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SOME GEOMETRIC PARAMETERS AND NORMAL STRUCTURE IN BANACH SPACES

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Abstract. In this paper we prove some sufficient conditions for the normal structure of a Banach space X in terms of the moduli of convexity $\delta(\epsilon)$ and $C(\epsilon)$, the modulus of smoothness $\rho_X(\epsilon)$, the modulus of squareness $J(X)$, the moduli of arc length $O(X)$ and $Q(X)$, and the coefficient of weak orthogonality $w(X)$. Some known results are improved and some of them are obtained in a different way.

1. Introduction

Let X be a Banach space, and let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \le 1\}$ be the unit sphere and unit ball of X respectively.

Definition 1.1. ([1]) A bounded, convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\| : y \in H\} < \mathrm{diam}(H)$, where $\mathrm{diam}(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have normal structure if every bounded, convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X that contains more than one point has normal structure. X is said to have uniform

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normal structure if there exists $0 < c < 1$ such that for any subset K as above, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} < c\,\mathrm{diam}(K)$.

For a reflexive Banach space X, the normal structure and weak normal structure coincide.

Let $\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x - y\| \geq \epsilon\}$ where $0 \leq \epsilon \leq 2$ be the modulus of convexity of X[2].

The following result regarding the relationship between normal structure and the modulus of convexity of X was proved in [6], [14] and [15].

Theorem 1.2. *For any Banach space X, $\delta_X(1+\epsilon) > \frac{\epsilon}{2}$ for some $0 \leq \epsilon \leq 1$ implies that X has uniform normal structure.*

The following result regarding the relationship between normal structure and a value of the modulus of convexity of X at a certain point was proved in [9].

Theorem 1.3. *A Banach space X with $\delta_X(\frac{1+\sqrt{5}}{2}) > \frac{3-\sqrt{5}}{2}$ has uniform normal structure.*

In [4] and [11], Gao and Lau introduced parameters $J(X) = \sup\{\|x + y\| \bigwedge \|x-y\| : x, y \in S(X)\}$, and $g(X) = \inf\{\|x+y\| \bigvee \|x-y\| : x, y \in S(X)\}$, and proved that $g(X) \cdot J(X) = 2$.

In [18], Sims introduced the following parameter

$$w(X) = \sup\{\lambda > 0 : \lambda \liminf_{n\to\infty} \|x_n + x\| \leq \liminf_{n\to\infty} \|x_n - x\|\}$$

where the supremum is taken over all the weakly null sequence x_n in X and all the elements x of X. It was proved that $\frac{1}{3} \leq w(X) \leq 1$ for all Banach space X.

In [12], Jimenez-Melado, Llorens-Fuster, and Saejung proved the following result regarding the relationship between normal structure and parameters $J(X)$ and $w(X)$.

Theorem 1.4. *For any Banach space X, $J(X) < 1 + w(X)$ implies X has normal structure.*

Let $\rho_X(\tau) = \sup\{\frac{\|x+y\|+\|x-y\|-2}{2} : x \in S(X), y \in \tau S(X)\}$, where $\tau \geq 0$ be the modulus of smoothness of X [2]. Then $\frac{\rho_X(\tau)}{\tau}$ is a decreasing function.

In [19], by a dual view of a theorem of Baillon, Turett proved that

Theorem 1.5. *If X is a Banach space with $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} < \frac{1}{2}$, then X has weak normal structure.*

The following result regarding the relationship between normal structure and the modulus of smoothness of X was proved in [8]:

Theorem 1.6. *A Banach space X with $\rho_X(\tau) < \frac{\tau}{2}$ for some $0 < \tau \leq 1$, or $\rho_X(\tau) < \tau - \frac{1}{2}$ for some $1 < \tau < \infty$ has uniform normal structure.*

The following result regarding the relationship between normal structure and a value of the modulus of smoothness of X at a certain point was proved in [9].

Theorem 1.7. *A Banach space X with $\rho_X(1) < \frac{\sqrt{5}-1}{2}$ has uniform normal structure.*

In this paper we demonstrate the relationships among parameters $\delta(\epsilon)$, $C(X)$, $\rho_X(\tau)$, $J(X)$, $O(X)$, $Q(X)$, and $\omega(X)$ of X, that imply uniform normal structure. The main results in [6], [8], [9], [10], [13], [14], [15] and [16] are either improved under a certain condition or obtained in a different way.

