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FAN-BROWDER TYPE FIXED POINT THEOREMS AND APPLICATIONS IN HADAMARD MANIFOLDS

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Abstract. In this paper, we provide two basic Fan-Browder type fixed point theorems for multimaps on geodesic convex sets in Hadamard manifolds. Also, an existence theorem of Nash equilibrium for an 1-person game in Hadamard manifolds is established.

1. Introduction

As we know, the concept of convexity and its various generalizations are important to quantitative and qualitative studies of nonlinear analysis and convex analysis. In 1961, using his generalization of the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem, Fan [4] established a very basic geometric lemma for multimaps and gave several applications. In 1968, Browder [1] obtained a fixed point theorem which is more convenient form of Fan's lemma, and using this theorem, he established a complete treatment of a wide applications of coincidence and fixed point theory, minimax theory, variational inequalities, monotone operators and game theory. Since then, this result is known as Fan-Browder fixed point theorem, and there have been numerous generalizations and applications in numerous areas of nonlinear analysis where the various generalized convexity concepts are equipped. For the literature, see Browder [1], Park [11] and the references therein.

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On the other hand, in the last two decades, several important concepts of nonlinear analysis have been extended from Euclid/Hilbert spaces to Riemannian manifold settings in order to go further in the studies of convex analysis, fixed point theory, variational problems, and related topics. In 2003, Németh [10] first proved a basic fixed point theorem for continuous maps on a compact geodesic convex subset of a Hadamard manifold, and he proved the existence of solutions for variational inequalities in a Hadamard manifold. Since then, using the Németh fixed point theorem, several authors investigate various applications of variational inequalities, minimax inequalities, and equilibrium problems in Hadamard manifolds, e.g., see [2,7-10,12]. Recently, there have been some researches on the Fan-Browder type fixed point theorem for multimaps in a geodesic convex subset of a Hadamard manifold as in [13,14]; but there are some typical problems of the concept of geodesic convex hull as remarked in [8]. Hence we could not find the exact Fan-Browder type fixed point theorems for multimaps on a geodesic convex subset of a Hadamard manifold yet.

In this paper, using a geodesic KKM theorem for closed valued multimaps due to Colao *et al.* [2], we prove a Fan-Browder type fixed point theorem for multimaps on a geodesic convex subset of a Hadamard manifold. As an application, we will give an existence theorem of Nash equilibrium for an 1-person game in Hadamard manifolds. Also, as an application of KKM theorem for open valued multimaps due to Kim [5], we will prove an analogous KKM theorem for geodesic convex valued multimaps in a Hadamard manifold, and next prove another Fan-Browder type fixed point theorem for multimaps on a geodesic convex subset in a Hadamard manifold. Those two Fan-Browder type fixed point theorems for geodesic convex sets in Hadamard manifolds can be basic tools for solving nonlinear problems in Hadamard manifolds, and there might have been numerous generalizations and applications in numerous areas of nonlinear analysis where various generalized geodesic convexity concepts are equipped.

2. Preliminaries

We begin with some standard notions and terminologies. If A is a subset of a vector space, we shall denote by 2^A the family of all subsets of A, and by $co\,A$ the convex hull of A. Denote the unit simplex in $\mathbb{R}^n$ by Δ_{n-1}, that is, $\Delta_{n-1} := \{(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i e_i \in [0,1]^n \mid \sum_{i=1}^n \lambda_i = 1\} = \langle e_1, \dots, e_n \rangle$; and simply write the i-th unit vector $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \Delta_{n-1}$.

Recall some basic definitions on Riemannian manifolds in [2, 7, 10, 12]. Let M be a complete finite dimensional Riemannian manifold with the Levi-Civita connection ∇ on M. Let $x \in M$ and let T_xM denote the tangent space at

x to M. For $x,\ y \in M$, let $\gamma : [0,1] \to M$ be a piecewise smooth curve joining x to y. Then, a curve γ is called a *geodesic* if $\gamma(0) = x$, $\gamma(1) = y$, and $\nabla_{\dot\gamma}\dot\gamma = 0$ for all $t \in [0,1]$. A geodesic $\gamma : [0,1] \to M$ joining x to y is *minimal* if its arc-length equals its Riemannian distance between x and y. And, M is called a *Hadamard manifold* if M is a simply connected complete Riemannian manifold of non-positive sectional curvature.

