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DEMICLOSEDNESS PRINCIPLE AND CONVERGENCE THEOREMS FOR LIPSCHITZIAN TYPE NONSELF-MAPPINGS IN CAT(0) SPACES

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Abstract. In this paper, we study the existence of fixed points, demiclosedness principle and the structure of fixed point sets for the class of nearly asymptotically nonexpansive nonself-mappings in CAT(0) spaces, and also we discuss the strong and Δ-convergence theorems for an iterative scheme introduced by Khan. Our results are improvements of the various well-known results of fixed point theory which is established in uniformly convex Banach spaces as well as CAT(0) spaces.

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1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let C be a nonempty subset of a metric space (X, d). Let $T : C \to C$ be a mapping. $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T.

Definition 1.1. Let C be a nonempty subset of a metric space (X, d) The mapping $T : C \to C$ is said to be

(i) *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive number $k_n > 0$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \text{for all } x, y \in C, \tag{1.1}$$

(ii) *uniformly* $k-$*Lipschitzian* if $k_n = k$ for all $n \in \mathbb{N}$,

(iii) *asymptotically nonexpansive* [17] with $\lim_{n\to\infty} k_n = 1$.

The existence theorem of fixed point of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [17] in 1972. They proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping of C has a fixed point. There are many papers dealing with the approximation of fixed points of nonexpansive and asymptotically nonexpansive mappings in uniformly convex Banach spaces through modified Mann and Ishikawa iteration processes (see [4, 30, 31, 38, 43, 45, 46, 47, 48] and references contained therein).

The class of nearly Lipschitzian mappings as an important generalization of the class of Lipschitzian mappings was introduced by Kim et al. ([32, 33]) and Sahu [39].

Definition 1.2. Let C be a nonempty subset of a metric space (X, d) and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x, y) + a_n) \quad \text{for all } x, y \in C. \tag{1.2}$$

The infimum of constants k_n in (1.2) is called the *nearly Lipschitz constant* of T^n and denoted by $\eta(T^n)$.

Definition 1.3. A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

(ii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \eta(T^n) = 1$,

(iii) *nearly uniformly* k-*Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(iv) *nearly uniformly* k-*contractive* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

The class of Lipschitzian mappings is larger than the classes of nonexpansive and asymptotically nonexpansive mappings. However, the theory of computation of fixed points of non-Lipschitzian mappings is equally important and interesting. There are few results in this direction appeared in the literature (see [25, 28, 39, 40, 45]).

In the recent years, CAT(0) spaces have attracted many authors as they played a very important role in different aspects of geometry (see [15, 21]). In 1976, Lim [35], introduced the concept of $\Delta-$convergence in a general metric space. In 2008, Kirk and Panyanak [22] specialized Lim's concept to CAT(0) space and proved that it is very similar to the weak convergence in Banach space setting. Since then notions of $\Delta-$convergence and strong convergence has been widely studied and a number of papers have appeared in literature to approximate fixed points via Mann [36], Ishikawa [18], S-iteratiion and SP- iteration schemes [3, 5, 29, 41, 42] for nonexpansive, asymptotically nonexpansive [17], nearly asymptotically nonexpansive [39], total asymptotically nonexpansive mappings [6, 11] in CAT(0) spaces (see [1, 2, 10, 14, 20, 23, 27, 37, 49, 50]).

It is well known that one of the fundamental and celebrated results in the theory for nonexpansive mappings is Browder's demiclosedness principle [9] which states that if C is a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at each $y \in X$, that is, for any sequence $\{x_n\}$ in C, if $x_n \to x$ weakly and $(I-T)x_n \to y$ strongly, then $(I-T)x = y$, where I is identity mapping of X. It is well known that the demiclosedness principle plays an important role in studying the asymptotic behavior for nonexpansive mappings (for details, see ([4, 34]). Due to importance of demiclosedness principle for the class of mappings which is essentially wider than that of nonexpansive mappings in the setting of Banach spaces and CAT(0) spaces has been studied by several authors (see [1, 2, 4, 10, 11, 24, 37, 38, 39, 40, 48, 49]).

In all above results, the operator T remains a self-mapping of nonempty closed convex subset in Banach spaces (or CAT(0) spaces). In 2003, Chidume *et al.* [11] introduced the concept of asymptotically nonexpansive nonself-mappings and proved the strong and weak convergence theorems for the modified Mann iteration scheme [43] in uniformly convex Banach spaces. Since then the notion of nonself-mapping essentially wider than that of asymptotically nonexpansive mappings has been widely studied and has been appeared in a number of papers (see [19, 26] and references therein) in uniformly convex Banach spaces.

The purpose of this paper, is to introduce the class of nearly asymptotically nonexpansive nonself-mappings which contain the class of asymptotically nonexpansive nonself-mappings and is contained in the class of nonself-mappings of asymptotically nonexpansive type. We prove the demiclosedness principle, existence of fixed points, structure of fixed point sets and approximation of fixed point of mappings of these classes in CAT(0) spaces. Our results improve various celebrated results of fixed point theory established in uniformly convex Banach spaces as well as CAT(0) spaces (see e.g., Abbas *et al.* [1], Chang *et al.* [10], Dhompongsa and Panyanak [14], Khan [26], Khan and Abbas [23], Sahu [40]).

