + (1 - \lambda(t)) F(x)$$

for all $x, y \in K$, and $t \in [0,1]$,

(2) *harmonically pre η-invex (incave)* on orthonormal harmonic η-invex set K if

$$F\left(\frac{1}{y + t\eta(x,y)}\right) \geq (\leq) tF(y) + (1 - t) F(x)$$

for all $x \in A(y)$, $y \in K$ and $t \in [0,1]$.

For strict case, the symbol $\geq$ $(\leq)$ is to be replaced by $>$ $(<)$.

Definition 2.6. The mapping $T : K \to X^*$ is said to be

(1) *harmonically η-monotone* on $K(\eta)$ if

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \leq 0$$

for all $x, y \in K(\eta)$. For strictly harmonically η-monotonicity case, equality holds in the above equation for $x = y$ only,

(2) *harmonically η-monotone* at $y \in K(\eta)$ if

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \leq 0$$

for all $x \in K(\eta)$.

Definition 2.7. The mapping $T : K \to X^*$ is said to be

(1) *generalized harmonically η-monotone* on $K(\eta)$ if

$$\left\langle T(x) - T(y), \frac{xy}{\eta(x,y)} \right\rangle \geq 0$$

for all $x, y \in K(\eta)$. For strictly generalized harmonically η-monotonicity case, equality holds in the above equation for $x = y$ only,

(2) *generalized harmonically η-monotone* at $y \in K(\eta)$ if

$$\left\langle T(x) - T(y), \frac{xy}{\eta(x,y)} \right\rangle \geq 0$$

for all $x \in K(\eta)$.

If $\eta(y,x) = -\eta(x,y)$, then harmonically η-monotonicity property of T reduces to generalized harmonically η-monotonicity property of T.

Definition 2.8. The mapping $T : K \to X^*$ is said to be a *harmonic pre-η-invex function* on $K_h(\eta)$ if for all x, y, and $x_t = y + t\eta(x,y) \in K(\eta)$, we have

$$\left\langle T(x_t), \frac{x_t y}{\eta(x_t, x_t)} \right\rangle \leq (1 - \lambda(t)) \left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \lambda(t) \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle,$$

where the mapping $\lambda : [0,1] \to \mathbb{R}_+$ satisfies the condition

$$\lim_{t \to 0} \lambda(t) = 0 \text{ and } \lim_{t \to 0} \frac{\lambda(t)}{t} = 1.$$

Definition 2.9. Let $K \subset \mathbb{R} \setminus \{0\}$ be a nonempty set and $F : K \to \mathbb{R} \setminus \{0\}$ be a map. The point $y \in K$ is said to be

(1) minimum point of F if $F(x) \geq F(y)$ for all $x \in K$,
(2) maximum point of F if $F(x) \leq F(y)$ for all $x \in K$.

Theorem 2.10. *Let $\eta : K \times K \to X$ be any map and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. If the mapping $T : K \to X^*$ is harmonically η-monotone on $K(\eta)$ and $y \in K(\eta)$ solves the problem* (GHVIP)*, then $y \in K(\eta)$ solves the problem* (GDHVIP)*.*

Proof. Since $y \in K(\eta)$ solves the problem (GHVIP), we have

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \geq 0$$

for all $x \in K(\eta)$. We have T is harmonically η-monotone on $K(\eta)$, implying T is harmonically η-monotone at $y \in K(\eta)$, *i.e.*,

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \leq 0$$

for all $x \in K(\eta)$, *i.e.*,

$$\begin{aligned} \left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle &= \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \\ &\leq 0 \end{aligned}$$

for all $x \in K(\eta)$. This means that $y \in K(\eta)$ solves the problem (GDHVIP). This completes the proof. $\square$

Theorem 2.11. *Let $\eta : K \times K \to X$ be a map and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. Assume that the following conditions hold:*

(a) *For all $x, y \in K(\eta)$,*

$$\left\langle T(x), \frac{xy}{\eta(x,x)} \right\rangle = 0,$$

(b) *The mapping $T : K \to X^*$ is harmonic pre-η-invex function on $K(\eta)$.*
If $y \in K(\eta)$ solves the problem (GDHVIP), *then $y \in K(\eta)$ solves the problem* (GHVIP).