Recall the following concept which generalize the convex condition in linear spaces to Riemannian manifolds:

Definition 2.1. A nonempty subset X of a Riemannian manifold M is said to be *geodesic convex* if for any $x, y \in X$, the geodesic joining x to y is contained in X. For an arbitrary subset C of M, the minimal geodesic convex subset which contains C is called the *geodesic convex hull* of C, and denoted by $Gco(C)$.

Then the simple definition of geodesic convex hull $\mathrm{Gco}(C)$ in a Riemannian manifold M overcomes the delicate problems of geodesic convexity remarked in [8]. It is clear that if S is geodesic convex, then $Gco\,(S) = S$. Note that as shown in [2], $Gco\,(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$, and $C_n = \{z \in \gamma_{x,y} \mid x, y \in C_{n-1}\}$ for each $n \in \mathbb{N}$. Here, $\gamma_{x,y} : [0,1] \to M$ denotes a geodesic joining x to y.

Remark 2.2. (1) For each $x \in M$, it is clear that $Gco\,\{x\} = \{x\}$ so that each singleton is geodesic convex. Note that when C is a geodesic convex subset of M, and for any finite subset $\{x_1, \dots, x_n\} \subseteq C$, we know that $Gco(\{x_1, \dots, x_n\}) \subseteq C$; however we do not know whether $Gco(\{x_1, \dots, x_n\}$ is closed or compact in general.
(2) Recently, various geodesic convexity assumptions, involving the exponential map along with affine maps, geodesics, and geodesic convex hulls, have been established on Hadamard manifolds by several authors, e.g., see [2, 7-10, 12-14]. In a recent paper [8], Kristály *et al.* pointed out that those conditions in [9, 13, 14] are mutually equivalent, and they hold if and only if the Hadamard manifold is isometric to the Euclidean space; and hence the corresponding results obtained in Hadamard manifolds are actually their well-known Euclidean counterparts. Therefore, it should be noted that we are very careful to apply the geodesic convex hull concept in Hadamard manifolds. In a finite dimensional Euclidean space, for the equivalent concepts for the minimal geodesic, see Theorem 2.1 in [8].

Next, we recall some notions and terminologies on the generalized Nash equilibrium for pure strategic games as in [3, 11]. Let $I = \{1, 2, \dots, n\}$ be a finite (or possibly countably infinite) set of players. A noncooperative *generalized game of normal form* is an ordered $3n$-triple $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ where for each player $i \in I$, X_i is a pure strategy space for the player i, and the set $X := \Pi_{i=1}^n X_i = X_{-i} \times X_i$, *joint strategy space*, is the Cartesian product of the individual strategy spaces, and the element of X_i is called a *strategy*. Here, we denote $X_{-i} := \Pi_{j \in I-\{i\}} X_j$. For an action profile $x = (x_1, \cdots, x_n) \in X = \Pi_{i \in I} X_i$, we shall write $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$, and we may simply write $x = (x_{-i}, x_i) \in X_{-i} \times X_i = X$. Let $f_i : X_{-i} \times X_i \to \mathbb{R}$ be a *payoff function* (or *utility function*), and $T_i : X \to 2^{X_i}$ be a *constraint correspondence* for the player i. Then, a strategy n-tuple $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ is called a *Nash equilibrium* for the generalized game $\mathcal{G}$ if for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y) \quad \text{for all} \quad y \in T_i(\bar{x}).$$

When I is singleton, then $\mathcal{G}$ is called an 1-*person game*.

Using the exact geodesic convex hull concept, Colao *et al.* [2] introduced the generalized concept of KKM maps in a Hadamard manifold as follows:

Definition 2.3. Let X be a nonempty geodesic convex subset of a Hadamard manifold M. A multimap $T : X \to 2^M$ is called a *geodesic KKM map* on X if for any finite subset $\{x_1, \ldots, x_n\} \subseteq X$,

$$Gco\big(\{x_1, \ldots, x_n\}\big) \subseteq \bigcup_{i=1}^n T(x_i).$$

It is clear that if T is a geodesic KKM map on X, then $x \in T(x)$ and hence $T(x)$ is nonempty for each $x \in X$.