2. Preliminaries

Let (X, d) be a metric space. A geodesic from x to y in X is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be a unique geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$, which will be denoted by $[x, y]$, and called the segment joining from x to y. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3},)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$ such a triangle exists (see [7, 8, 42]).

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom (see [7, 8, 42]):

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

If x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which will be denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2(x, \frac{y_1 \oplus y_2}{2}) \le \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.2}$$

The inequality (2.2) is the (CN) inequality of Bruhat and Tits [8]. In the sequel we need the following useful lemmas.

Lemma 2.1. ([8, 42]) *Let X be a CAT(0) space. Then, for all $x, y \in X$ and $t \in [0,1]$, there exists a unique point $(1-t)x \oplus ty \in [x,y]$ such that*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z), \tag{2.3}$$

for any $z \in X$.

Lemma 2.2. ([10]) *Let $\{a_n\}, \{\lambda_n\}$ and $\{c_n\}$ be three sequences of nonnegative numbers such that*

$$a_{n+1} \leq \lambda_n a_n + c_n, \tag{2.4}$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3. ([10]) *Let (X,d) be a complete CAT(0) space and $x \in X$. Let $\{t_n\}$ be a sequence in $[b,c]$ with $b, c \in (0,1)$ and $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \quad \limsup_{n\to\infty} d(y_n, x) \leq r,$$

and

$$\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n), x) = r,$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

First, we give the concept of Δ-convergence and some of its basic properties.

Let C be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X. Let $diam(C)$ denote the diameter of C. Consider a continuous functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x), \quad x \in X.$$

Then, the infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$.

A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\big\{r_a(x, \{x_n\}) : x \in C\big\},$$

the set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $AC(C, \{x_n\})$. This set may be empty, a singleton, or certain infinitely many points.

If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X, \{x_n\}) = r_a(\{x_n\})$ and $AC(X, \{x_n\}) = AC(\{x_n\})$, respectively. We know that $r_a(x, \{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$, for $x \in X$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, $[x, y] \subset C$.

It is known that uniformly convex Banach spaces and even CAT(0) spaces have the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets (see [12, 13]).

Next, we define $\Delta-$convergence of a sequence in a CAT(0) space.

Definition 2.4. ([13, 35]) Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X. Then $\{x_n\}$ is said to be $\Delta-$*convergent to* x in X if x is the unique asymptotic center of $\{x_m\}$ for every subsequence $\{x_m\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$.

Lemma 2.5. ([14]) *Let X be a complete CAT(0) space. Then we have the followings:*

(i) *Every bounded sequence in a complete CAT(0) space X has a $\Delta-$convergent subsequence.*
(ii) *If C is a closed convex subset of a complete CAT(0) space X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Recall that a bounded sequence $\{x_n\}$ in a complete CAT(0) space X is said to be *regular* if $r_a(X, \{x_n\}) = r_a(X, \{u_n\})$ for every subsequence of $\{u_n\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has regular subsequence. Since every regular sequence is $\Delta-$convergent, we see immediately that every bounded sequence in a complete CAT(0) space has a $\Delta-$convergent subsequence. Notice that (see e.g., [12], Proposition 7) for a given bounded sequence $\{x_n\}$ in a complete CAT(0) space X which is $\Delta-$convergent to x and given $y \in X$ with $y \neq x$, we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Clearly, X satisfies a condition which is well known in Banach space theory as the Opial property. We denote

$$w_w(x_n) = \bigcup AC(\{u_n\}),$$

where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

Next, we define the properties (D_1) and (D_2) which play important role in the approximation of fixed points of nonexpansive and asymptotically nonexpansive mappings in Banach spaces (see e.g., [4] and references therein).

Definition 2.6. Let C be a subset of CAT(0) space X and let $T : C \to C$ be a mapping. A sequence $\{x_n\}$ in C has:

(D_1) *the limit existence property* of T, if $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

(D_2) *the approximating fixed point property* of T, if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let (X, d) be a metric space and C be a nonempty subset of X. Recall that C is called a retract of X if there exists a continuous mapping P from X onto C such that $Px = x$, for all $x \in C$. A map $P : X \to C$ is said to be retraction if $P^2 = P$. It follows that if a map P is retraction, then $Py = y$ for all y in the range of P.

Definition 2.7. Let C be nonempty subset of a metric space (X, d). Let $P : X \to C$ be a nonexpansive retraction of X onto C.

