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UNIFIED COMPLEX COMMON FIXED POINT RESULTS VIA CONTRACTIVE CONDITIONS OF INTEGRAL TYPE WITH AN APPLICATION

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Abstract. The aim of this paper is to prove some unified common fixed point theorems under contractive conditions of integral type for two pairs of weakly compatible mappings satisfying the property (E.A) in complex valued symmetric spaces using an implicit relation. Some illustrative examples are also given which substantiate the usefulness of our utilized implicit relation. Our results generalize and improve some of the existing results in the literature and at the same time deduce new contractions of integral type in the context of complex fixed point theory. We apply one of our results to examine the existence and uniqueness of common solution for a system of Volterra-Hammerstein integral equations.

1. Introduction and Preliminaries

Banach contraction principle is one of the most powerful results in analysis. This principle has been extended and generalized in several ways to prove the existence and uniqueness of fixed points of mappings. One way of proving such generalized results is to vary spaces such as: metric spaces, metric like spaces, fuzzy metric spaces, cone metric spaces and several others.

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Recently, Azam *et al.* [6] initiated the concept of complex valued metric space and utilized the same to prove common fixed point theorems for two mappings satisfying rational inequalities. Since then, many authors studied the existence and uniqueness of the fixed point and common fixed point results of self mappings satisfying different contraction conditions in complex metric spaces. Though complex metric spaces form a special class of cone metric space, yet the definition of a cone metric space banks on the underlying Banach space which is not a division ring. Hence, rational expressions are not meaningful in cone metric spaces and henceforth many results involving rational contractions can not be generalized to cone metric spaces. So, with a view to prove results involving rational inequalities, Azam *et al.* [6] propounded the idea of complex metric spaces. In cone metric spaces the underlying metric assumes values in linear spaces where the linear space may be even infinite dimensional, whereas in the case of complex metric spaces the metric values belong to the set of complex numbers which is one dimensional vector space over the complex field. This is an instance which paves the way to consider complex metric spaces independently. With a view to have further improvement, we consider continuous complex symmetric (not necessarily satisfying triangle inequality).

Aamri and Moutawakil [1] introduced the notion of (E.A) property and Liu *et al.* [10] extend it to common (E.A) property. Verma and Pathak [21] redefined this property in complex-valued metric spaces. Thereafter, several authors proved a multitude of fixed point theorems using the concepts of weakly compatible mappings and (E.A) property.

In this paper, we introduce the concept of complex valued symmetric spaces which is larger than the class of complex valued metric spaces. Also, we redefine (E.A) property on such spaces.

Let $\mathbb{C}$ be the set of all complex numbers and $c_1, c_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$c_1 \precsim c_2 \Longleftrightarrow Re(c_1) \leq Re(c_2) \text{ and } Im(c_1) \leq Im(c_2).$$

Consequently, it follows that $c_1 \precsim c_2$, if one of the following conditions is satisfied:

(i) $Re(c_1) = Re(c_2),\ Im(c_1) = Im(c_2)$,
(ii) $Re(c_1) < Re(c_2),\ Im(c_1) = Im(c_2)$,
(iii) $Re(c_1) = Re(c_2),\ Im(c_1) < Im(c_2)$,
(iv) $Re(c_1) < Re(c_2),\ Im(c_1) < Im(c_2)$.

In particular, we write $c_1 = c_2$ if (i) holds and we write $c_1 \precnsim c_2$ if $c_1 \neq c_2$ and one of (ii), (iii) and (iv) is satisfied while $c_1 \prec c_2$ if only (iv) is satisfied.

Throughout this paper, $\succsim$ is the dual relation of $\precsim$ and I stands for the identity mapping. Also $\mathbb{C}_+ = \{c \in \mathbb{C} : 0 \precsim c\}$.

Remark 1.1. Note that the following assertions hold for all $c_1, c_2, c_3 \in \mathbb{C}$:

(1) $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ and $0 \precsim c_1 \Longrightarrow \alpha c_1 \precsim \beta c_1$,
(2) $0 \precsim c_1 \precnsim c_2 \Longrightarrow |c_1| < |c_2|$,
(3) $c_1 \precsim c_2, c_2 \prec c_3 \Longrightarrow c_1 \prec c_3$.

Definition 1.2. Let X be a nonempty set. Suppose that a mapping $d : X \times X \longrightarrow \mathbb{C}_+$ satisfies the following properties for all $x, y \in X$:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$.

Then d is called a complex valued symmetric and the pair (X, d) is called a complex valued symmetric space (shortly, complex symmetric space).

Let d be a complex symmetric on a nonempty set X and for $0 \prec \epsilon \in \mathbb{C}$ and $x \in X$, let $N(x, \epsilon) = \{y \in X : d(x, y) \prec \epsilon\}$. A topology τ_d on X is given by $U \in \tau_d$ if and only if for each $x \in U$, $N(x, \epsilon) \subseteq U$ for some $0 \prec \epsilon \in \mathbb{C}$.

Example 1.3. Let $X = [0, \infty)$ and define a mapping $d : X \times X \longrightarrow \mathbb{C}$ as follows:

$$d(x, y) = ie^{|x-y|} - i, \quad \forall\, x, y \in X.$$

Then the pair (X, d) is a complex symmetric space.

Definition 1.4. Let (X, d) be a complex symmetric space. A pair (P, Q) of self mappings on X is said to satisfy the property (E.A) if there is a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = u$, for some $u \in X$.

Example 1.5. Consider the complex symmetric space given in Example 1.3. Define $P, Q : X \longrightarrow X$ by

$$Px = 4x + 2 \text{ and } Qx = x + 17, \text{ for all } x \in X.$$

Consider the sequence $\{u_n = \frac{2}{n} + 5\}_{n\in\mathbb{N}}$. Then

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = 22.$$

Hence, the pair (P, Q) satisfies the property (E.A).

Definition 1.6. Let (X, d) be a complex symmetric space. Two pairs (P, f) and (Q, g) of self mappings on X are said to satisfy the common property (E.A) if there are two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} fu_n = \lim_{n\to\infty} Qv_n = \lim_{n\to\infty} gv_n = u,$$

for some $u \in X$.

Definition 1.7. Let P, Q, f and g be four self mappings on a complex symmetric space (X, d) and $u, v, t \in X$. If

(i) $Pu = u$, then u is said to be a fixed point of P,
(ii) $Pu = Qu = u$, then u is said to be a common fixed point of P and Q,
(iii) $Pu = fu$, then u is said to be a coincidence point of P and f,
(iv) $Pu = fu = t$, then t is called a point of coincidence of P and f,
(v) $Pu = fu = t$ and $Qv = gv = t$, then t is called a common point of coincidence of the pairs (P, f) and (Q, g).