Proof. Since $y \in K(\eta)$ solves the problem (GDHVIP), we have

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle \leq 0$$

for all $x \in K(\eta)$. Now at $y \in K(\eta)$, we have

$$\left\langle T(x_t), \frac{x_t y}{\eta(x_t,x_t)} \right\rangle = 0$$

for all x, y, and $x_t = y + t\eta(x,y) \in K(\eta)$. By harmonic pre-$\eta$-invexity of T on $K_h(\eta)$, we have

$$\begin{aligned} 0 &= \left\langle T(x_t), \frac{x_t y}{\eta(x_t,x_t)} \right\rangle \\ &\leq (1-\lambda(t)) \left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \lambda(t) \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \end{aligned}$$

for all $x \in K(\eta)$, *i.e.*,

$$\begin{aligned} \lambda(t) \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle &= -(1-\lambda(t)) \left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle \\ &\geq 0 \end{aligned}$$

for all $x \in K(\eta)$ and $t \in (0,1]$. Thus, we have

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \geq 0$$

for all $x \in K(\eta)$, that is, $y \in K(\eta)$ solves the problem (GHVIP). This completes the proof. $\square$

Definition 2.12. Let X be a topological vector space and K be a nonempty subset of X. A vector function $\eta : K \times K \to X$ is said to satisfy the *condition* C_0 if the followings hold:

(1) $\eta(z, x') + \eta(x', z) = 0$, where $z = x' + \eta(x, x')$,
(2) $\eta(x' + t\eta(x, x'), x') + t\eta(x, x') = 0, \quad \forall x, x' \in K$ and $\quad \forall t \in (0, 1)$.

Theorem 2.13. *Let $\eta : K \times K \to X$ be any map satisfying condition C_0 and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. Assume that*

(a) *$T : K \to X^*$ is a harmonically η-monotone mapping on $K(\eta)$,*
(b) *the map $y \mapsto \langle T(y), \frac{xy}{\eta(x,y)}\rangle$ of K into $L(X, Y)$ is continuous on the finite dimensional subspaces (or at least hemicontinuous).*

If $y \in K(\eta)$ solves the problem (GHVIP)*, then $y \in K(\eta)$ solves the problem* (GHCP).

Proof. Since $y \in K(\eta)$ solves the problem (GHVIP), we have

$$\left\langle T(y), \frac{xy}{\eta(x, y)} \right\rangle \geq 0$$

for all $x \in K(\eta)$. Since η satisfies condition C_0, we have $\eta(x_t, y) = -t\eta(x, y)$ for $x_t = y + t\eta(x, y) \in K(\eta)$. Substituting x by x_t in the above inequality and using condition C_0, we get

$$\left\langle T(y), \frac{x_t y}{\eta(x_t, y)} \right\rangle \geq 0,$$

that is,

$$-t \left\langle T(y), \frac{x_t y}{\eta(x, y)} \right\rangle \geq 0$$

implying

$$\left\langle T(y), \frac{x_t y}{\eta(x, y)} \right\rangle \leq 0,$$

for all $x \in K(\eta)$. Taking limit as $t \to 0$ and using the condition of hemicontinuity, we have

$$\left\langle T(y), \frac{yy}{\eta(x, y)} \right\rangle \leq 0,$$

for all $x \in K(\eta)$. Again substituting x by x_t in the above inequality and using condition C_0, we get

$$\left\langle T(y), \frac{yy}{\eta(x, y)} \right\rangle \geq 0,$$

for all $x \in K(\eta)$. Thus, we have

$$\left\langle T(y), \frac{yy}{\eta(x, y)} \right\rangle = 0,$$

for all $x \in K(\eta)$, that is, $y \in K(\eta)$ solves the problem GHCP. This completes the proof. □

3. Generalized F-harmonic variational inequalities

Let $F : K \to \mathbb{R} \setminus \{0\}$ be harmonically η-Lipschitz continuous near each $y \in K$. For any nonlinear map $T : K \to X^*$, the generalized harmonic variational inequalities are defined as follows:

(a) The generalized harmonic variational inequality problem associated with F (GHVIP$_F$) is to find: $y \in K$ such that

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + F(x) - F(y) \geq 0 \text{ for all } x \in K. \qquad (\text{GHVIP}_F)$$

(b) The generalized dual harmonic variational inequality problem associated with F (GDHVIP$_F$) is to find: $y \in K$ such that

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + F(x) - F(y) \leq 0 \text{ for all } x \in K. \qquad (\text{GDHVIP}_F)$$