(i) A nonself-mapping $T : C \to X$ is said to be *asymptotically nonexpansive* [11] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \quad \leq \quad k_n d(x, y), \tag{2.5}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(ii) A nonself-mapping $T : C \to X$ is said to be *uniformly $k-$Lipschitz-ian* if there exists a constant $k > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \quad \leq \quad k\ d(x, y) \tag{2.6}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(iii) A nonself-mapping $T : C \to X$ is said to be *nearly Lipschitzian* [26] with respect to $\{a_n\}$ for a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$, if each $n \in \mathbb{N}$, there exist a constants $k_n \geq 0$, such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \quad \leq \quad k_n(d(x, y) + a_n), \tag{2.7}$$

for all $x, y \in C$. The infimum of the constant k_n for which the above inequality holds, is denoted by $\eta(T(PT)^{n-1})$ and is called *nearly Lipschitz constant.*

(iv) For $n = 1$, above inequality (2.7) can be written as

$$d(T(PT)^{1-1}x, T(PT)^{1-1}y) \quad \leq \quad k_1(d(x, y) + a_1),$$

where we have to take a_1 as zero sequence. Thus in this case we have

$$d(T(PT)^{1-1}x, T(PT)^{1-1}y) \quad \leq \quad k_1 d(x, y).$$

(v) A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T(PT)^{n-1})\}$ is said to be *nearly asymptotically nonexpasive* [26], if

$$\eta(T(PT)^{n-1}) \geq 1,$$

for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T(PT)^{n-1}) = 1$.

Now, we define an S-iteration scheme for nonself-mappings in a CAT(0) space.

Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in C$ such that

$$\begin{cases} x_{n+1} = P\big[(1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n\big], \\ y_n = P\big[(1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n\big],\ n \in \mathbb{N}, \end{cases} \tag{2.8}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$.

Remark 2.8. If T is a self-map, then P becomes the identity map so that (2.5) and (2.7) coincide with (1.1) and (1.2). Moreover, (2.8) reduces to $S-$iteration scheme (see [3, 42]).

3. Existence theorem of fixed points

Now we are able to prove the existence of fixed point for a nearly asymptotically nonexpansive nonself-mapping in a complete CAT(0) space.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X and let $T : C \to X$ be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with sequence $\{a_n, \eta(T(PT)^{n-1})\}$. Then T has a fixed point in C. Moreover, the set $F(T)$ of fixed point of T is closed and convex.*

Proof. For a given $x_0 \in C$, we define

$$\Psi(u) = \limsup_{n\to\infty} d(T(PT)^{n-1}x_0, u), \tag{3.1}$$

for all $u \in C$, where P is a nonexpansive retraction of X onto C. Since T is a nearly asymptotically nonexpansive nonself-mapping, we have

$$d(T(PT)^{n+m-1}x_0, T(PT)^{m-1}u) \le \eta(T(PT)^{m-1})\big[d(T(PT)^{n-1}x_0, u) + a_m\big],$$

for any $n, m \in \mathbb{N}$. Taking $\limsup_{m\to\infty}$ in above inequality and using (3.1), we have that

$$\Psi(T(PT)^{m-1}u) \le \eta(T(PT)^{m-1})(\Psi(u) + a_m). \tag{3.2}$$

It is easy to know that the function $u \to \Psi(u)$ is a lower semi continuous. Since C is bounded closed and convex, there exists a point $w \in C$ such that $\Psi(w) = \inf_{u\in C} \Psi(u)$.

Letting $u = w$ in (3.2), for each $n \in \mathbb{N}$, we have

$$\Psi(T(PT)^{m-1}w) \le \eta(T(PT)^{m-1})(\Psi(w) + a_m),\ m \in \mathbb{N},$$

By using inequality (2.2) for any positive integers $n, m \in \mathbb{N}$, we obtain

$$
\begin{aligned}
& d^2\left(T(PT)^{n-1}x_0, \frac{T(PT)^{m-1}w \oplus T(PT)^{k-1}w}{2}\right) \\
& \leq \frac{1}{2}d^2\big(T(PT)^{n-1}x_0, T(PT)^{m-1}w\big) + \frac{1}{2}\big(T(PT)^{n-1}x_0, T(PT)^{k-1}w\big) \\
& \quad -\frac{1}{4}d^2\big(T(PT)^{m-1}w, T(PT)^{k-1}w\big).
\end{aligned}
$$

Taking $\limsup_{n\to\infty}$ of the both sides, we have

$$
\begin{aligned}
\Psi^2(w) & \leq \Psi^2\left[\frac{T(PT)^{m-1}w \oplus T(PT)^{k-1}w}{2}\right] \\
& \leq \frac{1}{2}\Psi^2(T(PT)^{m-1}w) + \frac{1}{2}\Psi^2(T(PT)^{k-1}w) \\
& \quad -\frac{1}{4}d^2(T(PT)^{m-1}w, T(PT)^{k-1}w) \\
& \leq \frac{1}{2}(\eta(T(PT)^{m-1}))^2(\Psi(w)+a_m)^2 \\
& \quad +\frac{1}{2}(\eta(T(PT)^{k-1}))^2(\Psi(w)+a_k)^2 \\
& \quad -\frac{1}{4}d^2(T(PT)^{m-1}w, T(PT)^{k-1}w).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d^2(T(PT)^{m-1}w, T(PT)^{k-1}w) & \leq 2\big(\eta(T(PT)^{m-1})\big)^2\big(\Psi(w)+a_m\big)^2 \\
& \quad +2\big(\eta(T(PT)^{k-1})\big)^2\big(\Psi(w)+a_k\big)^2 \\
& \quad -4\Psi^2(w).
\end{aligned}
$$

