Nonlinear Functional Analysis and Applications
Vol. 23, No. 1 (2018), pp. 129-140
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

THE STABILITY OF THE SOLUTION FOR A HYPERBOLIC PROBLEM ON $\mathbb{R}^N$

Perikles Papadopoulos[1], Niki Lina Matiadou[2] and Stavros Fatouros[3]

[1]Department of Electronics Engineering, School of Technological Applications
Piraeus University of Applied Sciences (Technological Education Institute of Piraeus)
GR 11244, Egaleo, Athens, Greece
e-mail: ppapadop@puas.gr

[2]Department of Electronics Engineering, School of Technological Applications
Piraeus University of Applied Sciences (Technological Education Institute of Piraeus)
GR 11244, Egaleo, Athens, Greece
e-mail: lmatiadou@yahoo.gr

[3]Department of Computer Systems Engineering, School of Technological Applications
Piraeus University of Applied Sciences (Technological Education Institute of Piraeus)
GR 11244, Egaleo, Athens, Greece
e-mail: fatouros@puas.gr

Abstract. We examine the generalized quasilinear Kirchhoff's string equation:

$$u_{tt} = -||A^{1/2}u||^2_H Au + f(u), \;\; x \in \mathbb{R}^N, \;\; t \geq 0,$$

with the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$. The purpose of our work is to study the stability of the solution for this equation.

1. Introduction

Our aim in this work is to study the following nonlocal quasilinear hyperbolic initial value problem:

$$u_{tt} = -||A^{1/2}u||^2_H Au \quad + \quad f(u), \;\; x \in \mathbb{R}^N, \;\; t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) \quad = \quad u_1(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

[0]Received June 1, 2017. Revised September 26, 2017.
[0]2010 Mathematics Subject Classification: 35A07, 35B30.
[0]Keywords: Quasilinear hyperbolic equations, Kirchhoff strings.
[0]Corresponding author: P. Papadopoulos(ppapadop@puas.gr).

with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$. The case of $N = 1$, Equation (1.1) describes the non-linear vibrations of an elastic string. We must remark here that Equation (1.1), also includes resembling phenomena of slowly varying wave speed (see [5]).

Kirchhoff in 1883 proposed the so called Kirchhoff string model in the study of oscillations of stretched strings and plates

$$ph\frac{\vartheta^2 u}{\vartheta t^2} + \delta\frac{\vartheta u}{\vartheta t} = \left\{p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\vartheta u}{\vartheta x}\right)^2 dx\right\}\frac{\vartheta^2 u}{\vartheta x^2} + f, \tag{1.3}$$

where we have $0 < x < L$, $t \geq 0$, and we have to mention that $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modules, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modules and f the external force (see [10]). When $p_0 = 0$ the equation is considered to be of degenerate type and the equation models an unstretched string or its higher dimensional generalization. Otherwise it is of nondegenerate type and the equation models a stretched string or its higher dimensional generalization.

The global existence and the uniqueness have been established in the energy class (see [22]). Once global existence is known, it is not difficult to show that solutions decay as $t \to \infty$. Furthermore, in the non-degenerate case a simple calculation of the energy shows that solutions decay at least exponentially.

In the degenerate case, however, estimates of the rate of decay requires far more delicate analysis. Much of the efforts have been focused on estimates from above (see [11], [14]). But it is difficult to obtain the estimates from below. In fact, except for some special cases (see [15], [16]), little has been known about the lower estimates. Also Ono in [17], proved global existence, asymptotic stability and blowing up results of solutions for some degenerate non-linear wave equation with a strong dissipation (see also [13], [18], [19]). Mizumachi (see [12]) studied the asymptotic behavior of solutions to the Kirchhoff equation with a viscous damping term with no external force.

In our previous work (see [20]), we prove global existence and blow-up results of an equation of Kirchhoff type in all of $\mathbb{R}^N$. Karachalios and Stavrakakis (see [7]-[9]) studied global existence, blow-up results and asymptotic behavior of solutions for semilinear wave equations with weak dissipation in all of $\mathbb{R}^N$.