Definition 1.8. A pair of self mappings (P, Q) on a complex symmetric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(PQu_n, QPu_n) = 0$, whenever $\{u_n\}$ is a sequence in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = u$, for some $u \in X$.

Evidently, in view of Definition 1.8, two self mappings P and Q on a complex symmetric space (X, d) are non-compatible if there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = u$, for some $u \in X$, but $\lim_{n\to\infty} d(PQu_n, QPu_n)$ is either non-zero or non-existent. Hence, two non-compatible self mappings on a complex symmetric space (X, d) enjoy the property (E.A).

Definition 1.9. A pair of self mappings (P, Q) on a complex symmetric space (X, d) is said to be weakly compatible if $PQx = QPx$ whenever $Px = Qx$, $x \in X$.

Definition 1.10. The required control functions are defined as follows:

(i) $\psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ is a continuous nondecreasing function with $\psi(z) = 0$ if and only if $z = 0$,
(ii) $\phi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ is a lower semicontinuous function with $\phi(z) = 0$ if and only if $z = 0$,
(iii) $\theta : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ is an upper semicontinuous function with $\theta(z) = 0$ if and only if $z = 0$.

By Ψ, Φ and Θ, we respectively denote the set of all $\psi' s$, the set of all $\phi' s$ and the set of all $\theta' s$. Here, it can be pointed out that (i) and (ii) are available in [2].

Before proving our results, let us point out a fallacy in [20], The max function for complex numbers with partial order relation $\precsim$ was defined in [20] as follows:

(i) $\max\{z_1, z_2\} = z_2 \Longleftrightarrow z_1 \precsim z_2$,
(ii) $z_1 \precsim \max\{z_2, z_3\} \Longrightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.

In fact, this function is not well defined since two complex numbers may or may not be comparable. For example $\max\{1+5i, 5+i\}$ does not exist. Here we correct this definition as follows:

Definition 1.11. The max function for complex numbers with partial order relation $\precsim$ is defined as follows (for all $z_1, z_2 \in \mathbb{C}$):

$$\max\{z_1, z_2\} = z_2 \Leftrightarrow |z_1| \leq |z_2|.$$

The purpose of this paper is to present unified common fixed point theorems under contractive conditions of integral type for two pairs of weakly compatible mappings satisfying the property (E.A) in complex symmetric spaces using an implicit relation. The proved results generalize and improve some of the existing results in the literature and at the same time deduce new contractions in the context of complex symmetric spaces. As an application of our main result, we apply our result to prove the existence and uniqueness of common solution for a system of Volterra-Hammerstein integral equations.

2. An implicit relation

In this section, we extend the idea of the implicit relations (due to Popa [12]) to complex symmetric spaces in order to prove unified common fixed point theorems of integral type in complex symmetric spaces.

Definition 2.1. Let $\Im$ be the set of all complex valued lower semi-continuous functions $\Xi : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ satisfying the following conditions (for all $0 \prec c \in \mathbb{C}$):

(Ξ_1) $\Xi(c, 0, 0, c, c, 0) \succ 0$,
(Ξ_2) $\Xi(c, 0, c, 0, 0, c) \succ 0$,
(Ξ_3) $\Xi(c, c, 0, 0, c, c) \succsim 0$.

Example 2.2. Define $\Xi : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ as follows:

(i) $\Xi(c_1, c_2, ..., c_6) = c_1 - \Sigma_{i=2}^{6} \lambda_i c_i$, where $\lambda_i \in \mathbb{R}_+, i = 2, 3, ..., 6$ such that $\Sigma_{i=2}^{6} \lambda_i < 1$,
(ii) $\Xi(c_1, c_2, ..., c_6) = c_1 - \lambda \max\{c_2, c_4, c_5\}, \lambda \in [0, 1)$,
(iii) $\Xi(c_1, c_2, ..., c_6) = c_1 - \lambda\Delta, \Delta \in \{c_2, c_3, c_4, \frac{1}{2}(c_3 + c_4)\}$ and $\lambda \in [0, 1)$,
(iv) $\Xi(c_1, c_2, ..., c_6) = c_1 - \lambda \max\{c_2, c_3, c_4, \frac{1}{2}(c_5 + c_6)\}, \lambda \in [0, 1)$,
(v) $\Xi(c_1, c_2, ..., c_6) = c_1 - \lambda_1 \frac{c_4(c_3+c_6)}{1+c_2+c_6} - \lambda_2 \frac{c_6 c_5(c_3+c_4)}{1+c_2+c_6} - \lambda_3(c_5 + c_6) - \lambda_4(c_3 + c_4) - \lambda_5 c_2$,
where $\lambda_i \in \mathbb{R}_+, i = 1, 2, ..., 5$ such that $\lambda_3 + \lambda_4 < 1$ and $2\lambda_3 + \lambda_5 < 1$,
(vi) $\Xi(c_1, c_2, ..., c_6) = c_1 - \lambda \frac{c_3 c_5 + c_4 c_6}{1+c_5+c_6}, \lambda \in [0, 2)$,

(vii) $\Xi(c_1, c_2, ..., c_6) = c_1 - \alpha_1(c_2)c_2 - \alpha_2(c_2)\frac{c_2c_4}{1+c_2} - \alpha_3(c_2)\frac{c_3c_4}{1+c_2} - \alpha_4(c_2)\frac{c_4c_6}{1+c_2}$, where $\alpha_i : \mathbb{C}_+ \longrightarrow [0, 1), i = 1, 2, 3, 4$ are given upper semi-continuous mappings,

(viii) $\Xi(c_1, c_2, ..., c_6) = \psi(c_1) - \psi\big(\frac{c_3c_4}{1+c_2}\big) + \phi\big(\frac{c_3c_4}{1+c_2}\big)$, $\psi \in \Psi$ and $\phi \in \Phi$,

(ix) $\Xi(c_1, c_2, ..., c_6) = \psi(c_1) - \psi\big(\frac{c_4c_6}{1+c_2}\big) - \theta\big(\frac{c_4c_6}{1+c_2}\big)$, $\psi \in \Psi$ and $\theta \in \Theta$,

(x) $\Xi(c_1, c_2, ..., c_6) = \psi(c_1) - \psi\big(\max\{c_2, c_3, c_4\}\big) + \psi\big(\max\{c_5, c_6\}\big)$, $\psi \in \Psi$,

