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FOURIER TRANSFORMATION OF REGULAR FUNCTIONS WITH VALUES IN GENERALIZED QUATERNIONS

Ji Eun Kim

Department of Mathematics, Dongguk University
Gyeongju-si 38066, Republic of Korea
e-mail: jeunkim@pusan.ac.kr

Abstract. We consider general base i_α and j_β which perform the roles of i and j do in quaternions. We give a representation and properties of a Fourier transformation of regular functions with values in generalized quaternions, referring the Fourier transformation using quaternions.

1. Introduction

Quaternion algebra is declared by Hamilton [6]. After his discovery of quaternions, quaternions recently have played a fundamental role in several areas of science. Adler [1] presented the quaternionic analogues of complex matrices converting a quaternion matrix to a pair of complex matrices. Agrawal [2] developed the algebra of dual-number-quaternions using properties of Hamilton operators and expressions for screw motion. Kim and Shon [9] gave a hyperholomorphic function and a split harmonic function with values in split quaternions and expressed polar coordinate forms for split quaternions. Kim and Shon [10] proposed a split regular function that has a split Cauchy-Riemann system in split quaternions and investigated properties of an inverse mapping theory with values in split quaternions. Cockle [4] gave coquaternions or para-quaternions and studied for manifolds which endowed

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with coquaternion structures in differential geometry.

In the analysis of systems of differential equations, the Fourier transform is being used a lot. The Fourier transform is a mapping of real-valued functions into complex-valued functions. It is used in the design of signal filters and control systems to transfer the time-domain of functions to the frequency domain. The quaternionic extension to the spectral transformations was introduced by Ell [5]. Hitzer [7] gave the quaternionic Fourier transform(QFT) applied to quaternion fields and investigated QFT properties useful for applications. Bahri *et al.* [3] established an uncertainty principle for the right-sided QFT which prescribed a lower bound on the product of the effective widths of quaternion-valued signals, by using the properties of the QFT. Pei *et al.* [11] introduced digital signal and image processing using reduced biquaternions which are an extension of the complex numbers, following the doubling procedure.

Using the Fourier transform which is composed of generalized quaternions, we can extend techniques for the properties of quaternionic spectral transformations and we can utilize a direct extension of the convolution product theorem of the Fourier transform to a two-dimensional convolution generalized quaternionic product. We consider that generalized quaternions are represented by base i_α and j_β and give some properties of regular functions with values in generalized quaternions. Also, we investigate the Fourier transformation of a generalized quaternionic function. We research representations and some characteristics of the regularity of functions on generalized quaternions. We give different forms of the generalized quaternions Fourier transform used to different Plancherel theorems. We propose non-commutative generalizations of the quaternionic Fourier transform with matrices and examples.

2. Preliminaries

The algebra of the elements of the form

$$z = x_0 \cdot 1 + x_1 i_\alpha + x_2 j_\beta + x_3 k$$

is said to be a generalized quaternion specially, α and β are non-zero scalar numbers and this algebra is a four dimensional non-commutative and associative real field with three bases $i_\alpha,\ j_\beta$ and k, where

$$x_r \in \mathbb{R}\ (r = 0, 1, 2, 3),\ i_\alpha^2 = \alpha,\ j_\beta^2 = \beta,\ k = i_\alpha j_\beta = -j_\beta i_\alpha.$$

We consider the set of generalized quaternions as follows:

$$\mathbb{G}_{\mathbb{H}} = \{z \mid z = x_0 \cdot 1 + x_1 i_\alpha + x_2 j_\beta + x_3 k\},$$

where the element 1 is the identity of $\mathbb{G}_{\mathbb{H}}$ which is isomorphic to $\mathbb{C}^2$. Based on the form of generalized quaternions, the conjugate number z^* of z in $\mathbb{G}_{\mathbb{H}}$ is given by

$$z^* = x_0 - x_1 i_\alpha - x_2 j_\beta - x_3 k.$$

Also, the norm $|z|$ of z is defined by

$$|z|^2 = z(z^*) = (z^*)z = x_0^2 - x_1^2\alpha - x_2^2\beta + x_3^2\alpha\beta$$