Since T is a nearly asymptotically nonexpansive nonself-mapping, taking $\limsup_{m,k\to\infty}$ on the both sides, we get

$$
\limsup_{m,k\to\infty} d(T(PT)^{m-1}w, T(PT)^{k-1}w) \leq 0,
$$

which implies that $\{T(PT)^{m-1}w\}$ is a Cauchy sequence in C. Since C is complete, it converges to some $w \in C$. Let $\lim_{m\to\infty} T(PT)^{m-1}w = w$. In the view, of the continuity of TP, we have

$$
w = \lim_{m\to\infty} TP(T(PT)^{m-1}w) = TPw = Tw.
$$

This means that T has a fixed point w. Next, we have to prove that $F(T)$ is closed and convex. Since T is continuous, $F(T)$ is closed. In order to prove that $F(T)$ is convex, it is enough to show that $\frac{x\oplus y}{2} \in F(T)$, whenever $x, y \in F(T)$. Let $w = \frac{x\oplus y}{2}$. By using inequality (2.2), we have

$$\begin{aligned} d^2(T(PT)^{n-1}w, w) &= d^2\left(T(PT)^{n-1}w, \frac{x\oplus y}{2}\right) \\ &\leq \frac{1}{2}d^2(x, T(PT)^{n-1}w) + \frac{1}{2}d^2(y, T(PT)^{n-1}w) \\ &- \frac{1}{4}d^2(x,y) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d^2(x, T(PT)^{n-1}w) &= d^2(T(PT)^{n-1}x, T(PT)^{n-1}w) \\ &\leq (\eta(T(PT)^{n-1}))^2\big(d(x,w)+a_n\big)^2 \\ &\leq (\eta(T(PT)^{n-1}))^2\left(d\left(x, \frac{x\oplus y}{2}\right)+a_n\right)^2 \\ &\leq (\eta(T(PT)^{n-1}))^2\left(\frac{1}{2}d(x,x)+\frac{1}{2}d(x,y)+a_n\right)^2 \\ &= (\eta(T(PT)^{n-1}))^2\left(\frac{1}{2}d(x,y)+a_n\right)^2. \end{aligned} \tag{3.4}$$

Similarly, we can get

$$d^2(y, T(PT)^{n-1}w) \leq (\eta(T(PT)^{n-1}))^2\left(\frac{1}{2}d(x,y)+a_n\right)^2. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) and simplifying, we have

$$\begin{aligned} d^2(w, T(PT)^{n-1}w) \leq\ & \frac{1}{2}\ (\eta(T(PT)^{n-1}))^2\left(\frac{1}{2}d(x,y)+a_n\right)^2 \\ &+\frac{1}{2}\ (\eta(T(PT)^{n-1}))^2\left(\frac{1}{2}d(x,y)+a_n\right)^2 \\ &-\frac{1}{4}d^2(x,y). \end{aligned}$$

Hence, letting $n \to \infty$, we have

$$\lim_{n\to\infty} T(PT)^{n-1}w = w.$$

In view of the continuity of TP, we have

$$w = \lim_{n\to\infty} TP(T(PT)^{n-1}w) = TPw = Tw.$$

It implies that $w = \frac{x\oplus y}{2} \in F(T)$. Since C is convex, $w = \frac{x\oplus y}{2} \in C$. Therefore, $Pw = w$ which implies that $w = Tw$, $w \in F(T)$. □

4. Demiclosedness principle for Δ-convergence

It is known that the demiclosedness principle plays a key role in studying the convergence theorems for various mappings. Now we introduce the demiclosedness principle for the Δ-convergence in a CAT(0) space.

First, we need the following proposition.

Proposition 4.1. *Let C be a nonempty closed convex subset of complete CAT(0) space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C and $T : C \to X$ be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with sequence $\{a_n, \eta(T(PT)^{n-1})\}$. If $\{y_n\}$ is a bounded sequence in C satisfying the property (D_2), then T has a fixed point.*

Proof. Since C is a nonempty closed convex subset of complete CAT(0) space X and $\{y_n\}$ be a bounded sequence in C, $AC(C, \{y_n\})$ is in C and consists of exactly one point ν (say). We now show that ν is a fixed point of T. Suppose that $T : C \to X$ is a uniformly continuous, nearly asymptotically nonexpansive nonself-mappings with sequences $\{a_n, \eta(T(PT)^{n-1})\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$, where P is a nonexpansive retraction of X onto C. By uniform continuity of TP, we have

$$\lim_{n\to\infty} d(T(PT)^i y_n, T(PT)^{i-1} y_n) = 0, \tag{4.1}$$

for $i = 0, 1, \cdots$. We define a sequence $\{z_m\}$ in C by $z_m = T(PT)^{m-1}\nu$, $m \in \mathbb{N}$. For integers $m, n \in \mathbb{N}$, we have

$$\begin{aligned} d(z_m, y_n) &= d(T(PT)^{m-1}\nu, T(PT)^{m-1}y_n) \\ &\quad + d(T(PT)^{m-1}y_n, T(PT)^{m-2}y_n) \\ &\quad + \cdots \\ &\quad + d(Ty_n, y_n) \\ &\le d(T(PT)^{m-1}\nu, T(PT)^{m-1}y_n) \\ &\quad + \sum_{i=1}^{m-1} d(T(PT)^i y_n, T(PT)^{i-1} y_n) \\ &\le \eta(T(PT)^{m-1})(d(\nu, y_n) + a_m) \\ &\quad + \sum_{i=1}^{m-1} d(T(PT)^i y_n, T(PT)^{i-1} y_n). \end{aligned} \tag{4.2}$$