(xi) $\Xi(c_1, c_2, ..., c_6) = c_1 - \theta\big(\max\{c_2, c_3, c_4, c_5, c_6\}\big)$, $\theta \in \Theta$ with $\theta(c) \prec c, \ \forall c \in \mathbb{C}_+$,

(xii) $\Xi(c_1, c_2, ..., c_6) = c_1 - \lambda_1\frac{c_3c_4}{1+c_2} - \lambda_2\frac{c_3c_5}{1+c_2} - \lambda_3\frac{c_4c_6}{1+c_2}, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$,

(xiii) $\Xi(c_1, c_2, ..., c_6) = c_1 - \Sigma_{i=2}^{6}\lambda_i c_i - \gamma\frac{c_2c_4+c_3c_5}{1+c_2+c_6}, \lambda_i \in \mathbb{R}$ such that $\Sigma_{i=2}^{6}\lambda_i < 1$ and $\gamma \in \mathbb{C}$.

3. Main results

On the lines of [22], consider $\Omega = \{\omega : \mathbb{R}^n \to \mathbb{C}\}$ wherein ω is a complex valued Lebesgue integrable mapping which is summable and non-vanishing on each measurable subset of $\mathbb{R}^n$, such that for each $0 \prec \epsilon \in \mathbb{C}$, $\int_0^\epsilon \omega(s)ds \succ 0$.

Throughout this presentation, we assume that the complex symmetric d is continuous.

Lemma 3.1. *Let (X, d) be a complex symmetric space and P, Q, f and g be self mappings on X. Suppose that*

(*a*) *the pair (P, f) (or (Q, g)) satisfies the property (E.A),*
(*b*) *$PX \subseteq gX$ (or $QX \subseteq fX$),*
(*c*) *Qy_n converges for every sequence $\{y_n\}$ in X whenever gy_n converges (or Py_n converges for every sequence $\{y_n\}$ in X whenever fy_n converges),*
(*d*) *for all $x, y \in X, \omega \in \Omega$ and $\Xi \in \Im$*

$$\Xi\Big(\int_0^{d(Px,Qy)}\omega(s)ds, \int_0^{d(fx,gy)}\omega(s)ds, \int_0^{d(Px,fx)}\omega(s)ds,$$
$$\int_0^{d(Qy,gy)}\omega(s)ds, \int_0^{d(Qy,fx)}\omega(s)ds, \int_0^{d(Px,gy)}\omega(s)ds\Big) \prec 0. \tag{3.1}$$

Then the pairs (P, f) and (Q, g) satisfy the common property (E.A).

Proof. If the pair (P, f) satisfies the property (E.A), then there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} fu_n = u$, for some $u \in X$. As $PX \subseteq gX$, for each u_n there is $v_n \in X$ such that $Pu_n = gv_n$. Hence, $\lim_{n\to\infty} gv_n = \lim_{n\to\infty} Pu_n = u$. Thus, we have

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} fu_n = \lim_{n\to\infty} gv_n = u.$$

Now, we claim that $Qv_n \to u$ as $n \to \infty$. On contrary, let us assume that $Qv_n \nrightarrow u$. Then $\int_0^{d(u,\lim_{n\to\infty} Qv_n)} \omega(s)ds \succ 0$. On setting $x = u_n$ and $y = v_n$ in (3.1), we have

$$\Xi\bigg(\int_0^{d(Pu_n,Qv_n)} \omega(s)ds, \int_0^{d(fu_n,gv_n)} \omega(s)ds, \int_0^{d(Pu_n,fu_n)} \omega(s)ds,$$
$$\int_0^{d(Qv_n,gv_n)} \omega(s)ds, \int_0^{d(Qv_n,fu_n)} \omega(s)ds, \int_0^{d(Pu_n,gv_n)} \omega(s)ds\bigg) \prec 0,$$

which on taking $n \to \infty$ gives rise

$$\Xi\bigg(\int_0^{d(u,\lim_{n\to\infty} Qv_n)} \omega(s)ds, 0, 0, \int_0^{d(\lim_{n\to\infty} Qv_n,u)} \omega(s)ds,$$
$$\int_0^{d(\lim_{n\to\infty} Qv_n,u)} \omega(s)ds, 0\bigg) \precsim 0,$$

which is a contradiction to (Ξ_1). Hence $Qv_n \to u$. Therefore,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} fu_n = \lim_{n\to\infty} Qv_n = \lim_{n\to\infty} gv_n = u.$$

Thus, (P, f) and (Q, g) satisfy the common property (E.A). □

Now, we present our main result which is new even in the context of symmetric spaces. In our main result we present a new situation in which the union of ranges of two functions is closed whereas the range of each one of them need not to be closed. For example if $f, g : [-2, 2] \longrightarrow [-2, 2]$ defined dy $fx = x,\ x \in [-2, 2),\ f(2) = 1$ and $gx = -x,\ x \in [-2, 2),\ g(2) = -1$, then $f([-2, 2]) = [-2, 2)$ and $g([-2, 2]) = (-2, 2]$ which are not closed. But $f([-2, 2]) \cup g([-2, 2]) = [-2, 2]$ is closed.

Now we are equipped to prove our main result as follows:

Theorem 3.2. *Let (X, d) be a complex symmetric space and P, Q, f and g be self mappings on X which satisfy inequality (3.1). Suppose that*

(a) *one of the pairs (P, f) and (Q, g) satisfies (E.A) property,*
(b) *$PX \subseteq gX$, $QX \subseteq fX$ and $fX \cup gX$ is closed subset of X,*
(c) *Qy_n converges for every sequence $\{y_n\}$ in X whenever gy_n converges (or Py_n converges for every sequence $\{y_n\}$ in X whenever fy_n converges).*

Then the pairs (P, f) and (Q, g) have a unique common point of coincidence. Moreover, P, Q, f and g have a unique common fixed point in X provided (P, f) and (Q, g) are weakly compatible.