In order to obtain existence results for various equilibrium problems in Hadamard manifolds, Colao *et al.* [2] provided an analogous KKM theorem in the setting of Hadamard manifolds which is essential in proving the main result of this paper as follows:

Lemma 2.4. ([2]) *Let X be a nonempty geodesic convex subset of a Hadamard manifold M, and $T : X \to 2^M$ be a geodesic KKM-map on X such that for each $x \in X$, $T(x)$ is closed, and $T(x_o)$ is compact for some $x_o \in X$. Then*

$$\bigcap_{x \in X} T(x) \neq \emptyset.$$

An open-valued KKM theorem for convex sets due to Kim [5] in linear spaces is also essential in proving an analogous open-valued KKM theorem in the setting of Hadamard manifolds as follows:

Lemma 2.5. ([5]) *Let $X = \{e_1, \ldots, e_n\}$ be the set of vertices of a simplex Δ_{n-1} in $E = \mathbb{R}^n$, and $T : X \to 2^E$ be a multimap such that*

(1) *for each $x \in X$, $T(x)$ is an open subset of E;*

(2) *for any finite subset* $\{e_1, \dots, e_k\} \subseteq X$, $co(\{e_1, \dots, e_k\}) \subseteq \bigcup_{i=1}^{k} T(e_i)$.
Then we have

$$\bigcap_{i=1}^{n} T(e_i) \neq \emptyset.$$

From now on, let M be a finite dimensional Hadamard manifold and X be a nonempty geodesic convex subset of a Hadamard manifold M. For the other standard notations and terminologies, we shall refer to Colao *et al.* [2], Németh [10], Udrişte [12], and the references therein.

3. Fan-Browder type fixed point theorems and applications

As remarked in [8], the proofs of Fan-Browder type fixed point theorems, Theorem 3.1 in [13], and Theorem 3.1 in [14], hold true only in the finite dimensional Euclidean spaces settings so that in this case, those theorems and proofs are reduced to the well-known results as in [1, 11]. Therefore, we should need new proofs for those theorems by using suitable and meaningful geodesic convex hull definition in Hadamard manifolds.

As an application of Lemma 2.4, we begin with the first Fan-Browder type fixed point theorem for geodesic convex sets in Hadamard manifolds as follows:

Theorem 3.1. *Let X be a nonempty geodesic convex subset of a Hadamard manifold M, and $S, T : X \to 2^X$ be two multimaps such that*

(1) *for each $y \in X$, $T^{-1}(y)$ is (possibly empty) open in X;*
(2) *for each $x \in X$, $T(x)$ is nonempty and $S(x)$ is geodesic convex, and $T(x) \subseteq S(x)$;*
(3) *there exists an $x_o \in X$ such that $X \setminus T^{-1}(x_o)$ is compact.*

Then S has a fixed point $\bar{x} \in X$, that is, $\bar{x} \in S(\bar{x})$.

Proof. First, note that if $T^{-1}(y) = X$ for some $y \in X$, then $y \in T(x) \subseteq S(x)$ for all $x \in X$ so that we obtain the conclusion. Therefore, we may assume $T^{-1}(y)$ is a proper subset of X for each $y \in X$.

Now we consider a multimap $F : X \to 2^X$ defined by

$$F(x) := X \setminus T^{-1}(x) \quad \text{for each } x \in X.$$

Then, by the assumption (1), each $F(x)$ is nonempty closed subset of X, and by the assumption (3), there exists an $x_o \in X$ such that $F(x_o)$ is compact. Here, we should note that $X = \bigcup_{y \in X} T^{-1}(y)$. Indeed, suppose that there exists some $x \in X$ such that $x \notin \bigcup_{y \in X} T^{-1}(y)$. Then, $x \notin T^{-1}(y)$ for all $y \in X$ so that $y \notin T(x)$ for all $y \in X$. This means that $T(x)$ must be an emptyset which contradicts the assumption (2). Therefore, we may assume that $X = \bigcup_{y\in X} T^{-1}(y)$; and hence

$$\bigcap_{x\in X} F(x) = \bigcap_{x\in X} X \setminus T^{-1}(x) = X \setminus \bigcup_{x\in X} T^{-1}(x) = \emptyset.$$

Therefore, by Lemma 2.4, F can not be a geodesic KKM map on X. That is, there exists a finite subset $\{y_1, \dots, y_m\} \subseteq X$ such that

$$Gco\big(\{y_1, \dots, y_m\}\big) \not\subseteq \bigcup_{i=1}^{m} F(y_i) \ = \ X \setminus \bigcap_{i=1}^{m} T^{-1}(y_i).$$