Then, by (4.1) and (4.2), we have

$$\begin{aligned} r_a(z_m, \{y_n\}) &= \limsup_{m\to\infty} d(z_m, y_n) \\ &\leq \limsup_{m\to\infty} \left[\eta(T(PT)^{m-1})(d(\nu, y_n) + a_m)\right] \\ &\leq r_a(\nu, \{y_n\}). \end{aligned}$$

Hence, we have

$$\limsup_{m\to\infty} r_a(z_m, \{y_n\}) \leq r_a(\nu, \{y_n\}). \tag{4.3}$$

By inequality (2.2), we have

$$d\left(y_n, \frac{\nu \oplus z_m}{2}\right)^2 \leq \frac{1}{2}d(y_n, \nu)^2 + \frac{1}{2}d(y_n, z_m)^2 - \frac{1}{4}d(\nu, z_m)^2, \tag{4.4}$$

for all $m, n \in \mathbb{N}$, and from (4.3) and (4.4), we obtained that

$$r_a\left(\frac{\nu \oplus z_m}{2}, \{y_n\}\right)^2 \leq \frac{1}{2}r_a(\nu, \{y_n\})^2 + \frac{1}{2}r_a(z_m, \{y_n\})^2 - \frac{1}{4}d(\nu, z_m)^2.$$

Since $AC(C, \{y_n\}) = \{\nu\}$, we have

$$\begin{aligned} r_a(\nu, \{y_n\})^2 &\leq r_a\left(\frac{\nu \oplus z_m}{2}, \{y_n\}\right)^2 \\ &\leq \frac{1}{2}r_a(\nu, \{y_n\})^2 + \frac{1}{2}r_a(z_m, \{y_n\})^2 - \frac{1}{4}d(\nu, z_m)^2, \end{aligned}$$

which implies that

$$\limsup_{m\to\infty} d(\nu, z_m)^2 \leq 2\limsup_{m\to\infty} \left[r_a(z_m, \{y_n\})^2 - r_a(\nu, \{y_n\})^2\right] = 0.$$

Thus, $T(PT)^{m-1}\nu \to \nu$.

In the view of the continuity of TP, we have

$$\begin{aligned} \nu &= \lim_{m\to\infty} T(PT)^m\nu = \lim_{m\to\infty} TP(T(PT)^{m-1}\nu) = TP\ \nu \\ &= T\nu. \end{aligned}$$

This completes the proof. □

Now, we are in a position to introduce and prove the demiclosedness principle (cf. [34]).

Theorem 4.2. *Let C be a nonempty closed convex subset of complete CAT(0) space X, $P : X \to C$ be a nonexpansive retraction of X onto C, and $T : C \to X$ be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with a sequence $\{a_n, \eta(T(PT)^{n-1})\}$. If $\{x_n\}$ is a bounded sequence in C which is Δ-convergent to x and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in C$ and*

$(I-T)x = 0$. *That is,* $I-T$ *is demiclosed at zero with respect to the* Δ-*convergence.*

Proof. Let $\{x_n\}$ be a bounded sequence in C and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. Then it is an approximating fixed point property of T and $\Delta-$convergent to x. One can see by Lemma 2.5 that $x \in C$. Note that $AC(\{x_n\}) = \{x\}$, so we have $r_a(x,\{x_n\}) = r_a(\{x_n\})$. By Proposition 4.1, we conclude that $(I-T)x = 0$. This completes the proof. □

5. Convergence theorems of the iterative schemes

In this section, we discuss, our iterative scheme (2.8) holds the properties (D_1) and (D_2) and prove the strong and Δ-convergence theorems for a nearly asymptotically nonexpansive nonself-mapping in $CAT(0)$ spaces.

Lemma 5.1. *Let* C *be a nonempty bounded closed convex subset of a complete CAT(0) space* X, $P : X \to C$ *be a nonexpansive retraction of* X *onto* C, *and* $T : C \to X$ *be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with a sequence* $\{a_n, \eta(T(PT)^{n-1})\}$ *such that*

$$\sum_{n=1}^{\infty} a_n < \infty \text{ and } \sum_{n=1}^{\infty}(\eta(T(PT)^{n-1}) - 1) < \infty.$$

Then the sequence $\{x_n\}$ *in* C *defined by (2.8) has the properties* (D_1) *and* (D_2), *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences of real numbers in* $(0,1)$ *such that* $0 < a \le \alpha_n, \beta_n \le b < 1$.