Proof. Since $PX \subseteq gX$, $QX \subseteq fX$ and either (P, f) or (Q, g) satisfies (E.A) property, due to Lemma 3.1 we have (P, f) and (Q, g) share the common property (E.A). Hence, there are two sequences $\{u_n\}$ and $\{v_n\}$ in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} fu_n = \lim_{n\to\infty} Qv_n = \lim_{n\to\infty} gv_n = u$, for some $u \in X$. Now, since $fX \cup gX$ is closed, $\lim_{n\to\infty} fu_n = u \in fX \cup gX$. If $u \in fX$, then there is $v \in X$ such that $fv = u$. We assert that $Pv = fv$. If not, then $\int_0^{d(Pv,fv)} \omega(s)ds \succ 0$. Setting $x = v$ and $y = v_n$ in (3.1), we have

$$\Xi\Big(\int_0^{d(Pv,Qv_n)} \omega(s)ds, \int_0^{d(fv,gv_n)} \omega(s)ds, \int_0^{d(Pv,fv)} \omega(s)ds, \int_0^{d(Qv_n,gv_n)} \omega(s)ds, \int_0^{d(Qv_n,fv)} \omega(s)ds, \int_0^{d(Pv,gv_n)} \omega(s)ds\Big) \prec 0,$$

which on taking $n \longrightarrow \infty$ gives rise

$$\Xi\Big(\int_0^{d(Pv,u)} \omega(s)ds, \int_0^{d(fv,u)} \omega(s)ds, \int_0^{d(Pv,fv)} \omega(s)ds, 0, \int_0^{d(u,fv)} \omega(s)ds, \int_0^{d(Pv,u)} \omega(s)ds\Big) \precsim 0,$$

which (in view of the fact that $fv = u$) reduces to

$$\Xi\Big(\int_0^{d(Pv,fv)} \omega(s)ds, 0, \int_0^{d(Pv,fv)} \omega(s)ds, 0, 0, \int_0^{d(Pv,fv)} \omega(s)ds\Big) \precsim 0,$$

a contradiction to Ξ_2. Hence, $Pv = fv$. Therefore, we have

$$Pv = fv = u, \tag{3.2}$$

that is, v is a coincidence point of the pair (P, f) and u is a point of coincidence of P and f. Since $PX \subseteq gX$, there exists $z \in X$ such that $gz = u$. We claim that $Qz = gz$. Setting $x = u_n$ and $y = z$ in (3.1), and using similar arguments as earlier, one can justify the claim. Thus,

$$Qz = gz = u, \tag{3.3}$$

that is, z is a coincidence point of the pair (Q, g) and u is a point of coincidence of Q and g. Therefore, u is a common point of coincidence of the pairs (P, f) and (Q, g).

Now, we prove that u is unique. For this, let us assume that u' is another common point of coincidence of the pairs (P, f) and (Q, g). Then $\int_0^{d(u,u')} \omega(s)ds \succ 0$ and there exist $v', z' \in X$ such that $Pv' = fv' = u'$ and $Qz' = gz' = u'$. On setting $x = v'$ and $y = z$ in (3.1), we have

$$\Xi\Big(\int_0^{d(Pv',Qz)} \omega(s)ds, \int_0^{d(fv',gz)} \omega(s)ds, \int_0^{d(Pv',fv')} \omega(s)ds, \int_0^{d(Qz,gz)} \omega(s)ds, \int_0^{d(Qz,fv')} \omega(s)ds, \int_0^{d(Pv',gz)} \omega(s)ds\Big) \prec 0,$$

which gives rise

$$\Xi\bigg(\int_0^{d(u',u)}\omega(s)ds,\int_0^{d(u',u)}\omega(s)ds,0,0,\int_0^{d(u,u')}\omega(s)ds,\int_0^{d(u',u)}\omega(s)ds\bigg)\prec 0,$$

which is a contradiction to Ξ_3. Hence, u is unique common point of coincidence of the pairs (P,f) and (Q,g).

Next, we prove that u is a common fixed point of the mappings P,Q,f and g. Since the pairs (P,f) and (Q,g) are weakly compatible, on using (3.2) and (3.3), we have

$$Pu = Pfv = fPv = fu, \tag{3.4}$$

$$Qu = Qgz = gQz = gu. \tag{3.5}$$

Now, we show that $Pu = u$. Suppose that $Pu \neq u$, then $\int_0^{d(Pu,u)}\omega(s)ds \succ 0$. Using (3.1) with $x = u$ and $y = z$, we have

$$\Xi\bigg(\int_0^{d(Pu,Qz)}\omega(s)ds,\int_0^{d(fu,gz)}\omega(s)ds,\int_0^{d(Pu,fu)}\omega(s)ds,$$
$$\int_0^{d(Qz,gz)}\omega(s)ds,\int_0^{d(Qz,fu)}\omega(s)ds,\int_0^{d(Pu,gz)}\omega(s)ds\bigg)\prec 0,$$

which, on using (3.3) and (3.4), reduces to

$$\Xi\bigg(\int_0^{d(Pu,u)}\omega(s)ds,\int_0^{d(Pu,u)}\omega(s)ds,0,0,$$
$$\int_0^{d(u,Pu)}\omega(s)ds,\int_0^{d(Pu,u)}\omega(s)ds\bigg)\prec 0,$$

which is a contradiction to (Ξ_3). Thus, $Pu = u$. This shows that u is a common fixed point of P and f. Similarly, on setting $x = v$ and $y = u$ in (3.1) and using (3.2) and (3.5) one can prove that u is also a common fixed point of Q and g. Therefore, u is a common fixed point of the mappings P,Q,f and g. The uniqueness of the common fixed point of the mappings P,Q,f and g is a direct consequence of the uniqueness of the common point of coincidence of the pairs (P,f) and (Q,g). The proof is similar if $u \in gX$, hence omitted. This completes the proof. □

Since two non-compatible self mappings satisfy the property (E.A), so as a consequence of Theorem 3.2 we conclude the following corollary.

Corollary 3.3. *Let P,Q,f and g be four self mappings defined on a complex symmetric space (X,d) which satisfy inequality (3.1). Suppose that (P,f) (or (Q,g)) are non-compatible and the pairs (P,f) and (Q,g) are weakly compatible. If $fX \cup gX$ is a closed subset of X, $PX \subseteq gX$, $QX \subseteq fX$ and Qy_n*

converges for every sequence $\{y_n\}$ in X whenever gy_n converges (or Py_n converges for every sequence $\{y_n\}$ in X whenever fy_n converges), then P, Q, f and g have a unique common fixed point in X.

As a consequence of Theorem 3.2 we have the following corollary for four finite families of self mappings defined on a complex symmetric space.