The presentation of this paper has as follows: In section 2 we discuss the space setting of the problem and the necessary embeddings for constructing the evolution triple. In section 3 we prove the existence and uniqueness of the solution for our problem. In section 4 we study the stability of the solution $u = 0$. In order to study the stability, we study the spectrum of the operator

$\widehat{A}$. In our problem we have an external force $f(u)$ and the stability of the solution depends on the sigh of $f'(0)$.

2. Formulation of the problem

The space $D(\mathcal{A})$ is going to be introduced and studied later in this section. We shall frequently use the following generalized version of Poincaré's inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \alpha \int_{\mathbb{R}^N} g u^2 dx, \tag{2.1}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$ and $g \in L^{N/2}(\mathbb{R}^N)$, where $\alpha =: k^{-2}||g||_{N/2}^{-1}$ (see [1, Lemma 2.1]).

We also need to make the following remarks: Let V, H be two Hilbert spaces, where

$$V \subset H, \quad \text{and} \quad V \text{ is dense in } \quad H. \tag{2.2}$$

We also have that

$$V \subset\subset H \quad \text{(this means that the embedding is compact).} \tag{2.3}$$

The scalar product and the norm in H are denoted by $(.,.)$, $||.||_H$, respectively. We identify H with its dual H' and H' with a dense subspace of the dual V' of V, thus

$$V \subset H \subset V', \tag{2.4}$$

where the injections are continuous and each space is dense in the following one.

Let $a(u,v)$ be a bilinear continuous form on V which is symmetric and coercive

$$\exists\, a_0 > 0, \quad a(u,u) \geq a_0 ||u||^2, \quad \forall\, u \in V. \tag{2.5}$$

With this form we associate the linear operator $\mathcal{A}$ from V into V' defined by

$$(\mathcal{A}u, v) = a(u,v), \qquad \forall\, u, v \in V.$$

Operator $\mathcal{A}$ is an isomorphism from V onto V' and it can also be considered as a selfadjoint unbounded operator in H with domain $D(\mathcal{A}) \subset V$,

$$D(\mathcal{A}) = \{v \in V,\ \mathcal{A}v \in H\}.$$

Due to (2.2) there exists an orthonormal basis of H, $\{w_j\}_{j\in N}$ which consists of eigenvectors of $\mathcal{A}$,

$$\begin{cases} \mathcal{A}w_j = \lambda_j w_j, & j = 1,2,..., \quad w_j \in H, \\ 0 < \lambda_1 \leq \lambda_2 \leq ..., & \lambda_j \to \infty, \quad \text{as } j \to \infty. \end{cases} \tag{2.6}$$

A weak solution of our problem must satisfy the following definition.

Definition 2.1. A weak solution of the problem (1.1)-(1.2) is a function u such that

(i) $u \in L^2[0,T;D(\mathcal{A})], \ \ u_t \in L^2[0,T;V], \ \ u_{tt} \in L^2[0,T;H]$,

(ii) for all $v \in C_0^\infty([0,T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_H \, d\tau \ + \ \int_0^T \left(||\mathcal{A}^{1/2}u(t)||_H^2 \int_{\mathbb{R}^N} \mathcal{A}^{1/2}u(\tau)\mathcal{A}^{1/2}v(\tau)dxd\tau \right) - \int_0^T (f(u(\tau)), v(\tau))_H \, d\tau = 0,$$

where $f(s) = |s|^a s$,

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x), \ \ u_0 \in D(\mathcal{A}), \qquad u_t(x,0) = u_1(x), \ \ u_1 \in V.$$

3. Global existence

In this section we prove the existence of solution for problem (1.1)-(1.2), under small initial data.

Theorem 3.1. (Local Existence) *Let* $f(u)$ *be a* C^1*-function such that* $|f(u)| \leq k_1|u|^{a+1}, \ |f'(u)| \leq k_2|u|^a, \ \ 0 \leq a \leq (N+2)/(N-2)$ *and* $N \geq 3$. *Consider that* $(u_0, \ u_1) \in D(A) \times V$ *and satisfy the nondegenerate condition*

$$||A^{1/2}u_0||^2 > 0. \tag{3.1}$$

Then there exists $T_* = T(||Au_0||, ||A^{1/2}u_1||) > 0$ *such that problem* (1.1)-(1.2) *admits a unique local weak solution u satisfying*