Proof. First, we show that $\lim_{n\to\infty} d(x_n, p)$ exists, for each $p \in F(T)$. Let $p \in F(T)$. From (2.8), we have

$$\begin{aligned} d(y_n, p) &= d\big(P\big[(1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n\big], p\big) \\ &\le d((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(T(PT)^{n-1}x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n \eta(T(PT)^{n-1})[d(x_n, p) + a_n] \\ &\le \eta(T(PT)^{n-1})d(x_n, p) + \beta_n \eta(T(PT)^{n-1})a_n. \end{aligned} \tag{5.1}$$

From (2.8) and (5.1), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(P[(1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n], p) \\ &\le (1-\alpha_n)d(T(PT)^{n-1}x_n, p) + \alpha_n d(T(PT)^{n-1}y_n, p) \\ &\le \eta(T(PT)^{n-1})\big[(1-\alpha_n)(d(x_n, p) + a_n) + \alpha_n(d(y_n, p) + a_n)\big] \end{aligned}$$

$$\begin{aligned} &\leq \eta(T(PT)^{n-1})\big[(1-\alpha_n)d(x_n,p)+\alpha_n\eta(T(PT)^{n-1})d(x_n,p) \\ &\quad +(1+\eta(T(PT)^{n-1}))a_n\big] \\ &\leq L_n d(x_n,p)+\rho_n, \end{aligned} \tag{5.2}$$

where $L_n = (\eta(T(PT)^{n-1}))^2$ and $\rho_n = \eta(T(PT)^{n-1})\big[\eta(T(PT)^{n-1})+1\big]a_n$. Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty}(L_n-1) &= \sum_{n=1}^{\infty}\Big(\eta(T(PT)^{n-1})-1\Big)\Big(\eta(T(PT)^{n-1})+1\Big) \\ &\leq (1+\eta)\sum_{n=1}^{\infty}\Big(\eta(T(PT)^{n-1})-1\Big) \\ &< \infty \end{aligned}$$

and $\sum_{n=1}^{\infty}\rho_n < \infty$. Applying Lemma 2.2, we have $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. This is the proof for the property (D_1).

Next, we prove that the sequence $\{x_n\}$ has the property (D_2). It follows from the first part that $\lim_{n\to\infty} d(x_n,p)$ exists. Let $\lim_{n\to\infty} d(x_n,p) = r \geq 0$. Since

$$d(T(PT)^{n-1}x_n,p) \leq \eta(T(PT)^{n-1})\big[d(x_n,p)+a_n\big],$$

for all $n \in \mathbb{N}$. Taking $\limsup_{n\to\infty}$ on the both sides, we have

$$\limsup_{n\to\infty} d(T(PT)^{n-1}x_n,p) \leq r. \tag{5.3}$$

From (5.1), it implies that

$$\limsup_{n\to\infty} d(y_n,p) \leq r. \tag{5.4}$$

Hence, from (5.4), we have

$$\begin{aligned} \limsup_{n\to\infty} d(T(PT)^{n-1}y_n,p) &\leq \limsup_{n\to\infty} \eta(T(PT)^{n-1})(d(y_n,p)+a_n) \\ &\leq r. \end{aligned} \tag{5.5}$$

Moreover, $r = \lim_{n\to\infty} d(x_{n+1},p)$ means that

$$\begin{aligned} r &= \lim_{n\to\infty}\Big\{d\big(P[(1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n],p\big)\Big\} \\ &\leq \lim_{n\to\infty}\Big\{(1-\alpha_n)\limsup_{n\to\infty} d(T(PT)^{n-1}x_n,p) \\ &\quad +\alpha_n \limsup_{n\to\infty} d(T(PT)^{n-1}y_n,p)\Big\}. \end{aligned}$$

Using (5.3) and (5.5), we have

$$r \leq \lim_{n\to\infty}((1-\alpha_n)r + \alpha_n r) = r.$$

Thus,

$$\lim_{n\to\infty}\left\{d(P[(1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n], p)\right\} = r,$$

for $r > 0$. Hence, it follows from Lemma 2.3, we have

$$\lim_{n\to\infty} d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) = 0. \tag{5.6}$$

Now

$$\begin{aligned} d(x_{n+1}, p) &= d(P[(1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n], p) \\ &\leq (1-\alpha_n)d(T(PT)^{n-1}x_n, p) + \alpha_n d(T(PT)^{n-1}y_n, p) \\ &\leq d(T(PT)^{n-1}x_n, p) + \alpha_n d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n). \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of above inequality, we have

$$r \leq \liminf_{n\to\infty} d(T(PT)^{n-1}x_n, p). \tag{5.7}$$

Hence, from (5.3) and (5.7), we have

$$\lim_{n\to\infty} d(T(PT)^{n-1}x_n, p) = r. \tag{5.8}$$

Next, we compute

$$\begin{aligned} d(T(PT)^{n-1}x_n, p) &\leq d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) \\ &\quad + d(T(PT)^{n-1}y_n, p) \\ &\leq d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n) \\ &\quad + \eta(T(PT)^{n-1})(d(y_n, p) + a_n). \end{aligned}$$