Lemma 1.8. *([3]) Let X be a Banach space without weak normal structure, then for any $0 < \epsilon < 1$, there exists a sequence $\{x_n\} \subseteq S(X)$ with $x_n \stackrel{w}{\to} 0$, and*

$$1 - \epsilon < \|x_{n+1} - x\| < 1 + \epsilon$$

for sufficiently large n, and any $x \in \mathrm{co}\{x_k\}_{k=1}^n$.

Lemma 1.9. *Let X be a Banach space without weak normal structure. Then, for any $0 < \epsilon < 1$, there exists a sequence $\{x_n\} \subseteq S(X)$ satisfying*

(i) $1 - \epsilon \leq \|x_n - x_1\| \leq 1 + \epsilon, \forall n > 1$;

(ii) $\|x_n + x_1\| \leq \frac{1+\epsilon}{w(X)-\epsilon}, \forall n > 1$.

Proof. It follows directly from the definition of $w(X)$ and Lemma 1.8. □

2. Main results

Theorem 2.1. *For a Banach space X, if $\delta(1 + w(X)) > \frac{1-w(X)}{2}$, then X has normal structure.*

Proof. $\delta(1+w(X)) > \frac{1-w(X)}{2}$ implies $\delta(2^-) > 0$, so X is uniformly nonsquare, hence X is reflexive, therefore weak normal structure and normal structure coincide [5].

Let $\epsilon > 0$ be such that $1 - w(X) + \epsilon < 1 + w(X) - \epsilon$. We let x_1 and $\{x_n\}$ be as in Lemma 1.9, and let $u_n = x_n - x_1$ and $v_n = (w(X) - \epsilon)(x_n + x_1)$. Then $\|u_n\| \leq 1 + \epsilon$ and $\|v_n\| \leq 1 + \epsilon$ for all $n > 1$. Since $0 \in \overline{\mathrm{co}}^w\{x_n\}_{n=1}^\infty = \overline{\mathrm{co}}\{x_n\}_{n=1}^\infty$, we can also assume by Lemma 1.8 that $\|x_n - \frac{1-w(X)+\epsilon}{1+w(X)-\epsilon} \cdot x_1\|=$

$\|x_n - (\frac{1-w(X)+\epsilon}{1+w(X)-\epsilon} \cdot x_1 + \frac{2w(X)-2\epsilon}{1+w(X)-\epsilon} \cdot 0)\| \geq 1-\epsilon$, for larger n. Then

$$\begin{aligned}\|u_n + v_n\| &= \|(1+w(X)-\epsilon)x_n - (1-w(X)+\epsilon)x_1\| \\ &\geq (1+w(X)-\epsilon)\left\|x_n - \frac{1-w(X)+\epsilon}{1+w(X)-\epsilon} \cdot x_1\right\| \\ &\geq (1+w(X)-\epsilon)(1-\epsilon),\end{aligned}$$

and

$$\|u_n - v_n\| = \|(1-w(X)+\epsilon)x_n - (1+w(X)-\epsilon)x_1\|.$$

Since $(1-w(X)+\epsilon)x_n - (1+w(X)-\epsilon)x_1 \overset{w}{\to} -(1+w(X)-\epsilon)x_1$, we can take n big enough such that

$$\|u_n - v_n\| \geq \|-(1+w(X)-\epsilon)x_1\| - \epsilon = 1 + w(X) - 2\epsilon.$$

It then follows from the definition of $\delta_X(\cdot)$ that

$$\delta_X(\|u_n - v_n\|) \leq 1 - \frac{\|u_n + v_n\|}{2} \leq 1 - \frac{1}{2}((1+w(X)-\epsilon)(1-\epsilon)).$$

Letting $\epsilon \to 0$ gives $\delta(1+w(X)) \leq \frac{1-w(X)}{2}$, which is a contradiction. So, if $\delta(1+w(X)) > \frac{1-w(X)}{2}$, X has normal structure. □

Remark 2.2. (1) If $w(X) > \frac{1}{2}$, then $\frac{w(X)}{2} > \frac{1-w(X)}{2}$. Therefore, Theorem 2.1 improves Theorem 1.2 for the case $w(X) > \frac{1}{2}$.

(2) Similarly, Theorem 2.1 improves Theorem 1.3 for the case $w(X) > \frac{\sqrt{5}-1}{2}$.