Then, we have an element $w \in Gco\big(\{y_1, \dots, y_m\}\big)$ such that $w \notin \bigcup_{i=1}^{m} F(y_i)$ so that $w \in Gco\big(\{y_1, \dots, y_m\}\big)$ with $w \in \bigcap_{i=1}^{m} T^{-1}(y_i)$. Therefore, $w \in T^{-1}(y_i)$ so that $y_i \in T(w)$ for each $1 \leq i \leq m$. By the assumption (2), $\{y_1, \dots, y_m\} \subseteq T(w) \subseteq S(w)$, and $S(w)$ is geodesic convex so that we have $Gco\big(\{y_1, \dots, y_m\}\big) \subseteq S(w)$. Therefore,

$$w \in Gco\big(\{y_1, \dots, y_m\}\big) \subseteq S(w)$$

which completes the proof. □

Remark 3.2. (1) In Theorem 3.1, if X is compact, then the assumption (3) is automatically satisfied. In this case, when $S = T$, Theorem 3.1 is the exact form of the Fan-Browder fixed point theorem in a Hadamard manifold. Following the methods in [13, 14], we can further generalize Theorem 3.1 under weaker assumptions. Indeed, we can relax the assumptions of Theorem 3.1 with transfer open conditions, e.g., see [11, 13].
(2) By using the various forms of equivalencies in [11], we can show that Theorem 3.1 is actually equivalent to Lemma 2.4. Therefore, we can also obtain various theorems which are equivalent to Theorem 3.1 as shown in [11].

As an application of Theorem 3.1, we will prove an existence of Nash equilibrium for an 1-person game of compact geodesic convex settings in a Hadamard manifold as follows:

Theorem 3.3. *Let $\mathcal{G} = (X; T, f)$ be an 1-person game such that X is a nonempty compact geodesic convex subset of a Hadamard manifold M. Suppose $f : X \times X \to \mathbb{R}$ is a function on $X \times X$, and $T : X \to 2^X$ is a multimap such that*

(1) *the set $\{(x, y) \in X \times X \mid f(x, x) > f(x, y)\}$ is open;*
(2) *T has open graph in $X \times X$, and $T(x)$ is nonempty for each $x \in X$;*
(3) *$\{y \in X \mid f(x, x) > f(x, y)\} \cap T(x)$ is a geodesic convex subset of X for each $x \in X$.*

Then there is an $x_o \in X$ *such that*

$$f(x_o, x_o) \le f(x_o, y) \quad \text{for each } \ y \in T(x_o).$$

Furthermore, if $f(x_o, x_o) > f(x_o, y)$ *for each* $y \notin T(x_o)$, *then* $x_o \in X$ *is a Nash equilibrium for the game* $\mathcal{G}$, *that is,*

$$x_o \in T(x_o) \quad \text{and} \quad f(x_o, x_o) \le f(x_o, y) \quad \text{for each } \ y \in T(x_o).$$

Proof. First, we define a multimap $S : X \to 2^X$ by

$$S(x) := \{y \in X \mid f(x, x) > f(x, y)\} \cap T(x) \quad \text{for each } \ x \in X.$$

By the assumption (3), each $S(x)$ is a geodesic subset of X. For each $y \in X$, we have

$$\begin{aligned} S^{-1}(y) &= \{x \in X \,|\, y \in S(x)\} \\ &= \{x \in X \,|\, f(x, x) > f(x, y)\} \cap T^{-1}(y). \end{aligned}$$

By the assumption (2), T^{-1} has also open graph in $X \times X$ so that $S^{-1}(y)$ is open for each $y \in X$. Since X is compact, and each $S^{-1}(y)$ is open in X, the set $X \setminus S^{-1}(y)$ is always compact. Therefore, if $S(x)$ is nonempty for each $x \in X$, then the multimap S satisfies all the assumptions of Theorem 3.1 in case of $S = T$ so that there exists a fixed point $\bar{y} \in X$ for S, *i.e.*, $\bar{y} \in S(\bar{y})$. This implies that $f(\bar{y}, \bar{y}) > f(\bar{y}, \bar{y})$ which is impossible. Therefore, we have that $S(x_o)$ should be empty for some $x_o \in X$. Since $T(x_o)$ is nonempty, we can obtain the conclusion

$$f(x_o, x_o) \le f(x_o, y) \quad \text{for each } \ y \in T(x_o).$$

Furthermore, by the assumption, if $x_o \notin T(x_o)$, then $f(x_o, x_o) > f(x_o, x_o)$ which is a contradiction. Therefore, we obtain that $x_o \in T(x_o)$ which completes the proof. □