Taking $\liminf_{n\to\infty}$ and using (5.8) which yields that

$$r \leq \liminf_{n\to\infty} d(y_n, p). \tag{5.9}$$

Hence, from (5.4) and (5.9), we have

$$\lim_{n\to\infty} d(y_n, p) = r. \tag{5.10}$$

That is,

$$\begin{aligned} r &= \lim_{n\to\infty}\left\{d(P[(1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n], p)\right\} \\ &\leq \lim_{n\to\infty}\left\{d((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n, p)\right\} \\ &\leq \lim_{n\to\infty}\left\{(1-\beta_n)r + \beta_n r\right\} = r. \end{aligned}$$

Again, by using Lemma 2.3, we have

$$\lim_{n\to\infty} d(T(PT)^{n-1}x_n, x_n) = 0. \tag{5.11}$$

Now, applying (5.11), we have

$$\begin{aligned} d(y_n, x_n) &\leq d(P[(1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n], x_n) \\ &\leq \beta_n d(T(PT)^{n-1}x_n, x_n) \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{5.12}$$

Also, we observe that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(P[(1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n], x_n) \\ &\leq d((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n, x_n) \\ &\leq d(T(PT)^{n-1}x_n, x_n) + \alpha_n d(T(PT)^{n-1}y_n, T(PT)^{n-1}x_n) \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{5.13}$$

Therefore, we have

$$d(x_{n+1}, y_n) \leq d(x_{n+1}, x_n) + d(y_n, x_n) \to 0, \text{ as } n \to \infty. \tag{5.14}$$

Furthermore, since

$$\begin{aligned} d(x_{n+1}, T(PT)^{n-1}y_n) &\leq d(x_{n+1}, x_n) + d(x_n, T(PT)^{n-1}x_n) \\ &\quad + d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_n), \end{aligned}$$

by using (5.6), (5.11) and (5.13), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, T(PT)^{n-1}y_n) = 0. \tag{5.15}$$

Finally, we make use of the fact that every nearly asymptotically nonexpansive mapping is nearly k-Lipschitzian, then we get

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, T(PT)^{n-1}x_n) + d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_{n-1}) \\ &\quad + d(T(PT)^{n-1}y_{n-1}, Tx_n) \\ &\leq d(x_n, T(PT)^{n-1}x_n) + d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_{n-1}) \\ &\quad + d(T(PT)^{1-1}(PT)^{n-1}y_{n-1}, T(PT)^{1-1}x_n) \\ &\leq d(x_n, T(PT)^{n-1}x_n) + d(T(PT)^{n-1}x_n, T(PT)^{n-1}y_{n-1}) \\ &\quad + k_1 d((PT)^{n-1}y_{n-1}, x_n) \\ &\leq d(x_n, T(PT)^{n-1}x_n) + \eta(T(PT)^{n-1})(d(x_n, y_{n-1}) + a_n) \\ &\quad + k_1 d(PT(PT)^{n-2}y_{n-1}, x_n) \\ &\leq d(x_n, T(PT)^{n-1}x_n) + \eta(T(PT)^{n-1})(d(x_n, y_{n-1}) + a_n). \\ &\quad + k_1 d(T(PT)^{n-2}y_{n-1}, x_n) \end{aligned}$$

Letting $n \to \infty$ and using (5.12), (5.14) and (5.15), we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{5.16}$$

Hence, the sequence $\{x_n\}$ has an approximating fixed point property for T, i.e, $\{x_n\}$ has a property (D_2). □

Now we will give a Δ-convergence theorem in a CAT(0) space.

Theorem 5.2. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X, $P : X \to C$ be a nonexpansive retraction of X onto C, and $T : C \to X$ be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with a sequence $\{a_n, \eta(T(PT)^{n-1})\}$ such that*

$$\sum_{n=1}^{\infty} a_n < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} (\eta(T(PT)^{n-1}) - 1) < \infty.$$

Let $\{x_n\}$ be a sequence in C defined by (2.8), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $(0, 1)$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then $\{x_n\}$ is Δ-convergent to an element of $F(T)$.

Proof. First, we show that $w_w(\{x_n\}) \subseteq F(T)$. Let $u \in w_w(\{x_n\})$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $AC(C, \{u_n\}) = \{u\}$. By Lemma 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$. By Theorem 4.2, $v \in F(T)$. By Lemma 5.1, $\lim_{n\to\infty} d(x_n, v)$ exists. We now claim that $u = v$. Suppose, on the contrary, that $u \neq v$. Then by uniqueness of asymptotic center, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \leq \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) = \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction. Thus, $u = v \in F(T)$ and hence $w_w(\{x_n\}) \subseteq F(T)$.

To show that $\{x_n\}$ is Δ-convergent to a fixed point of T, it suffices to show that $w_w(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$. Let $AC(C, \{u_n\}) = \{u\}$ and $AC(C, \{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$.

Finally, we claim that $x = v$. Suppose that if not, then by the existence of $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic center, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction and hence $x = v \in F(T)$. Therefore, $w_w(\{x_n\}) = \{x\}$. This means that $\{x_n\}$ is Δ-convergent to an element of $F(T)$. □

Next, we will give two strong convergence theorems in CAT(0) spaces.