Let $C_X(\epsilon) = \sup\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x-y\| \leq \epsilon\}$ where $0 \leq \epsilon \leq 2$ be the another modulus of convexity of X. It is not hard to see that $C(\epsilon)$ is an non-decreasing and continuous function of ϵ on $[0, 2)$ (see [7]).

By using the similar argument as in the proof of theorem 2.1, we can prove the following theorem.

Theorem 2.3. *For a Banach space X, if $C_X\left(\frac{2}{1+w(X)}\right) < \frac{w(X)}{1+w(X)}$, then X has normal structure.*

Lemma 2.4. *([11]) For a Banach space X, $J(X) = \sup\{\epsilon : \delta_X(\epsilon) \leq 1 - \frac{\epsilon}{2}\}$.*

Lemma 2.5. *For a Banach space X, $g(X) = \sup\{\epsilon : C_X(\epsilon) \leq 1 - \frac{\epsilon}{2}\}$.*

Proof. The proof is similar to the proof of Lemma 2.4 in [11] so it is omitted. □

Remark 2.6. By Lemma 2.4 and Lemma 2.5, it is easy to see that

$$\begin{aligned}
&J(X) < 1 + w(X) \\
&\Leftrightarrow \delta(1 + w(X)) > \frac{1 - w(X)}{2} \\
&\Leftrightarrow C\left(\frac{2}{1 + w(X)}\right) < \frac{w(X)}{1 + w(X)}.
\end{aligned}$$

So Theorem 2.1 and Theorem 2.3 are equivalent to Theorem 1.4, but proved in a different way.

Theorem 2.7. *For a Banach space X, if $\rho_X(\tau) < \tau \cdot w(X)$ for $\tau \leq 1$; or $\rho_X(\tau) < \tau + w(X) - 1$ for $\tau > 1$ then X has normal structure.*

Proof. We first prove that $\rho_X(\tau) < \tau \cdot w(X)$ for $\tau \leq 1$ implies that X has normal structure. Let $\tau \leq 1$ and $\epsilon > 0$ be such that $1 - \tau(w(X) - \epsilon) < 1 + \tau(w(X) - \epsilon)$. Let x_1 and $\{x_n\}$ be as in Lemma 1.9, and let $u_n = x_n - x_1$ and $v_n = (w(X) - \epsilon)(x_n + x_1)$ again. We now estimate $\|u_n \pm \tau v_n\|$.

We can assume by Lemma 1.8 that x_n and x_1 also satisfy

$$\begin{aligned}
&\left\| x_n - \frac{1 - \tau(w(X) - \epsilon)}{1 + \tau(w(X) - \epsilon)} x_1 \right\| \\
&= \left\| x_n - \left(\frac{1 - \tau(w(X) - \epsilon)}{1 + \tau(w(X) - \epsilon)} \cdot x_1 + \frac{2\tau(w(X) - \epsilon)}{1 + \tau(w(X) - \epsilon)} \cdot 0 \right) \right\| \\
&\geq 1 - \epsilon
\end{aligned}$$

for sufficiently large n. Then, for such n,

$$\begin{aligned}
\|u_n + \tau v_n\| &= \|(1 + \tau(w(X) - \epsilon))x_n - (1 - \tau(w(X) - \epsilon))x_1\| \\
&\geq (1 + \tau(w(X) - \epsilon)) \left\| x_n - \frac{1 - \tau(w(X) - \epsilon)}{1 + \tau(w(X) - \epsilon)} \cdot x_1 \right\| \\
&\geq (1 + \tau(w(X) - \epsilon))(1 - \epsilon)
\end{aligned}$$

and

$$\|u_n - \tau v_n\| = \|(1 - \tau(w(X) - \epsilon)) \cdot x_n - (1 + \tau(w(X) - \epsilon)) \cdot x_1\|.$$

Since

$$(1 - \tau(w(X) - \epsilon)) \cdot x_n - (1 + \tau(w(X) - \epsilon)) \cdot x_1 \xrightarrow{w} -(1 + \tau(w(X) - \epsilon)) \cdot x_1,$$

we can take n big enough such that

$$\|u_n - \tau v_n\| \geq \| - (1 + \tau(w(X) - \epsilon)) \cdot x_1\| - \epsilon = 1 + \tau(w(X) - \epsilon) - \epsilon.$$

From the definition of $\rho_X(\cdot)$, we have

$$\begin{aligned}\rho_X(\tau) &\geq \frac{(1+\tau(w(X)-\epsilon))(1-\epsilon)+1+\tau(w(X)-\epsilon)-\epsilon-2}{2} \\ &= \frac{(2\tau(w(X)-\epsilon))-\epsilon-\epsilon(1+\tau(w(X)-\epsilon))}{2}.\end{aligned}$$

Let $\epsilon \to 0$ gives $\rho_X(\tau) \geq \tau w(X)$ which is a contradiction. So, if $\rho_X(\tau) < \tau \cdot w(X) \leq 1$ for some $0 \leq \tau \leq 1$, X has normal structure.