(c) The generalized harmonic variational inequality problem associated with ξ (GHVIP$_\xi$) is to find: $y \in K \setminus \{0\}$ such that $\xi \in \partial F(y)$ and

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + \langle \xi, \eta(x,y) \rangle \geq 0 \text{ for all } x \in K. \qquad (\text{GHVIP}_\xi)$$

(d) The generalized dual harmonic variational inequality problem associated with ξ (GDHVIP$_\xi$) is to find: $y \in K$ such that $\xi \in \partial F(y)$ and

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \langle \xi, \eta(y,x) \rangle \geq 0 \text{ for all } x \in K. \qquad (\text{GDHVIP}_\xi)$$

The following theorems proves the equivalence between the problems GHVIP$_F$ and GDHVIP$_F$, GHVIP$_F$ and GHVIP$_\xi$.

Theorem 3.1. *Let $\eta : K \times K \to X$ be a map and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. Let $F : K \to \mathbb{R}$ and $T : K \to X^*$ be two mappings. Let T be harmonically η-monotone on $K(\eta)$ and $y \in K(\eta)$ be a maximal point of F. If $y \in K(\eta)$ solves the problem GHVIP$_F$, then $y \in K(\eta)$ solves the problem GDHVIP$_F$.*

Proof. Since $y \in K(\eta)$ is the maximal point of F, we have

$$F(x) - F(y) \leq 0$$

for all $x \in K(\eta)$. Since T is harmonically η-monotone on $K(\eta)$, T is harmonically η-monotone at $y \in K(\eta)$, that is,

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle \leq 0$$

for all $x \in K(\eta)$. Hence we have

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + 2\left[F(x) - F(y)\right] \leq 0$$

for all $x \in K(\eta)$, that is,

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + F(x) - F(y) \leq -\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle - F(x) + F(y)$$

for all $x \in K(\eta)$. Since $y \in K(\eta)$ solves the problem GHVIP$_F$, we have

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + F(x) - F(y) \geq 0$$

for all $x \in K(\eta)$, this implies that

$$-\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle - F(x) + F(y) \geq 0$$

for all $x \in K(\eta)$. That is,

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + F(x) - F(y) \leq 0$$

for all $x \in K(\eta)$. This means that $y \in K(\eta)$ solves the problem GDHVIP$_F$. This completes the proof. □

Theorem 3.2. *Let $\eta : K \times K \to X$ be a map and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. Assume that the conditions hold:*

(a) *For all $x, y \in K(\eta)$,*

$$\left\langle T(x), \frac{xy}{\eta(x,x)} \right\rangle = 0.$$

(b) *The mapping $T : K \to X^*$ is harmonic pre-η-invex on $K(\eta)$.*

If $y \in K(\eta)$ is the minimum point of F which solves the problem GDHVIP$_F$, then $y \in K(\eta)$ solves the problem GHVIP$_F$.

Proof. Since $y \in K(\eta)$ is the minimum point of F, we have

$$F(x) - F(y) \geq 0$$

for all $x \in K(\eta)$. Again $y \in K(\eta)$ solves the problem GDHVIP$_F$, *i.e.*,

$$\left\langle T(x), \frac{xy}{\eta(y,x)} \right\rangle + F(x) - F(y) \leq 0$$

for all $x \in K(\eta)$. Now at $y \in K(\eta)$, we have

$$\left\langle T(x_t), \frac{x_t y}{\eta(x_t,x_t)} \right\rangle = 0$$

for all x, y, and $x_t = y + t\eta(x, y) \in K(\eta)$. By harmonic pre-$\eta$-invexity of T on $K(\eta)$, we have

$$\begin{aligned}
0 &= \left\langle T(x_t), \frac{x_t y}{\eta(x_t, x_t)} \right\rangle \\
&\leq \left\langle T(x_t), \frac{x_t y}{\eta(x_t, x_t)} \right\rangle + F(x) - F(y) \\
&\leq (1-\lambda(t)) \left\langle T(x), \frac{xy}{\eta(y, x)} \right\rangle + \lambda(t) \left\langle T(y), \frac{xy}{\eta(x, y)} \right\rangle + F(x) - F(y) \\
&= (1-\lambda(t)) \left[\left\langle T(x), \frac{xy}{\eta(y, x)} \right\rangle + F(x) - F(y) \right] \\
&\quad + \lambda(t) \left[\left\langle T(y), \frac{xy}{\eta(x, y)} \right\rangle + F(x) - F(y) \right]
\end{aligned}$$