Corollary 3.4. *Let $\{P_i\}_1^l, \{Q_j\}_1^n, \{f_k\}_1^m$ and $\{g_r\}_1^s$ be four finite pairwise commuting families of self mappings defined on a complex symmetric space (X, d). Let $P = P_1P_2\cdots P_l$, $Q = Q_1Q_2\cdots Q_n$, $f = f_1f_2\cdots f_m$ and $g = g_1g_2\cdots g_s$ satisfying the inequality (3.1). Assume that*

- (a) *one of the pairs (P, f) and (Q, g) satisfies the property (E.A),*
- (b) *$PX \subseteq gX$, $QX \subseteq fX$ and $fX \cup gX$ is closed subset of X,*
- (c) *Qy_n converges for every sequence $\{y_n\}$ in X whenever gy_n converges (or Py_n converges for every sequence $\{y_n\}$ in X whenever fy_n converges).*

Then the component maps of the families $\{P_i\}_1^l, \{Q_j\}_1^n, \{f_k\}_1^m$ and $\{g_r\}_1^s$ have a unique common fixed point.

Proof. On the lines of Theorem 2.2 due to Imdad *et al.* [8] and using Theorem 3.2 one can prove this result. □

On the lines of Theorem 3.2 one can prove the following results which can be viewed as generalizations of Theorems 3.1, 3.2 and 3.3 due to Ali and Imdad [3], Theorem 3.1 of Manro [11], Theorems 3.1, 3.4 of Aliouche [5] and Theorem 4.4 of Popa and Patriciu [13].

Theorem 3.5. *Let (X, d) be a complex symmetric space and P, Q, f and g be self mappings on X satisfying inequality (3.1). Suppose that*

- (a) *the pairs (P, f) and (Q, g) enjoy the common property (E.A),*
- (b) *fX and gX are closed subsets of X, (or)*
- (b') *$\overline{PX} \subseteq gX$ and $\overline{QX} \subseteq fX$, (or)*
- (b'') *PX and QX are closed provided $PX \subseteq gX$ and $QX \subseteq fX$.*

Then the pairs (P, f) and (Q, g) have a unique common point of coincidence. Moreover, if the pairs (P, f) and (Q, g) are weakly compatible, then P, Q, f and g have a unique common fixed point in X.

Theorem 3.6. *Let (X, d) be a complex symmetric space and P, Q, f and g be self mappings on X satisfying inequality (3.1). Suppose that*

- (a) *the pair (P, f) (or (Q, g)) has the property (E.A),*
- (b) *$PX \subseteq gX$ (or $QX \subseteq fX$) and fX (or gX) is closed subset of X,(or)*
- (b') *$PX \subseteq gX$, $QX \subseteq fX$ and one of PX, QX, fX and gX is closed subset of X,*

(c) *Qy_n converges for every sequence $\{y_n\}$ in X whenever gy_n converges (or Py_n converges for every sequence $\{y_n\}$ in X whenever fy_n converges).*

Then the pairs (P, f) and (Q, g) have a unique common point of coincidence. Moreover, if the pairs (P, f) and (Q, g) are weakly compatible, then P, Q, f and g have a unique common fixed point in X.

Corollary 3.7. *Let (X, d) be a complex symmetric space and P, Q, f and g be self mappings on X. Suppose that there exists $\Xi \in \Im$ such that for all $x, y \in X$,*

$$\Xi\big(d(Px, Qy), d(fx, gy), d(Px, fx), d(Qy, gy), d(Qy, fx), d(Px, gy)\big) \precsim 0,$$

if

(a) *the pair (P, f) (or (Q, g)) has the property (E.A),*
(b) *$PX \subseteq gX$, $QX \subseteq fX$ and $fX \cup gX$ is closed subset of X,(or)*
(b′) *$PX \subseteq gX$ (or $QX \subseteq fX$) and fX (or gX) is closed subset of X,*
(c) *Qy_n converges for every sequence $\{y_n\}$ in X whenever gy_n converges (or Py_n converges for every sequence $\{y_n\}$ in X whenever fy_n converges).*

Then the pairs (P, f) and (Q, g) have a unique common point of coincidence. Moreover, if the pairs (P, f) and (Q, g) are weakly compatible, then P, Q, f and g have a unique common fixed point in X.

Proof. On setting $\omega(s) = 1$ for all $s \in \mathbb{R}^n$ in Theorems 3.2 and 3.6 we get this corollary. □

Corollary 3.8. *Let (X, d) be a complex symmetric space and P, Q, f and g be self mappings on X. Suppose that there exists $\Xi \in \Im$ such that for all $x, y \in X$,*

$$\Xi\big(d(Px, Qy), d(fx, gy), d(Px, fx), d(Qy, gy), d(Qy, fx), d(Px, gy)\big) \precsim 0,$$

if

(a) *the pairs (P, f) and (Q, g) enjoy the common property (E.A),*
(b) *fX and gX are closed subsets of X,(or)*
(b′) *$\overline{PX} \subseteq gX$ and $\overline{QX} \subseteq fX$, (or)*
(b″) *PX and QX are closed provided $PX \subseteq gX$ and $QX \subseteq fX$.*

Then the pairs (P, f) and (Q, g) have a unique common point of coincidence. Moreover, if the pairs (P, f) and (Q, g) are weakly compatible, then P, Q, f and g have a unique common fixed point in X.

Proof. On setting $\omega(s) = 1$ for all $s \in \mathbb{R}^n$ in Theorem 3.5 we get this corollary. □

In view of Example 2.2, we have the following corollary which covers, generalizes and improves several known results beside yielding new contraction conditions in the context of complex fixed point theory (e.g. $A_7 - A_{13}$).

Corollary 3.9. *The conclusions of Theorems* 3.2, 3.5 *and* 3.6 *remain true if (for all* $x, y \in X, \omega \in \Omega$*) implicit relation* (3.1) *is replaced by any one of the following:*

$$(A_1) \int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda_1 \int_0^{d(fx,gy)} \omega(s)ds + \lambda_2 \int_0^{d(Px,fx)} \omega(s)ds + \lambda_3 \int_0^{d(Qy,gy)} \omega(s)ds + \lambda_4 \int_0^{d(Qy,fx)} \omega(s)ds + \lambda_5 \int_0^{d(Px,gy)} \omega(s)ds,$$

where $\lambda_i \in \mathbb{R}_+, i = 1, 2, \cdots, 5$ *such that* $\Sigma_{i=1}^5 \lambda_i < 1$.

$$(A_2) \int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda \max\left\{ \int_0^{d(fx,gy)} \omega(s)ds, \int_0^{d(Qy,gy)} \omega(s)ds, \int_0^{d(Qy,fx)} \omega(s)ds \right\},$$

where $\lambda \in [0, 1)$.