$$u \in C(0,T_*;D(A)) \ \text{and} \ u_t \in C(0,T_*;V).$$

Proof. For given constants $T_* > 0, \ R > 0,$ we introduce the two parameter space of solutions

$$X_{T_*,R} =: \left\{ v \in C\left(0,T_*;D(A)\right) : v_t \in C_w^0\left(0,T_*;V\right), v(0) = u_0, \ v_t(0) = u_1, \ e(v) \leq R^2, \ t \in [0,T_*] \right\},$$

where $e(v(t)) =: ||Av(t)||_H^2 + ||A^{1/2}v'(t)||_H^2$, the norm in the space $X_0 =: D(A) \times V$. Also u_0 satisfies the nondegenerate condition (3.1). It is easy to see that the set $X_{T_*,R}$ is a complete metric space under the distance $d(u, \ v) =: \sup_{0 \leq t \leq T_*} e_1(u(t) - v(t))$, where

$$e_1(v) = ||v'(t)||_H^2 + ||A^{1/2}v||_H^2,$$

the norm in the space $X_1 =: V \times H$. We have that $X_0 \subset X_1$ compactly, that is, $e_1(u(t)) \leq e(u(t))$.

Next, we introduce the nonlinear mapping S in the following way. Given $v \in X_{T_*,R}$, we define $u = Sv$ to be the unique solution of the linear wave equation

$$\begin{aligned} u''(t) &+ ||A^{1/2}v(t)||_H^2 Au(t) = f(v), \\ u(0) &= u_0, \quad u'(0) = u_1. \end{aligned} \tag{3.2}$$

In the sequel we shall show that there exist $T_* > 0$ and $R > 0$ such that the following two conditions are valid

$$\text{(i) } S \text{ maps } X_{T_*,R} \text{ into itself,} \tag{3.3}$$

$$\text{(ii) } S \text{ is a contraction with respect to the metric } d(.,.). \tag{3.4}$$

Set $2M_0 =: ||A^{1/2}u_0||_H^2 > 0$ and

$$T_0 =: \sup\left\{t \in [0\,, +\infty); ||A^{1/2}v(s)||_H^2 > M_0, \ \text{ for } \ 0 \le s \le t\right\}.$$

Then

$$T_0 > 0 \ \text{ and } \ ||A^{1/2}v(t)||_H^2 \ge M_0, \ \text{ on } \ [0\,, T_0]. \tag{3.5}$$

(i) To check (3.3), we multiply (3.2) by $2Au_t$ and integrate over $\mathbb{R}^N$ to get

$$\begin{aligned} &2\int_{\mathbb{R}^N} Au_t u_{tt} dx + 2\int_{\mathbb{R}^N} ||A^{1/2}v(t)||_H^2 Au_t Au dx \\ &= 2\int_{\mathbb{R}^N} f(v) Au_t dx. \end{aligned} \tag{3.6}$$

So, we have t

$$\begin{aligned} &\frac{d}{dt}||A^{1/2}u_t(t)||_H^2 + ||A^{1/2}v(t)||_H^2 \frac{d}{dt}(||Au(t)||_H^2) \\ &= (\frac{d}{dt}||A^{1/2}v(t)||_H^2)||Au(t)||_H^2 + 2(f(v), Au_t(t)) \end{aligned}$$

Finally, we obtain

$$\frac{d}{dt}e_2^*(u(t)) = (\frac{d}{dt}||A^{1/2}v(t)||_H^2)||Au(t)||_H^2 + 2\left(f(v)\ , Au_t(t)\right), \tag{3.7}$$

where we set

$$e_2^*(u(t)) =: ||A^{1/2}u_t(t)||_H^2 + ||A^{1/2}v(t)||_H^2||Au(t)||_H^2.$$

Note that

$$e_2^*(u) \ge ||A^{1/2}u_t||_H^2 + M_0||Au(t)||_H^2 \ge c_1^{-2}e(u(t)) \ge c_1^{-2}e_1(u(t)) \tag{3.8}$$

with $c_1 = \left(\max\left\{1\ , M_0^{-1}\right\}\right)^{1/2}$. To proceed further, we observe that