Remark 3.4. (1) In Theorem 3.3, if f is a continuous function on $X \times X$, then the assumption (1) is clearly satisfied. Whenever X is a non-compact geodesic convex subset of a Hadamard manifold M, then we shall need the additional assumption without affecting the conclusion:

($*$) there exists an $y_o \in X$ such that $X \setminus \big(\{x \in X \,|\, f(x, x) > f(x, y_o)\} \cap T^{-1}(y_o)\big)$ is compact.

(2) It should be noted that by following the generalization method from 1-person game into generalized games with infinite agents in Ding *et al.* [3], Theorem 3.3 can be further generalized into compact generalized games in Hadamard manifolds.

In Theorem 3.3, when $T(x) := X$ for each $x \in X$, then a variational inequality is obtained as a corollary:

Corollary 3.5. *Suppose that X is a nonempty compact geodesic convex subset of a Hadamard manifold M. Suppose $f : X \times X \to \mathbb{R}$ is a continuous function on $X \times X$ such that $\{y \in X \mid f(x,x) > f(x,y)\}$ is a geodesic convex subset of X for each $x \in X$. Then there is an $x_o \in X$ such that*

$$f(x_o, x_o) \leq f(x_o, y) \quad \text{for each } \; y \in X.$$

By modifying an example in [7], we give a simple example of geodesic convex 1-person game which is suitable for Theorem 3.3, but the previous equilibrium existence theorems due to Ding *et al.* [3] and Yang-Pu [14] for compact games can not be applied:

Example 3.6. Let $\mathcal{G} = (X; T, f)$ be a compact geodesic convex 1-person game such that the pure strategic space X is defined by

$$X := \left\{ (\cos t, \, \sin t) \in \mathbb{R}^2 \mid \frac{\pi}{4} \leq t \leq \frac{3\pi}{4} \right\}.$$

Then, X is compact but not a convex subset of $\mathbb{R}^2$ in the usual sense. However, as remarked in [7], if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, then the set X is geodesic convex with respect to the metric $g_{\mathbb{H}}$ being the image of a geodesic segment from $(\mathbb{H}^2, g_{\mathbb{H}})$.

Let the payoff function $f : X \times X \to \mathbb{R}$, and a continuous constraint correspondence $T : X \to 2^X$ are defined as follows: for each $\big((x_1, x_2), (y_1, y_2)\big) \in X \times X$, $f\big((x_1, x_2), (y_1, y_2)\big) := (1 - x_1)(y_1^2 - y_2^2)$; and $T(x_1, x_2) := X$ for each $(x_1, x_2) \in X$. Then it is easy to see that the action set X is compact and geodesic convex, and payoff function f is continuous. Also, we have

(1) T has open graph in $X \times X$, and $T(x_1, x_2)$ is nonempty for each $(x_1, x_2) \in X$; and
(2) $\{(y_1, y_2) \in X \mid 0 = f\big((x_1, x_2), (x_1, x_2)\big) > f\big((x_1, x_2), (y_1, y_2)\big)\} \cap T(x_1, x_2) = \{ (y_1, y_2) \in X \mid 0 > y_1^2 - y_2^2 \} = X \setminus \{(\cos t, \, \sin t) \in \mathbb{R}^2 \mid t = \frac{\pi}{4}, \frac{3\pi}{4}\}$ is clearly a geodesic convex subset of X for each $(x_1, x_2) \in X$.

Therefore, we can apply Theorem 3.3 to the game $\mathcal{G} = (X; T, f)$; then we obtain an equilibrium point $(0, 1) \in X$ for $\mathcal{G}$ such that $(0, 1) \in T(0, 1)$ and

$$0 = f\big((0,1), (0,1)\big) \leq f\big((0,1), (y_1, y_2)\big) \quad \text{for all } \; (y_1, y_2) \in T(0,1) = X.$$