$$(A_3) \int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda\Delta,$$

where

$$\Delta \in \left\{ \frac{1}{2} \int_0^{d(Px,fx)} \omega(s)ds + \frac{1}{2} \int_0^{d(Qy,gy)} \omega(s)ds, \int_0^{d(fx,gy)} \omega(s)ds, \int_0^{d(Px,fx)} \omega(s)ds, \int_0^{d(Qy,gy)} \omega(s)ds \right\}$$

and $\lambda \in [0, 1)$.

$$(A_4) \int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda \max\left\{ \int_0^{d(fx,gy)} \omega(s)ds, \int_0^{d(Px,fx)} \omega(s)ds, \int_0^{d(Qy,gy)} \omega(s)ds, \frac{1}{2} \int_0^{d(Qy,fx)} \omega(s)ds + \frac{1}{2} \int_0^{d(Px,gy)} \omega(s)ds \right\},$$

where $\lambda \in [0,1)$.

$$(A_5) \quad \int_0^{d(Px,Qy)} \omega(s)ds$$
$$\precsim \lambda_1 \frac{\int_0^{d(Qy,gy)} \omega(s)ds\big(\int_0^{d(Px,fx)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds\big)}{1+\int_0^{d(fx,gy)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds}$$
$$+\lambda_2 \frac{\int_0^{d(Px,gy)} \omega(s)ds \int_0^{d(Qy,fx)} \omega(s)ds \int_0^{d(Px,fx)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds}$$
$$+\lambda_2 \frac{\int_0^{d(Px,gy)} \omega(s)ds \int_0^{d(Qy,fx)} \omega(s)ds \int_0^{d(Qy,gy)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds}$$
$$+\lambda_3\left(\int_0^{d(Qy,fx)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds\right)$$
$$+\lambda_4\left(\int_0^{d(Px,fx)} \omega(s)ds + \int_0^{d(Qy,gy)} \omega(s)ds\right)$$
$$+\lambda_5 \int_0^{d(fx,gy)} \omega(s)ds,$$

where $\lambda_i \in \mathbb{R}_+, i = 1,2,...,5$ *such that* $\lambda_3 + \lambda_4 < 1$ *and* $2\lambda_3 + \lambda_5 < 1$.

$$(A_6) \quad \int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda \frac{\int_0^{d(Px,fx)} \omega(s)ds \int_0^{d(Qy,fx)} \omega(s)ds}{1+\int_0^{d(Qy,fx)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds}$$
$$+\lambda \frac{\int_0^{d(Qy,gy)} \omega(s)ds \int_0^{d(Px,gy)} \omega(s)ds}{1+\int_0^{d(Qy,fx)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds}$$

where $\lambda \in [0,2)$.

$$(A_7) \quad \int_0^{d(Px,Qy)} \omega(s)ds$$
$$\precsim \alpha_1\left(\int_0^{d(fx,gy)} \omega(s)ds\right) \int_0^{d(fx,gy)} \omega(s)ds$$
$$+\alpha_2\left(\int_0^{d(fx,gy)} \omega(s)ds\right) \frac{\int_0^{d(fx,gy)} \omega(s)ds \int_0^{d(Qy,gy)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds}$$
$$+\alpha_3\left(\int_0^{d(fx,gy)} \omega(s)ds\right) \frac{\int_0^{d(Px,fx)} \omega(s)ds \int_0^{d(Qy,gy)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds}$$

$$+\alpha_4\bigg(\int_0^{d(fx,gy)}\omega(s)ds\bigg)\frac{\int_0^{d(Qy,gy)}\omega(s)ds\int_0^{d(Px,gy)}\omega(s)ds}{1+\int_0^{d(fx,gy)}\omega(s)ds},$$

where $\alpha_i:\mathbb{C}_+\longrightarrow[0,1), i=1,2,3,4$ *are given upper semi-continuous mappings.*

$$(A_8)\ \psi\bigg(\int_0^{d(Px,Qy)}\omega(s)ds\bigg)\precsim\psi\bigg(\frac{\int_0^{d(Px,fx)}\omega(s)ds\int_0^{d(Qy,gy)}\omega(s)ds}{1+\int_0^{d(fx,gy)}\omega(s)ds}\bigg)-\phi\bigg(\frac{\int_0^{d(Px,fx)}\omega(s)ds\int_0^{d(Qy,gy)}\omega(s)ds}{1+\int_0^{d(fx,gy)}\omega(s)ds}\bigg),$$

where $\psi\in\Psi$ *and* $\phi\in\Phi$*.*

$$(A_9)\ \psi\bigg(\int_0^{d(Px,Qy)}\omega(s)ds\bigg)\precsim\psi\bigg(\frac{\int_0^{d(Qy,gy)}\omega(s)ds\int_0^{d(Px,gy)}\omega(s)ds}{1+\int_0^{d(fx,gy)}\omega(s)ds}\bigg)+\theta\bigg(\frac{\int_0^{d(Qy,gy)}\omega(s)ds\int_0^{d(Px,gy)}\omega(s)ds}{1+\int_0^{d(fx,gy)}\omega(s)ds}\bigg),$$

where $\psi\in\Psi$ *and* $\theta\in\Theta$*.*

(A_{10})

$$\psi\bigg(\int_0^{d(Px,Qy)}\omega(s)ds\bigg)\precsim\psi\bigg(\max\bigg\{\int_0^{d(fx,gy)}\omega(s)ds,\int_0^{d(Px,fx)}\omega(s)ds,\int_0^{d(Qy,gy)}\omega(s)ds\bigg\}\bigg)-\psi\bigg(\max\bigg\{\int_0^{d(Qy,fx)}\omega(s)ds,\int_0^{d(Px,gy)}\omega(s)ds\bigg\}\bigg),$$

where $\psi\in\Psi$*.*

$$(A_{11})\int_0^{d(Px,Qy)}\omega(s)ds\precsim\theta\bigg(\max\bigg\{\int_0^{d(fx,gy)}\omega(s)ds,\int_0^{d(Px,fx)}\omega(s)ds,\int_0^{d(Qy,gy)}\omega(s)ds,\int_0^{d(Qy,fx)}\omega(s)ds,\int_0^{d(Px,gy)}\omega(s)ds\bigg\}\bigg),$$

where $\theta \in \Theta$ *with* $\theta(z) \prec z, \quad \forall z \in \mathbb{C}_+$.