$$\begin{aligned}(\frac{d}{dt}||A^{1/2}v||_H^2)||Au(t)||_H^2 &= 2\int_{\mathbb{R}^N} Avv_t dx\ ||Au(t)||_H^2 \\ &\leq 2\left(||Av||_H^2\right)^{1/2}\left(||v_t||_H^2\right)^{1/2}||Au(t)||_H^2 \\ &\leq 2Rk(||v_t||_H^2)^{1/2}e(u(t)) \\ &\leq 2RkRc_1^2 e_2^*(u(t)) \leq c_2 R^2 e_2^*(u(t)), \qquad (3.9)\end{aligned}$$

with $c_2 = 2kc_1^2$, where k is the constant of the embedding $V \subset H$. We also have that

$$\begin{aligned}2(f(v), Au_t) &= 2\int_{\mathbb{R}^N} f'(v)A^{1/2}vA^{1/2}u_t dx \\ &\leq 2k_2||v||_{Na}^a||A^{1/2}v||_{2N/N-2}||A^{1/2}u_t||_H,\end{aligned}$$

where we used Holder inequality with $p^{-1} = 1/N,\ q^{-1} = N-2/2N,\ r^{-1} = 1/2$. From the embeddings $D(A) \subset V \subset H$ and using Sobolev-Poincare inequality, we get

$$||v||_{Na}^a \leq c_*^a||Av||_H^a \leq c_*^a R^a,\ ||A^{1/2}v||_{2N/N-2} \leq c_*||Av||_H \leq c_* R.$$

Thus, we obtain that

$$\begin{aligned}2\left(f(v)\ , Au_t(t)\right) &\leq 2k_2 c_*^a R^a c_* R e(u(t))^{1/2} \\ &\leq 2k_2 c_*^{a+1} R^{a+1} c_1 e_2^*(u(t))^{1/2}. \qquad (3.10)\end{aligned}$$

Using estimates (3.9)-(3.10), we get from (3.7) that

$$\frac{d}{dt}e_2^*(u(t)) \leq c_2 R^2 e_2^*(u(t)) + c_3 R^{a+1} e_2^*(u(t))^{1/2},$$

where $c_3 = 2k_2 c_*^{a+1} c_1$. Hence, from Gronwall's inequality we get

$$e_2^*(u(t)) \leq \left\{e_2^*(u(0))^{1/2} + c_3 R^{a+1} T_*\right\}^2 e^{c_2 R^2 T_*}.$$

Thus from estimation (3.8) we obtain

$$\begin{aligned}e_1(u) &\leq e(u(t)) \\ &\leq c_1^2\left\{(||A^{1/2}u_1||_H^2 + ||A^{1/2}u_0||_H^2||Au_0||_H^2)^{1/2} + c_3 R^{a+1} T_*\right\}^2 e^{c_2 R^2 T_*} \\ &=: C_1(T_*, R), \qquad (3.11)\end{aligned}$$

for any $t \in [0, T_*]$, with $T_* \leq T_0$. Since we have that function $u \in L^\infty(0, T_*; D(A)) \cap W^{1,\infty}(0, T_*; V)$ and $u(t)$ satisfies Eq.(3.2), it follows that $u'' \in L^\infty(0, T_*; H)$ and hence, $u \in C_w^0([0, T_*]; D(A)) \cap C_w^1([0, T_*]; V)$. Thus, for the map S to verify condition (3.3) it will be enough the parameters T_*, R satisfy

$$C_1(T_*, R) < R^2, \qquad (3.12)$$

which is true for T_* and the norms small enough.
(ii) We take $v_1, v_2 \in X_{T_*,R}$ and denote by $u_1 = Sv_1, u_2 = Sv_2$. Hereafter we suppose that (3.12) is valid, *i.e.*, $u_1, u_2 \in X_{T_*,R}$. We set $w = u_1 - u_2$. The function w satisfies the following relation

$$\begin{aligned} w_{tt} + \|A^{1/2}v_1\|_H^2 Aw &= -\left\{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\right\} Au_2 \\ &\quad + f(v_1) - f(v_2), \\ w(0) = 0, &\qquad w_t(0) = 0. \end{aligned} \tag{3.13}$$