Using geodesics for a geodesic convex subset X of a Hadamard manifold M, for any finite subset $\{x_1, \ldots, x_n\} \subseteq X$, Colao *et al.* [2] inductively define the subsets D and D_i of X by

$$D(\{x_1,\ldots,x_n\}) := \bigcup_{i=1}^{n} D_i;$$

where $D_1 := \{x_1\}$ and $D_j := \{z \in \gamma_{x_j,y} \mid y \in D_{j-1}\}$ for each $2 \le j \le n$ in which $\gamma_{x_j,y}$ is the geodesic joining x_j to y. Then it is easy to see that $D(\{x_1,\ldots,x_n\})$ is a closed subset of $Gco(\{x_1,\ldots,x_n\})$. Moreover, any element $y_k \in D_k \subseteq D(\{x_1,\ldots,x_n\})$ can be written in the form $y_k \in \gamma(t_k)$ where $t_k \in [0,1]$ and γ is the geodesic joining x_k to some $y_{k-1} \in D_{k-1}$.

Using the above notions, we will prove an analogous KKM theorem for open-valued geodesic KKM maps in a Hadamard manifold which is comparable with Lemma 2.4 as follows:

Theorem 3.7. *Let X be a nonempty geodesic convex subset of a Hadamard manifold M, and $T : X \to 2^M$ be a geodesic KKM map such that for each $x \in X$, $T(x)$ is an open subset of M. Then the family of sets $\{T(x) \mid x \in X\}$ has the finite intersection property.*

Proof. For any finite subset $\{x_1,\ldots,x_n\} \subseteq X$, we shall show $\bigcap_{i=1}^n T(x_i) \neq \emptyset$. Suppose the contrary, *i.e.*, $\bigcap_{i=1}^n T(x_i) = \emptyset$. For each x_i, we associate a corresponding i-th unit vertex e_i of the simplex $\Delta_{n-1} = \langle e_1,\ldots,e_n\rangle$. Let the mapping $S : \Delta_{n-1} \to D(\{x_1,\ldots,x_n\})$ be defined by the induction as follows: If $\lambda_1 \in \langle e_1,e_2\rangle$, then $S(\lambda_1) := \gamma_1(t_1)$, where t_1 is the unique element in $[0,1]$ such that $\lambda_1 := t_1e_2 + (1-t_1)e_1$ and γ_1 is the geodesic joining x_1 to x_2. Given $1 < k \le n$, if $\lambda_k \in \langle e_1,\ldots,e_k\rangle \setminus \langle e_1,\ldots,e_{k-1}\rangle$, then $\lambda_k := t_ke_k + (1-t_k)\lambda_{k-1}$ for some $t_k \in (0,1]$ and $\lambda_{k-1} \in \langle e_1,\ldots,e_{k-1}\rangle$; and we define $S(\lambda_k) := \gamma_k(t_k)$, where γ_k is the geodesic joining x_k to $S(\lambda_{k-1})$. Then, by the definition of $D(\{x_1,\ldots,x_n\})$, we can see that $S(\Delta_{n-1})$ coincides with $D(\{x_1,\ldots,x_n\})$, and as shown in the proof of Lemma 3.1 in [2], we can see that S is a continuous mapping on Δ_{n-1}. Since T is a geodesic KKM map on X, $\{x_i\} = Gco(\{x_i\}) \subseteq T(x_i)$ and $x_i \in D_i \subseteq D(\{x_1,\ldots,x_n\})$. By the assumption, $T(x_i) \cap D(\{x_1,\ldots,x_n\})$ is a nonempty open subset in $D(\{x_1,\ldots,x_n\})$ for each $1 \le i \le n$.

Now we consider the multimap $O : \{e_1,\ldots,e_n\} \to 2^{\Delta_{n-1}}$ defined by

$$O(e_i) \equiv O_i := S^{-1}\big(T(x_i) \cap D(\{x_1,\ldots,x_n\})\big) \quad \text{for each} \quad 1 \le i \le n.$$

Then, each O_i is the open subset of Δ_{n-1}, and we can see that $\{O_i \mid 1 \le i \le n\}$ satisfies the whole assumptions of Lemma 2.5. Indeed, let $\lambda = \sum_{j=1}^k t_{i_j} e_{i_j} \in co(\{e_{i_1},\ldots,e_{i_k}\})$, with $\sum_{j=1}^k t_{i_j} = 1, \forall\, t_{i_j} \in [0,1]$. Since T is a geodesic KKM-map on X, we have

$$S(\lambda) \in D(\{x_{i_1}, \ldots, x_{i_k}\}) \subseteq Gco\big(\{x_{i_1}, \ldots, x_{i_k}\}\big) \subseteq \bigcup_{j=1}^{k} T(x_{i_j}).$$