Theorem 5.3. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X, $P : X \to C$ be a nonexpansive retraction of X onto C, and $T : C \to X$ be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with a sequence $\{a_n, \eta(T(PT)^{n-1})\}$ such that*

$$\sum_{n=1}^{\infty} a_n < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} (\eta(T(PT)^{n-1}) - 1) < \infty.$$

Let $\{x_n\}$ be a sequence in C defined by (2.8) and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers in $(0, 1)$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0,$$

where $D(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.

Proof. Necessity is obvious.
Conversely, suppose that $\liminf_{n\to\infty} D(x_n, F(T)) = 0$. From (5.2), we have

$$D(x_{n+1}, F(T)) \le (1 + (L_n - 1))D(x_n, F(T)) + \rho_n, \ n \in \mathbb{N}.$$

By applying Lemma 2.2, $\lim_{n\to\infty} D(x_n, F(T))$ exists. It follows that

$$\lim_{n\to\infty} D(x_n, F(T)) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. In order to prove that, we set $u_n = (L_n - 1)$, then above inequality become

$$D(x_{n+1}, F(T)) \le (1 + u_n)D(x_n, F(T)) + \rho_n, \ n \in \mathbb{N}.$$

Using arguments similar to those given in Lemma 5 [16] and Theorem 4.3 [1], we can easily obtain the following inequality

$$d(x_{n+m}, p) \quad \le \quad L\left[d(x_n, p) + \sum_{j=n}^{\infty} \rho_j\right],$$

for every $p \in F(T)$ and for all $m, n \in \mathbb{N}$, where $L = e^{M\left(\sum_{j=n}^{n+m-1} u_j\right)} > 0$ and $M > 0$. Since, $\sum_{n=1}^{\infty} u_n < \infty$, we have

$$L^* = e^{M\left(\sum_{n=1}^{\infty} u_n\right)} \geq L = e^{M\left(\sum_{j=n}^{n+m-1} u_j\right)} > 0.$$

Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} D(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \rho_n < \infty$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$D(x_n, F(T)) \quad < \quad \frac{\epsilon}{4L^*} \;\text{ and }\; \sum_{j=n_0}^{\infty} \rho_j < \frac{\epsilon}{6L^*}.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\epsilon}{4L^*}$. Thus, there must exist $p^* \in F(T)$ such that

$$d(x_{n_0}, p^*) \quad < \quad \frac{\epsilon}{3L^*}.$$

Hence for $n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2L^*\left[d(x_{n_0}, p^*) + \sum_{j=n_0}^{\infty} b_j\right] \\ &< 2L^*\left(\frac{\epsilon}{3L^*} + \frac{\epsilon}{6L^*}\right) \\ &= \epsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in closed subset C of a complete CAT(0) space X, and so it must converge strongly to a point q in C.

Now, $\lim_{n\to\infty} D(x_n, F(T)) = 0$ gives that $D(q, F(T)) = 0$. From Theorem 4.2 $F(T)$ is closed, we have $q \in F(T)$. This completes the proof. □

Recall that a mapping T from a subset of a metric space (X, d) into itself with $F(T) \neq \phi$ is said to *satisfy condition* (A) (see [44]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for $t \in (0, \infty)$ such that

$$d(x, Tx) \quad \geq \quad f(D(x, F(T))),$$

for all $x \in C$.

Theorem 5.4. *Let C be a bounded closed convex subset of a complete CAT(0) space X, $P : X \to C$ be a nonexpansive retraction, and $T : C \to X$ be a uniformly continuous, nearly asymptotically nonexpansive nonself-mapping with a sequence $\{a_n, \eta(T(PT)^{n-1})\}$ such that*

$$\sum_{n=1}^{\infty} a_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (\eta(T(PT)^{n-1}) - 1) < \infty.$$

Let $\{x_n\}$ be a sequence in C defined by (2.8) and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers in $(0,1)$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Suppose that T satisfies condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 5.1, we observe that sequence $\{x_n\}$ has an approximating fixed point property for T, *i.e.*, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Further, by condition (A),

$$\lim_{n\to\infty} d(x_n, Tx_n) \geq \lim_{n\to\infty} f(D(x_n, F(T))).$$

It follows that $\lim_{n\to\infty} D(x_n, F(T)) = 0$. Therefore, the desired result follows from Theorem 5.3. □

Remark 5.5.
(i) Our Theorem 5.2 extends Theorems 4.2, 4.4 of Abbas *et al.* [1] from nearly asymptotically nonexpansive mappings to nearly asymptotically nonexpansive nonself-mappings.
(ii) Theorem 5.2 extends Theorem 1 of Khan [26] from uniformly convex Banach spaces to CAT(0) spaces.
(iii) Theorems 5.2, 5.4 extend corresponding results of Abbas *et al.* [2], Chang *et al.* [10], Dhompongsa and Panyanak [14], Khan *et al.* [23], Karapinar, *et al.* [24], Kang *et al.* [19, 20], Kim *et al.* [27], Nanjaras and Panyanak [37], to more general class of nonself-mappings.
(iv) In view of Remark 2.8, we conclude that our Theorems 5.2, 5.4 extend corresponding results of [38, 43, 46, 47, 48], for faster iteration scheme.

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