Finally, we observe that

$$\begin{aligned}&\rho(\tau) < \tau w(X) \text{ for some } 0 < \tau < 1 \\ &\Leftrightarrow 1+\rho(\tau) < 1+\tau w(X) \text{ for some } 0 < \tau < 1 \\ &\Leftrightarrow \tau(1+\rho(\frac{1}{\tau})) < 1+\tau w(X) \text{ for some } 0 < \tau < 1 \\ &\Leftrightarrow \rho(\frac{1}{\tau}) < \frac{1}{\tau} + w(X) - 1 \text{ for some } 0 < \tau < 1 \\ &\Leftrightarrow \rho_X(\tau') < \tau' + w(X) - 1 \text{ for some } \tau' > 1.\end{aligned}$$

Consequently, if $\rho_X(\tau) < \tau + w(X) - 1$ for some $\tau > 1$, then X has normal structure. □

Remark 2.8. (1) Compare to Theorem 1.5, if $w(X) > \frac{1}{2}$, then Theorem 1.5 is improved.

(2) Compare to theorem 1.6, if $0 < \tau \leq 1$ and $w(X) > \frac{1}{2}$, we have $\tau w(X) \geq \frac{\tau}{2}$, then Theorem 1.6 is improved; if $1 < \tau \leq \frac{1}{2(1-w(X))}$ and $w(X) > \frac{1}{2}$, we have $\tau w(X) \geq \tau - \frac{1}{2}$, then Theorem 1.6 is improved too.

(3) Compare to theorem 1.6, if $w(X) > \frac{1}{2}$, we have $\tau + w(X) - 1 \geq \tau - \frac{1}{2}$, then Theorem 1.6 is improved. Therefore, for a Banach space X with $w(X) > \frac{1}{2}$, Theorem 1.5 and Theorem 1.6 are improved.

(4) Compare to theorem 1.7, if $w(X) > \frac{\sqrt{5}-1}{2}$, Theorem 1.7 is improved.

(5) In the case $0 < \rho \leq 1$, conditions $\rho(\tau) < \tau w(X)$ and $\rho'(0) < w(X)$ are equivalent and the latter condition for $0 < \rho \leq 1$ implies X has normal structure is proved in [13] and [16].

We now present some more sufficient condition for normal structure in terms of the moduli of arc length. We recall some definition. A continuous mapping $x(t)$ from a closed interval $[a, b]$ to a Banach space X is called a curve in X and denoted by $C = \{x(t) : a \leq t \leq b\}$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces. For a curve $C = \{x(t) : a \leq t \leq b\}$, let P stand for a partition $a = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_n = b$ of the interval $[a, b]$

and $l(C,P) = \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|$. The points $x(t_i)$, where $i = 0, 1, 2, \ldots, n$, are called partition points on C, then the length $l(C)$ of a curve C, is defined as the least upper bound of $l(C,P)$ for all possible partitions P of $[a,b]$, that is,

$$l(C) = \sup_P \{l(C,P)\}.$$

If $l(C)$ is finite, then the curve C is called rectifiable.

Let $l_a^t(C)$ denote the length of curve $C = x(t)$ from a to t. For a rectifiable curve $C = x(t), a \leq t \leq b$, the arc length $l_a^t(C)$ is a continuous function of t.

For a normed linear space X, if X_2 is a two dimensional subspace of X, then $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation (see [5] or [17]).

Let $O(X) = \inf\{l(S(Y)) : Y \in F_2(X)\}$, and $Q(X) = \sup\{l(S(Y)) : Y \in F_2(X)\}$, where $F_2(X)$ denotes the family of all two dimensional subspaces of X [7].

The following result shows a relationship between normal structure and the modulus of arc length.

Corollary 2.9. *For a Banach space X, if $Q(X) < 4 + 4w(X)$ then X has normal structure.*

Proof. If X fails to have normal structure, then by theorem 2.7 $Q(X) \geq 4 + 4\rho(1) \geq 4 + 4w(X)$, which is a contradiction. □

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