Then there exists $j \in \{1, \ldots, k\}$ where $S(\lambda) \in T(x_{i_j}) \cap D(\{x_{i_1}, \ldots, x_{i_k}\})$ so that we have $\lambda \in S^{-1}\big(T(x_{i_j}) \cap D(\{x_{i_1}, \ldots, x_{i_k}\})\big) = O_{i_j}$. Therefore, we have $co(\{e_{i_1}, \ldots, e_{i_k}\}) \subseteq \bigcup_{j=1}^{k} O_{i_j}$, and hence $\{O_i \mid 1 \leq i \leq n\}$ satisfies the whole assumptions of Lemma 2.5. Therefore, there exists $\hat{\lambda} \in \Delta_{n-1} = \langle e_1, \ldots, e_n \rangle$ such that $\hat{\lambda} \in \bigcap_{i=1}^{n} O_i$ and hence $S(\hat{\lambda}) \in \bigcap_{i=1}^{n} T(x_i)$ which is a contradiction. This completes the proof. □

As an application of Theorem 3.7, we can prove the second Fan-Browder type fixed point theorem for geodesic convex sets in a Hadamard manifold as follows:

Theorem 3.8. *Let X be a nonempty geodesic convex subset of a Hadamard manifold M and $S, T : X \to 2^X$ be two multimaps such that*

(1) *for each $x \in X$, $S(x)$ is geodesic convex, and $T(x) \subseteq S(x)$;*
(2) *for each $y \in X$, $T^{-1}(y)$ is a closed subset of X;*
(3) *there exists a finite subset $\{x_1, \ldots, x_n\}$ of X such that $X \subseteq \bigcup_{i=1}^{n} T^{-1}(x_i)$.*

Then S has a fixed point $\bar{x} \in X$, i.e., $\bar{x} \in S(\bar{x})$.

Proof. If $T^{-1}(y) = X$ for some $y \in X$ then $y \in T(x) \subseteq S(x)$ for all $x \in X$ so that we obtain the conclusion. Hence we may assume that each $T^{-1}(x)$ is a proper subset of X. Now we consider a multimap $F : X \to 2^X$ defined by

$$F(x) := X \setminus T^{-1}(x) \quad \text{for each } x \in X.$$

By the assumption (2), each $F(x)$ is nonempty open in X. Since $T(x) \subseteq S(x)$ for each $x \in X$, it is clear that $T^{-1}(y) \subseteq S^{-1}(y)$ for each $y \in X$. By the assumption (3), X is contained in $\bigcup_{i=1}^{n} T^{-1}(x_i)$ so that we have

$$\bigcap_{i=1}^{n} F(x_i) = X \setminus \bigcup_{i=1}^{n} T^{-1}(x_i) = \emptyset.$$

Therefore, by Theorem 3.7, F can not be a geodesic KKM map on X. That is, there exists a finite subset $\{y_1, \ldots, y_m\} \subseteq X$, and $\bar{y} \in Gco\big(\{y_1, \ldots, y_m\}\big)$ such that $\bar{y} \notin \bigcup_{i=1}^{m} F(y_i)$. Since $\bar{y} \in X \setminus \bigcup_{i=1}^{m} F(y_i) = \bigcap_{i=1}^{m} T^{-1}(y_i)$, we have $\bar{y} \in T^{-1}(y_i)$ for each $1 \leq i \leq m$ so that $\{y_1, \ldots, y_m\} \subseteq T(\bar{y}) \subseteq S(\bar{y})$. By the assumption (1), $S(\bar{y})$ is a geodesic convex subset of X, and hence

$$\bar{y} \in Gco\left(\{y_1, \ldots, y_m\}\right) \subseteq Gco\left(S(\bar{y})\right) = S(\bar{y})$$

which completes the proof. □

Remark 3.9. Finally, it should be noted that those two Fan-Browder type fixed point theorems in this paper can be very basic tools for solving nonlinear problems of geodesic convex settings in Hadamard manifolds. By following the methods in [1-3, 9-14], it is possible to prove a number of generalizations of Theorem 3.1 and Theorem 3.8, respectively. Then we can establish several applications in numerous areas of nonlinear analysis where various generalized geodesic convex concepts are equipped.

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