and the inverse z^{-1} of z is

$$z^{-1} = \frac{z^*}{|z|^2} \ (z \neq 0).$$

For two generalized quaternions $z = x_0 + x_1 i_\alpha + x_2 j_\beta + x_3 k$ and $w = y_0 + y_1 i_\alpha + y_2 j_\beta + y_3 k$, where $y_m \in \mathbb{R}$ $(m = 0, 1, 2, 3)$, we give the addition as follows:

$$z + w = (x_0 + y_0) + (x_1 + y_1)i_\alpha + (x_2 + y_2)j_\beta + (x_3 + y_3)k$$

and the multiplication is given by

$$\begin{aligned} zw = {} & x_0y_0 \cdot 1 + x_1y_1\alpha + x_2y_2\beta - x_3y_3\alpha\beta + (x_1y_0 + x_0y_1 + x_3y_2\beta - x_2y_3\beta)i_\alpha \\ & + (x_2y_0 - x_3y_1\alpha + x_0y_2 + x_1y_3\alpha)j_\beta + (x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3)k. \end{aligned}$$

From the properties of z and z^*, we give differential operators as follows:

$$D := \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\alpha^{-1} i_\alpha + \frac{\partial}{\partial x_2}\beta^{-1} j_\beta - \frac{\partial}{\partial x_3}\alpha^{-1}\beta^{-1} k\Big)$$

and

$$D^* = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}\alpha^{-1} i_\alpha - \frac{\partial}{\partial x_2}\beta^{-1} j_\beta + \frac{\partial}{\partial x_3}\alpha^{-1}\beta^{-1} k\Big),$$

where α^{-1} satisfies $\alpha\alpha^{-1} = 1$ and β^{-1} satisfies $\beta\beta^{-1} = 1$. If $\alpha,\ \beta = 0$, that is, z is a dual quaternion, then the differential operators are defined by the settings that can be applied in dual quaternions (see [8]).

Let Ω be an open subset of $\mathbb{R}^4$. Let $f : \Omega \to \mathbb{G}_{\mathbb{H}}$ be a function with values in $\mathbb{G}_{\mathbb{H}}$ such that

$$f(z) = f(x_0, x_1, x_2, x_3) = u_0 \cdot 1 + u_1 i_\alpha + u_2 j_\beta + u_3 k$$

for $z = x_0 \cdot 1 + x_1 i_\alpha + x_2 j_\beta + x_3 k$ in Ω, is called a generalized quaternionic function, where

$$u_r = u_r(x_0, x_1, x_2, x_3) : \Omega \subset \mathbb{R}^4 \to \mathbb{R}$$

are real-valued functions.

Remark 2.1. By the properties of the differential operators D and D^*, we have the following results:

$$\begin{aligned} Df &= \frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} + \Big(\frac{\partial u_0}{\partial x_1}\alpha^{-1} + \frac{\partial u_1}{\partial x_0} - \frac{\partial u_2}{\partial x_3}\alpha^{-1} - \frac{\partial u_3}{\partial x_2}\Big) i_\alpha \\ &\quad + \Big(\frac{\partial u_0}{\partial x_2}\beta^{-1} + \frac{\partial u_1}{\partial x_3}\beta^{-1} + \frac{\partial u_2}{\partial x_0} + \frac{\partial u_3}{\partial x_1}\Big) j_\beta \\ &\quad + \Big(-\frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1} - \frac{\partial u_1}{\partial x_2}\beta^{-1} + \frac{\partial u_2}{\partial x_1}\alpha^{-1} + \frac{\partial u_3}{\partial x_0}\Big) k \end{aligned}$$

and

$$\begin{aligned} D^*f &= \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + \Big(\frac{\partial u_1}{\partial x_0} - \frac{\partial u_0}{\partial x_1}\alpha^{-1} + \frac{\partial u_2}{\partial x_3}\alpha^{-1} + \frac{\partial u_3}{\partial x_2}\Big) i_\alpha \\ &\quad + \Big(\frac{\partial u_2}{\partial x_0} - \frac{\partial u_0}{\partial x_2}\beta^{-1} - \frac{\partial u_1}{\partial x_3}\beta^{-1} - \frac{\partial u_3}{\partial x_1}\Big) j_\beta \\ &\quad + \Big(\frac{\partial u_3}{\partial x_0} - \frac{\partial u_2}{\partial x_1}\alpha^{-1} + \frac{\partial u_0}{\partial x_3}\alpha^{-1}\beta^{-1} + \frac{\partial u_1}{\partial x_2}\beta^{-1}\Big) k. \end{aligned}$$