for all $x \in K(\eta)$, that is,

$$\begin{aligned}
&\lambda(t) \left[\left\langle T(y), \frac{xy}{\eta(x, y)} \right\rangle + F(x) - F(y) \right] \\
&= -(1-\lambda(t)) \left[\left\langle T(x), \frac{xy}{\eta(y, x)} \right\rangle + F(x) - F(y) \right] \\
&\geq 0
\end{aligned}$$

for all $x \in K(\eta)$ and $t \in (0, 1]$. Thus, we have

$$\left\langle T(y), \frac{xy}{\eta(x, y)} \right\rangle + F(x) - F(y) \geq 0$$

for all $x \in K(\eta)$. This means that $y \in K(\eta)$ solves the problem GHVIP$_F$. This completes the proof. □

Theorem 3.3. *Let $\eta : K \times K \to X$ be a map and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. Let $F : K \to \mathbb{R}$ be a mapping and $T : K \to X^*$ be a mapping with the condition:*

$$\left\langle T(y), \frac{x_t y}{\eta(x_t, y)} \right\rangle \leq \left\langle T(y), \frac{\lambda(t) xy}{\eta(x, y)} \right\rangle$$

for all $x_t \in K(\eta)$, $t \in [0, 1]$. If $y \in K(\eta)$ solves the problem GHVIP$_F$, then $y \in K(\eta)$ solves the problem GHVIP$_\xi$.

Proof. Since $y \in K(\eta)$ solves the problem GHVIP$_F$, we have

$$\left\langle T(y), \frac{xy}{\eta(x, y)} \right\rangle + F(x) - F(y) \geq 0$$

for all $x \in K(\eta)$, *i.e.*,

$$F(x) - F(y) \geq -\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle$$

for all $x \in K(\eta)$. Replacing x by $x_t = y + t\eta(x,y)$, we have

$$\begin{aligned} F(x_t) - F(y) &\geq -\left\langle T(y), \frac{x_t y}{\eta(x_t,y)} \right\rangle \\ &= -\left\langle T(y), \frac{\lambda(t) xy}{\eta(x,y)} \right\rangle, \end{aligned}$$

$$F(y + t\eta(x,y)) - F(y) \geq -\lambda(t) \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle$$

for all $x \in K(\eta)$ and $t \in (0,1)$. Dividing both sides by $\lambda(t)$ and taking limit as $t \to 0$, we have

$$\langle \xi, \eta(x,y) \rangle \geq -\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle$$

for all $x \in K(\eta)$ and $\xi \in \partial F(y)$ (the subdifferential of F at y). Thus

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + \langle \xi, \eta(x,y) \rangle \geq 0$$

for all $x \in K(\eta)$, that is, $y \in K(\eta)$ solves the problem GHVIP$_\xi$. This completes the proof. □

Theorem 3.4. *Let $\eta : K \times K \to X$ be a map and $K(\eta) \subset K \subset X$ be a harmonically η-invex set. Let $F : K \to \mathbb{R}$ be a nonsmooth ξ-η mapping on $K(\eta)$. Let $T : K \to X^*$ be any mapping. If $y \in K(\eta)$ solves the problem GHVIP$_\xi$, then $y \in K(\eta)$ solves the problem GHVIP$_F$.*

Proof. Since $y \in K(\eta)$ solves the problem GHVIP$_\xi$, we have

$$\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + \langle \xi, \eta(x,y) \rangle \geq 0$$

for all $x \in K(\eta)$. Since $F : K \to \mathbb{R}$ is nonsmooth ξ-η mapping on $K(\eta)$, for $y \in K(\eta)$ we have

$$F(x) - F(y) \geq \langle \xi, \eta(x,y) \rangle$$

for all $x \in K(\eta)$. Hence we have

$$\begin{aligned} &\left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + F(x) - F(y) \\ &\geq \left\langle T(y), \frac{xy}{\eta(x,y)} \right\rangle + \langle \xi, \eta(x,y) \rangle \\ &\geq 0 \end{aligned}$$

for all $x \in K(\eta)$. This means that $y \in K(\eta)$ solves the problem GHVIP_F. This completes the proof. $\square$

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