$$(A_{12}) \quad \int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda_1 \frac{\int_0^{d(Px,fx)} \omega(s)ds \int_0^{d(Qy,gy)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds} + \lambda_2 \frac{\int_0^{d(Px,fx)} \omega(s)ds \int_0^{d(Qy,fx)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds} + \lambda_3 \frac{\int_0^{d(Qy,gy)} \omega(s)ds \int_0^{d(Px,gy)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds},$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

(A_{13})

$$\int_0^{d(Px,Qy)} \omega(s)ds \precsim \lambda_2 \int_0^{d(fx,gy)} \omega(s)ds + \lambda_3 \int_0^{d(Px,fx)} \omega(s)ds + \lambda_4 \int_0^{d(Qy,gy)} \omega(s)ds + \lambda_5 \int_0^{d(Qy,fx)} \omega(s)ds + \lambda_6 \int_0^{d(Px,gy)} \omega(s)ds + \gamma \frac{\int_0^{d(fx,gy)} \omega(s)ds \int_0^{d(Qy,gy)} \omega(s)ds + \int_0^{d(Px,fx)} \omega(s)ds \int_0^{d(Qy,fx)} \omega(s)ds}{1+\int_0^{d(fx,gy)} \omega(s)ds + \int_0^{d(Px,gy)} \omega(s)ds},$$

where $\lambda_i \in \mathbb{R}$ *such that* $\Sigma_{i=2}^{6} \lambda_i < 1$ *and* $\gamma \in \mathbb{C}$.

Proof. The proof of each contraction condition in this corollary follows from Theorems 3.2, 3.5 and 3.6 in view of Example 2.2. □

Remark 3.10. The majority of results corresponding to contraction conditions given in Corollary 3.9 generalize and improve versions of multitude existing results. Theorem 3.6 corresponding to contraction condition:

(1) (A_1) improves Theorem 3.2 of Sarwar *et al.* [16].
(2) (A_2) generalizes Theorem 1 of Aliouche [4] and also corrects, generalizes and improves Theorem 2.1 of [21]. Especially, taking $\omega(s) = 1, \ \forall s \in \mathbb{R}^n$ we get the corrected form of Theorem 2.1 of Verma and Pathak [21].
(3) (A_3) generalizes and improves Theorem 3.1 of [17]. Particulary, taking $\omega(s) = 1, \ \forall s \in \mathbb{R}^n$ we get Theorem 3.1 of Shukla and Pagey [17] except when $u_{xy} = \frac{1}{2} d(Qy, fx) d(Px, gy)$.
(4) (A_4) corrects, generalizes and improves Theorem 3.1 of [9]. Especially, taking $\omega(s) = 1, \ \forall s \in \mathbb{R}^n$ we get the corrected form of Theorem 3.1 of J. Kumar and Y. Kumar [9].

(5) (A_5) generalizes and improves Theorem 3.1 of [18]. Particulary, taking $\omega(s) = 1,\ \forall s \in \mathbb{R}^n$ we get Theorem 3.1 of Shukla and Pagey [18].
(6) (A_6) generalizes and improves Corollary 2 of [15]. Especially, taking $\omega(s) = 1,\ \forall s \in \mathbb{R}^n$ we get Corollary 2 of Sarwar and Zada [15].

4. An application to integral equations

Our plan in this section is to apply Theorem 3.2 (corresponding to contraction condition (A_9)) to prove the existence and uniqueness of a common solution for the following system of Volterra-Hammerstein integral equations:

$$x(t) = h_i(t) + a\int_0^s \alpha(t,z)k_i(z,x(z))dz + b\int_0^\infty \beta(t,z)q_i(z,x(z))dz, \tag{4.1}$$

for all $t \in (0,\infty)$, where $a,b \in \mathbb{R}$, $x, h_i \in C(L(0,\infty),\mathbb{R})$, α, β, k_i and $q_i, i = 1,2,3,4$ are real valued measurable functions with respect to both variables on $(0,\infty)$.

For simplification, we use the following symbols:

$$\Omega_i\big(x(t)\big) = \int_0^s \alpha(t,z)k_i(z,x(z))dz, \quad \mho_i\big(x(t)\big) = \int_0^\infty \beta(t,z)q_i(z,x(z))dz,$$

$$\Gamma_{xy}(t) = \|h_1(t) + a\Omega_1\big(x(t)\big) + b\mho_1\big(x(t)\big) - h_2(t) - a\Omega_2\big(y(t)\big) - b\mho_2\big(y(t)\big)\|e^i,$$

$$\Lambda_{xy}(t) = \|h_2(t) + a\Omega_2\big(y(t)\big) + b\mho_2\big(y(t)\big) - h_4(t) - a\Omega_4\big(y(t)\big) - b\mho_4\big(y(t)\big)\|e^i,$$

$$\Upsilon_{xy}(t) = \|h_1(t) + a\Omega_1\big(x(t)\big) + b\mho_1\big(x(t)\big) - h_4(t) - a\Omega_4\big(y(t)\big) - b\mho_4\big(y(t)\big)\|e^i,$$

$$\top_{xy}(t) = \|h_3(t) + a\Omega_3\big(x(t)\big) + b\mho_3\big(x(t)\big) - h_4(t) - a\Omega_4\big(y(t)\big) - b\mho_4\big(y(t)\big)\|e^i,$$

$X = C(L(0,\infty),\mathbb{R}),$ space of all real valued measurable functions on$(0,\infty)$.

Define four mappings $g_i : X \longrightarrow X, i = 1,2,3,4$ as follows:

$$g_i x(t) = h_i(t) + a\Omega_i\big(x(t)\big) + b\mho_i\big(x(t)\big), \quad \forall x \in X. \tag{4.2}$$

One can note that the system (4.1) of Volterra-Hammerstein integral equations have a unique common solution if and only if the four self mappings g_1, g_2, g_3 and g_4 given in (4.2) have a unique common fixed point.

Assume that the following assumptions hold (for all $t \in (0,\infty), x \in X$):

$(\mathbf{p}_1)$ $h_4(t) - h_1(t) + a\Big[\Omega_4\Big(g_1x(t) + h_4(t)\Big) - \Omega_1\big(x(t)\big)\Big] + b\Big[\mho_4\Big(g_1x(t) + h_4(t)\Big) - \mho_1\big(x(t)\big)\Big] = 0,$

$(\mathbf{p}_2)$ $h_3(t) - h_2(t) + a\Big[\Omega_3\Big(g_2x(t) + h_3(t)\Big) - \Omega_2\big(x(t)\big)\Big] + b\Big[\mho_3\Big(g_2x(t) + h_3(t)\Big) - \mho_2\big(x(t)\big)\Big] = 0,$

$(\mathbf{p}_3)$ $g_1^2x(t) = g_3^2x(t)$ and $g_2^2x(t) = g_4^2x(t)$.