3. Generalized quaternionic Fourier transform (GQFT)

We extend the Fourier transform to the algebra of generalized quaternions. Since generalized quaternions are non-commutative, there are three different types of GQFT of two dimensional generalized quaternion-valued signals as follows:

Definition 3.1. Let

$$\int_{\mathbb{R}^2} |f(\chi)| \; d^2\chi$$

exist. Then a generalized quaternionic Fourier transform of a function f, denoted by $\mathcal{F}\{f\}$, is given by:
for $\mathcal{F}\{f\} : \mathbb{R}^2 \to \mathbb{G}_{\mathbb{H}}$,
Type 1.

$$\mathcal{F}_1\{f\}(\omega) = \int_{\mathbb{R}^2} \exp(-j_\beta \; \omega_2\chi_2) \exp(-i_\alpha \; \omega_1\chi_1) f(\chi) \; d^2\chi,$$

Type 2.

$$\mathcal{F}_2\{f\}(\omega) = \int_{\mathbb{R}^2} \exp(-i_\alpha \; \omega_1\chi_1) f(\chi) \exp(-j_\beta \; \omega_2\chi_2) \; d^2\chi,$$

Type 3.

$$\mathcal{F}_3\{f\}(\omega) = \int_{\mathbb{R}^2} f(\chi) \exp(-j_\beta \; \omega_2\chi_2) \exp(-i_\alpha \; \omega_1\chi_1) \; d^2\chi,$$

where $\chi = (\chi_1, \chi_2)$, $\omega = (\omega_1, \omega_2)$, $d^2\chi = d\chi_1 d\chi_2$ and the exponential product

$$\exp(-i_\alpha\ \omega_1\chi_1)\exp(-j_\beta\ \omega_2\chi_2)$$

is called the generalized quaternion Fourier kernel.

We will use 'GQFT' instead of 'generalized quaternionic Fourier transform' to make the expression of the word simply. We always consider that $\int_{\mathbb{R}^2} |f(\chi)|\ d^2\chi$ exists. So, after that, we don't mention this condition again. Also, since the calculating processes and styles of types in Definition 3.1 are similar to each other, we deal with only Type 1 in the rest of paper when we prove theorems.

Remark 3.2. Using the Euler formula for the generalized quaternion Fourier kernel, we have

$$\exp(-i_\alpha\ \omega_1\chi_1) = \cosh(\sqrt{\alpha}\ \omega_1\chi_1) - \sinh(\sqrt{\alpha}\ \omega_1\chi_1)$$

and

$$\exp(-j_\beta\ \omega_2\chi_2) = \cosh(\sqrt{\beta}\ \omega_2\chi_2) - \sinh(\sqrt{\beta}\ \omega_2\chi_2).$$

Hence, we can write the following form

$$\begin{aligned}\mathcal{F}_1\{f\}(\omega) &= \int_{\mathbb{R}^2}\{\cosh(\sqrt{\alpha}\ \omega_1\chi_1)\cosh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi \\ &\quad + \int_{\mathbb{R}^2}\{\sinh(\sqrt{\alpha}\ \omega_1\chi_1)\sinh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi \\ &\quad + \int_{\mathbb{R}^2}\{\cosh(\sqrt{\alpha}\ \omega_1\chi_1)\sinh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi \\ &\quad + \int_{\mathbb{R}^2}\{\sinh(\sqrt{\alpha}\ \omega_1\chi_1)\cosh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi.\end{aligned}$$

That is, we can represent

$$\begin{aligned}\mathcal{F}_1\{f\}(\omega) &= \int_{\mathbb{R}^2}\{\cosh(\sqrt{\alpha}\ \omega_1\chi_1 + \sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi \\ &\quad - \int_{\mathbb{R}^2}\{\sinh(\sqrt{\alpha}\ \omega_1\chi_1 + \sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi.\end{aligned}$$