Multiplying equation (3.13) by $2w_t$ and integrating over $\mathbb{R}^N$ we have the following equation

$$\begin{aligned} &2\int_{\mathbb{R}^N} w_t w_{tt} dx + 2\int_{\mathbb{R}^N} \|A^{1/2}v_1\|_H^2 Aw w_t dx \\ &= -2\left\{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\right\}\int_{\mathbb{R}^N} Au_2 w_t dx \\ &\quad +2\int_{\mathbb{R}^N} \left(f(v_1) - f(v_2)\right) w_t dx. \end{aligned} \tag{3.14}$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt} e^*_{v_1}(w) &= \frac{d}{dt}\|A^{1/2}v_1\|_H^2 \|A^{1/2}w\|_H^2 \\ &\quad -2\left\{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\right\} (Au_2, w_t) \\ &\quad +2(f(v_1) - f(v_2), w_t) \\ &\equiv I_1(t) + I_2(t) + I_3(t), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} I_1(t) &= \frac{d}{dt}\|A^{1/2}v_1\|_H^2 \|A^{1/2}w\|_H^2, \\ I_2(t) &= -2\left\{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\right\} (Au_2, w_t), \\ I_3(t) &= 2(f(v_1) - f(v_2), w_t), \end{aligned}$$

and we also set $e^*_{v_1}(w(t)) =: \|w_t(t)\|^2 + \|A^{1/2}v_1(t)\|_H^2 \|A^{1/2}w(t)\|_H^2$. Note that the following estimations are valid

$$e^*_{v_1}(w) \geq \|w_t\|^2 + M_0\|A^{1/2}w(t)\|_H^2 \geq c_1^{-2} e_1(w). \tag{3.16}$$

As in (3.9), we observe that

$$I_1(t) \leq c_2 R^2 e^*_{v_1}(w) \tag{3.17}$$

and

$$\begin{aligned} I_2(t) &\leq 2\left(||A^{1/2}v_1||_H + ||A^{1/2}v_2||_H\right)||A^{1/2}(v_1-v_2)||_H \;||Au_2||\;||w_t(t)|| \\ &\leq 2(R+R)e_1(v_1-v_2)^{\frac{1}{2}}\; R\; e_1(w(t))^{1/2} \\ &\leq 4R^2e_1(v_1-v_2)^{1/2}c_1e^*_{v_1}(w) = c_4R^2e_1(v_1-v_2)^{1/2}e^*_{v_1}(w), \qquad (3.18) \end{aligned}$$

where $c_4 = 4c_1$. Next, applying the generalized Poincaré's inequality (2.1) and the embeddings $D(A) \subset V \subset H$, we obtain the following

$$\begin{aligned} I_3(t) &\leq 2k_1\alpha^{-1}\left(||A^{1/2}v_1||^a_H + ||A^{1/2}v_2||^a_H\right)\;||A^{1/2}(v_1-v_2)||_H\;||w_t|| \\ &\leq c_6R^ae_1(v_1-v_2)^{1/2}e^*_{v_1}(w(t))^{1/2}, \qquad (3.19) \end{aligned}$$

where $c_6 = 4k_1\alpha^{-1}c_1$. From estimates (3.17)-(3.19), we obtain the following estimate for the relation (3.15)

$$\frac{d}{dt}e^*_{v_1}(w) \leq c_2R^2e^*_{v_1}(w) + \left(c_4R^2 + c_6R^a\right)e_1(v_1-v_2)^{1/2}e^*_{v_1}(w)^{1/2}.$$

Gronwall's inequality and the fact that $e^*_{v_1}(w(0)) = 0$ imply that

$$e^*_{v_1}(w) \leq \left(c_4R^2 + c_6R^a\right)^2 T_*^2 e^{c_2R^2T_*}\sup_{0\leq t\leq T_*} e_1\left(v_1(t) - v_2(t)\right). \qquad (3.20)$$