Theorem 4.1. *The system of Volterra-Hammerstein integral equations given in* (4.1) *under assumptions* $(\mathbf{p}_1)-(\mathbf{p}_3)$ *have a unique solution if*

(i) *there is a sequence* $\{u_n\}$ *in* X *such that* $\lim_{n\to\infty} g_1u_n = \lim_{n\to\infty} g_3u_n = u \in X$,

(ii) $\Gamma_{xy}(t) \precsim \frac{\Lambda_{xy}(t)\Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)}$, *for each* $x, y \in X$, $t \in (0,\infty)$,

(iii) $g_3X \cup g_4X$ *is closed subspace of* X *and*

(iv) g_2y_n *converges for every sequence* $\{y_n\}$ *in* X *whenever* g_4y_n *converges.*

Proof. Define a mapping $d : X \times X \longrightarrow \mathbb{C}_+$ by

$$d(x,y) = \max_{t\in(0,\infty)} \|x(t)-y(t)\|e^i \quad \text{for all} \quad x,y \in X.$$

Then (X,d) is a complex symmetric space. Let $x, y \in X$. Then (for each $t \in (0,\infty)$) we have

$$\begin{aligned} d(g_1x, g_2y) &= \max_{t\in(0,\infty)} \Gamma_{xy}(t), \\ d(g_2y, g_4y) &= \max_{t\in(0,\infty)} \Lambda_{xy}(t), \\ d(g_1x, g_4y) &= \max_{t\in(0,\infty)} \Upsilon_{xy}(t), \\ d(g_3x, g_4y) &= \max_{t\in(0,\infty)} \top_{xy}(t). \end{aligned}$$

Now, from assumption (ii) for each $x, y \in X$ and $t \in (0,\infty)$, we have

$$\begin{aligned} \Gamma_{xy}(t) &\precsim \frac{\Lambda_{xy}(t)\Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)} \\ &\precsim \frac{\max_{t\in(0,\infty)} \Lambda_{xy}(t) \max_{t\in(0,\infty)} \Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)}, \end{aligned}$$

which implies that

$$\max_{t\in(0,\infty)} \Gamma_{xy}(t) \precsim \frac{\max_{t\in(0,\infty)} \Lambda_{xy}(t) \max_{t\in(0,\infty)} \Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)},$$

implying thereby

$$\begin{aligned} \psi\Big(\max_{t\in(0,\infty)} \Gamma_{xy}(t)\Big) &\precsim \psi\Big(\frac{\max_{t\in(0,\infty)} \Lambda_{xy}(t) \max_{t\in(0,\infty)} \Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)}\Big), \\ &\precsim \psi\Big(\frac{\max_{t\in(0,\infty)} \Lambda_{xy}(t) \max_{t\in(0,\infty)} \Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)}\Big) \\ &\quad +\theta\Big(\frac{\max_{t\in(0,\infty)} \Lambda_{xy}(t) \max_{t\in(0,\infty)} \Upsilon_{xy}(t)}{1+\max_{t\in(0,\infty)} \top_{xy}(t)}\Big), \end{aligned}$$

which yields that

$$\psi\big(d(g_1x, g_2y)\big) \precsim \psi\Big(\frac{d(g_2y, g_4y)d(g_1x, g_4y)}{1 + d(g_3x, g_4y)}\Big) + \theta\Big(\frac{d(g_2y, g_4y)d(g_1x, g_4y)}{1 + d(g_3x, g_4y)}\Big),$$

where $\psi \in \Psi$ and $\theta \in \Theta$.

Now, we prove that $g_1X \subseteq g_4X$. Let $x \in X$. Then for each $t \in (0, \infty)$ we have

$$\begin{aligned} g_4\big(g_1x(t) + h_4(t)\big) &= h_4(t) + a\Omega_4\big(g_1x(t) + h_4(t)\big) + b\mho_4\big(g_1x(t) + h_4(t)\big) \\ &= g_1x(t) - g_1x(t) + h_4(t) + a\Omega_4\big(g_1x(t) + h_4(t)\big) \\ &\quad + b\mho_4\big(g_1x(t) + h_4(t)\big) \\ &= g_1x(t) - h_1(t) - a\Omega_1\big(x(t)\big) - b\mho_1\big(x(t)\big) + h_4(t) \\ &\quad + a\Omega_4\big(g_1x(t) + h_4(t)\big) + b\mho_4\big(g_1x(t) + h_4(t)\big) \\ &= g_1x(t) + h_4(t) - h_1(t) + a\big[\Omega_4\big(g_1x(t) + h_4(t)\big) \\ &\quad - \Omega_1\big(x(t)\big)\big] + b\big[\mho_4\big(g_1x(t) + h_4(t)\big) - \mho_1\big(x(t)\big)\big]. \end{aligned}$$

On using $(\mathbf{p}_1)$, we get that $g_4\big(g_1x(t)+h_4(t)\big) = g_1x(t)$ for each $t \in (0, \infty)$. This shows that $g_1X \subseteq g_4X$. Similarly, using $(\mathbf{p}_2)$ one can prove that $g_2X \subseteq g_3X$.

Next, we show that the pairs (g_1, g_3) and (g_2, g_4) are weakly compatible. Assume that $g_1x = g_3x$ for some $x \in X$. Then

$$g_1x(t) = g_3x(t) \quad \text{for all} \quad t \in (0, \infty). \tag{4.3}$$

On using (4.3) and $(\mathbf{p}_3)$, we have

$$g_1g_3x(t) = g_1g_1x(t) = g_1^2x(t) = g_3^2x(t) = g_3g_3x(t) = g_3g_1x(t) \quad \forall t \in (0, \infty).$$

Therefore, $g_1g_3x = g_3g_1x$ whenever $g_1x = g_3x$. Proving that g_1 and g_3 are weakly compatible. Similarly, one can prove that g_2 and g_4 are weakly compatible. Thus, all conditions of Theorem 3.2 [corresponding to contraction condition (A_9) with $\omega(s) = 1\ \forall s \in \mathbb{R}^n$] are satisfied. So that, there exists a unique common fixed point of g_1, g_2, g_3 and g_4 in X and, hence, the system (4.1) of Volterra-Hammerstein integral equations have a unique solution. □

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