Remark 3.3. For the generalized quaternion Fourier kernel, when α and β are negative, we have

$$\exp(-i_\alpha\ \omega_1\chi_1)\exp(-j_\beta\ \omega_2\chi_2) \neq \exp(-j_\beta\ \omega_2\chi_2)\exp(-i_\alpha\ \omega_1\chi_1).$$

For example, when $\alpha = \beta = -1$, since

$$\begin{aligned}&\exp(-i_\alpha\ \omega_1\chi_1)\exp(-j_\beta\ \omega_2\chi_2)\\&=(\cos(\omega_1\chi_1)-i_\alpha\sin(\omega_1\chi_1))(\cos(\omega_2\chi_2)-j_\beta\sin(\omega_2\chi_2))\end{aligned}$$

and

$$\begin{aligned}&\exp(-j_\beta\ \omega_2\chi_2)\exp(-i_\alpha\ \omega_1\chi_1)\\&=(\cos(\omega_2\chi_2)-j_\beta\sin(\omega_2\chi_2))(\cos(\omega_1\chi_1)-i_\alpha\sin(\omega_1\chi_1)),\end{aligned}$$

we obtain

$$\exp(-i_\alpha\ \omega_1\chi_1)\exp(-j_\beta\ \omega_2\chi_2)\neq\exp(-j_\beta\ \omega_2\chi_2)\exp(-i_\alpha\ \omega_1\chi_1).$$

Otherwise, we have

$$\exp(-i_\alpha\ \omega_1\chi_1)\exp(-j_\beta\ \omega_2\chi_2)=\exp(-j_\beta\ \omega_2\chi_2)\exp(-i_\alpha\ \omega_1\chi_1).$$

Hence, we can write the following form

$$\begin{aligned}\mathcal{F}_1\{f\}(\omega) &= \int_{\mathbb{R}^2}\{\cosh(\sqrt{\alpha}\ \omega_1\chi_1)\cosh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi\\&\quad+\int_{\mathbb{R}^2}\{\sinh(\sqrt{\alpha}\ \omega_1\chi_1)\sinh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi\\&\quad+\int_{\mathbb{R}^2}\{\cosh(\sqrt{\alpha}\ \omega_1\chi_1)\sinh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi\\&\quad+\int_{\mathbb{R}^2}\{\sinh(\sqrt{\alpha}\ \omega_1\chi_1)\cosh(\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi.\end{aligned}$$

That is, we represent

$$\begin{aligned}\mathcal{F}_1\{f\}(\omega) &= \int_{\mathbb{R}^2}\{\cosh(\sqrt{\alpha}\ \omega_1\chi_1+\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi\\&\quad-\int_{\mathbb{R}^2}\{\sinh(\sqrt{\alpha}\ \omega_1\chi_1+\sqrt{\beta}\ \omega_2\chi_2)\}f(\chi)\ d^2\chi.\end{aligned}$$

Example 3.4. Consider the quaternionic distribution signal (see [3])

$$f(\chi)=\exp(i_\alpha\lambda\chi_1)\exp(j_\beta\mu\chi_2).$$

If $\omega \neq \omega_0$, then the GQFT of f is

$$\begin{aligned}\mathcal{F}_1\{f\}(\omega) &= \int_{\mathbb{R}^2}\exp(-j_\beta\ \omega_2\chi_2)\exp(-i_\alpha\ \omega_1\chi_1)\exp(i_\alpha\lambda\chi_1)\exp(j_\beta\mu\chi_2)\ d\chi_1 d\chi_2\\&=2\pi\int_{\mathbb{R}}\exp(-j_\beta\ \omega_2\chi_2)\exp(j_\beta\mu\chi_2)\ d\chi_2\\&=(2\pi)^2,\end{aligned}$$

where $\omega_0 = (\lambda, \mu)$. Also, if $\omega = \omega_0$, then $\mathcal{F}_1\{f\}(\omega) = 0$. Therefore, we obtain the result as follows:

$$\mathcal{F}_1\{f\}(\omega) = (2\pi)^2 \delta(\omega - \omega_0).$$

Definition 3.5. Suppose that $f \in L^1(\mathbb{R}^2; \mathbb{G}_{\mathbb{H}})$ and $\mathcal{F}_r\{f\}(\omega) \in L^2(\mathbb{R}^2; \mathbb{G}_{\mathbb{H}})$ $(r = 1, 2, 3)$. Then $\mathcal{F}_r\{f\}(\omega)$ is an invertible transform and each inverse transformation of $\mathcal{F}_r\{f\}(\omega)$ is as follows:
Type 1.

$$\mathcal{F}_1^{-1}[\mathcal{F}_1\{f\}(\omega)] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(j_\beta\ \omega_2\chi_2) \exp(i_\alpha\ \omega_1\chi_1) \mathcal{F}_1\{f\}(\omega)\ d^2\chi;$$

Type 2.

$$\mathcal{F}_2^{-1}[\mathcal{F}_2\{f\}(\omega)] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(i_\alpha\ \omega_1\chi_1) \mathcal{F}_2\{f\}(\omega) \exp(j_\beta\ \omega_2\chi_2)\ d^2\chi;$$

Type 3.

$$\mathcal{F}_3^{-1}[\mathcal{F}_3\{f\}(\omega)] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_3\{f\}(\omega) \exp(j_\beta\ \omega_2\chi_2) \exp(i_\alpha\ \omega_1\chi_1)\ d^2\chi.$$

Theorem 3.6. *For two generalized quaternionic functions, the GQFT of* $f,\ g \in L^1(\mathbb{R}^2; \mathbb{G}_{\mathbb{H}})$ *is a linear operator, that is,*

$$\mathcal{F}_r\{p\ f + q\ g\}(\omega) = p\ \mathcal{F}_r\{f\}(\omega) + q\ \mathcal{F}_r\{g\}(\omega) \quad (r = 1, 2, 3),$$

where p *and* q *in* $\mathbb{G}_{\mathbb{H}}$ *are generalized quaternion constants.*

Proof. From the Definition of the GQFT, we have the following equations:

$$\begin{aligned}
\mathcal{F}_1\{p\ f + q\ g\}(\omega) &= \int_{\mathbb{R}^2} \exp(-j_\beta\ \omega_2\chi_2) \exp(-i_\alpha\ \omega_1\chi_1)\{p\ f(\chi) + q\ g(\chi)\}\ d^2\chi \\
&= \int_{\mathbb{R}^2} \exp(-j_\beta\ \omega_2\chi_2) \exp(-i_\alpha\ \omega_1\chi_1)\{p\ f(\chi) + q\ g(\chi)\}\ d^2\chi \\
&= \int_{\mathbb{R}^2} \exp(-j_\beta\ \omega_2\chi_2) \exp(-i_\alpha\ \omega_1\chi_1) p\ f(\chi)\ d^2\chi \\
&\quad + \int_{\mathbb{R}^2} \exp(-j_\beta\ \omega_2\chi_2) \exp(-i_\alpha\ \omega_1\chi_1) q\ g(\chi)\ d^2\chi \\
&= p\ \mathcal{F}_1\{f\}(\omega) + q\ \mathcal{F}_1\{g\}(\omega).
\end{aligned}$$

Therefore, we obtain the result. □

Now, we give the GQFT Plancherel theorem for generalized quaternionic functions.

Theorem 3.7. (GQFT Plancherel) *Let $f,\ g \in L^2(\mathbb{R}^2;\mathbb{G}_{\mathbb{H}})$ be generalized quaternion module functions. Then the generalized quaternionic inner product of $f,\ g$ is given by the inner product of the corresponding GQFTs $\mathcal{F}_r\{f\}(\omega)$ and $\mathcal{F}_r\{g\}(\omega)$ $(r = 1, 2, 3)$ as follows:*

$$\begin{cases} \textit{if}\ \alpha,\ \beta\ <\ 0, \\ \quad \dfrac{1}{(2\pi)^2}\ < \mathcal{F}_r\{f\}(\omega), \mathcal{F}_3\{g\}(\omega) >; \\ \textit{otherwise}, \\ \quad \dfrac{1}{(2\pi)^2}\ < \mathcal{F}_r\{f\}(\omega), \mathcal{F}_r\{g\}(\omega) > . \end{cases}$$

Proof. For $f,\ g \in L^2(\mathbb{R}^2;\mathbb{G}_{\mathbb{H}})$, the inner product is expressed by the following equation:

$$\begin{aligned} &< f, g > \\ &= \int_{\mathbb{R}^2} f(z)\ g(z)^*\ d^2z \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \exp(j_\beta\ \omega_2\chi_2) \exp(i_\alpha\ \omega_1\chi_1) \mathcal{F}_1\{f\}(\omega)\ d^2\chi \Big)\ g(z)^*\ d^2z \\ &= \frac{1}{(2\pi)^2}\ \mathcal{F}_1\{f\}(\omega) \int_{\mathbb{R}^2} \exp(j_\beta\ \omega_2\chi_2) \exp(i_\alpha\ \omega_1\chi_1)\ g(z)^*\ d^2z\ d^2\chi \\ &= \begin{cases} \text{if}\ \alpha,\ \beta\ <\ 0, \\ \quad \dfrac{1}{(2\pi)^2}\ \mathcal{F}_1\{f\}(\omega) \Big(\displaystyle\int_{\mathbb{R}^2} g(z) \exp(-j_\beta\ \omega_2\chi_2) \exp(-i_\alpha\ \omega_1\chi_1)\ d^2\chi \Big)^*\ d^2z; \\ \text{otherwise}, \\ \quad \dfrac{1}{(2\pi)^2}\ \mathcal{F}_1\{f\}(\omega) \Big(\displaystyle\int_{\mathbb{R}^2} \exp(-j_\beta\ \omega_2\chi_2) \exp(-i_\alpha\ \omega_1\chi_1) g(z)\ d^2\chi \Big)^*\ d^2z \end{cases} \\ &= \begin{cases} \text{if}\ \alpha,\ \beta\ <\ 0, \\ \quad \dfrac{1}{(2\pi)^2}\ < \mathcal{F}_1\{f\}(\omega), \mathcal{F}_3\{g\}(\omega) >; \\ \text{otherwise}, \\ \quad \dfrac{1}{(2\pi)^2}\ < \mathcal{F}_1\{f\}(\omega), \mathcal{F}_1\{g\}(\omega) > . \end{cases} \end{aligned}$$

Similarly, for Type 2 in the Definition 3.5, we have

$$
\begin{aligned}
&< f, g > \\
&= \int_{\mathbb{R}^2} f(z)\ g(z)^*\ d^2 z \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \exp(i_\alpha\ \omega_1 \chi_1) \mathcal{F}_2\{f\}(\omega) \exp(j_\beta\ \omega_2 \chi_2)\ d^2 \chi \Big)\ g(z)^*\ d^2 z \\
&= \begin{cases} \text{if } \alpha,\ \beta\ <\ 0, \\ \qquad \dfrac{1}{(2\pi)^2}\ < \mathcal{F}_2\{f\}(\omega), \mathcal{F}_3\{g\}(\omega) >; \\ \text{otherwise}, \\ \qquad \dfrac{1}{(2\pi)^2}\ < \mathcal{F}_2\{f\}(\omega), \mathcal{F}_2\{g\}(\omega) >, \end{cases}
\end{aligned}
$$

and for Type 3 in the Definition 3.5, we get

$$
\begin{aligned}
&< f, g > \\
&= \int_{\mathbb{R}^2} f(z)\ g(z)^*\ d^2 z \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} \mathcal{F}_3\{f\}(\omega) \exp(j_\beta\ \omega_2 \chi_2) \exp(i_\alpha\ \omega_1 \chi_1)\ d^2 \chi \Big)\ g(z)^*\ d^2 z \\
&= \frac{1}{(2\pi)^2}\ < \mathcal{F}_3\{f\}(\omega), \mathcal{F}_3\{g\}(\omega) > .
\end{aligned}
$$

Therefore, the result is obtained. □

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