Therefore from (3.11) and (3.20) we get

$$d(u_1\,,u_2) \leq C_2(T_*,R)\; d(v_1\,,v_2), \qquad (3.21)$$

where the constant $C_2(T_*,R)$ depending on T_* and R is

$$C_2(T_*,R) =: c_1^2\left\{c_4R^2 + c_6R^a\right\}^2 T_*^2e^{c_2R^2T_*}.$$

For small enough $T_* > 0$, we have that $C_2(T_*,R) < 1$. From the above argument, by applying the Banach contraction mapping principle we know that the problem (1.1)-(1.2) admits a unique solution $u(t)$ in the class

$$C^0_w([0,T_*];D(A)) \cap C^1_w([0,T_*];V).$$

Moreover, we see that $u \in L^\infty(0,T_*;D(A)) \cap W^{1,\infty}(0,T_*;V)$ and $f(u) \in L^\infty(0,T_*;V)$. Therefore, it follows from the continuity argument for wave equations (see [23]) that this solution u belongs to

$$C^0([0,T_*];D(A)) \cap C^1([0,T_*];V).$$

This completes the proof. □

Remark 3.2. In the above Theorem 3.1, if we assume that $u_0 \in D(A)$, $u_1 \in V$ and f is a nonlinear C^1 function, then it is easy to see following the same steps, that the solution u belongs to

$$C^0([0,T_*];V) \cap C^1([0,T_*];H). \qquad (3.22)$$

In that case, because of the inequalities

$$e_1(u(t)) \le e(u(t)) \le R^2,$$

we find that u is a solution such that

$$u \in L^\infty(0, T_*; V),\ u' \in L^\infty(0, T_*; H).$$

The continuity properties (3.22), are proved with the methods indicated in ([23], Sections II.3 and II.4).

Corollary 3.3. (Global Existence) *We assume that* $(u_0,\ u_1) \in D(A) \times V$. *Then, there exists a unique solution of problem* (1.1)-(1.2) *such that*

$$u \in C([0,\ +\infty)\ ;\ V), \quad u_t \in C([0,\ +\infty)\ ;\ H).$$

Proof. Following the same steps as in [20] we obtain the global existence result. □

4. The linearized system

In this section we study the stability of the solution $u = 0$. The linearized equation of the system around solution u is the following

$$v_{tt} = -||A^{1/2}u||_H Av + f'(u)v. \tag{4.1}$$

In the case where $u = 0$, equation (4.1) becomes

$$v_{tt} - f'(0)v = 0. \tag{4.2}$$

We set $v_t = w$ and we get from (4.2) that

$$\begin{cases} w_t = f'(0)v, \\ v_t = w, \end{cases}$$

or,

$$\begin{bmatrix} w \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -f'(0) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Finally we obtain

$$\bar{u_t} + \hat{A}\bar{u} = 0, \tag{4.3}$$

where $\bar{u_t} = (w, v)^T$ and $\hat{A} = \begin{bmatrix} 0 & -f'(0) \\ -1 & 0 \end{bmatrix}$. So, in order to study the stability of the solution, we study the spectrum of the operator $\hat{A}$.

For later use we give the following definition and theorems. (for the proofs we refer to ([6], Theorem 5.1.1 and Theorem 5.1.3)).

Definition 4.1. The continuous spectrum of an operator $\mathcal{A}$ is the set $\sigma_c(\mathcal{A})$ of complex numbers λ for which $\lambda I - \mathcal{A}$ is one to one and has a dense range which is not equal to X, where X is a Banach space.

Theorem 4.2. *Let $\mathcal{A}$ be a sectorial linear operator in a Banach space X, and let $f : U \to X$ where U is a cylindrical neighborhood in $\mathbb{R}^N \times X^a$, (for some $a < 1$). Also let x_0 be an equilibrium point. Suppose*

$$f(t,\ x_0 + z) = f(t,\ x_0) + Bz + g(t,\ z),$$

where B is a bounded linear map from X^a to X and $||g(t,\ z)|| = O(||z||_a)$ as $||z||_a \to 0$, uniformly in t, and $f(t,x)$ is locally Holder continuous in t, locally Lipschitzian in x, on U. If the spectrum of $\mathcal{A} - B$ lies in $\{\mathbb{R}\,\lambda > \beta\}$, for some $\beta > 0$, or equivalently if the linearization

$$\frac{dz}{dt} + \mathcal{A}z = Bz,$$

is uniformly asympotically stable, then the original equation has the solution x_0 unifomly asympotically stable in X^a.

Theorem 4.3. *Let $\mathcal{A}$, f be as in Theorem 4.2. Assume also $\mathcal{A}x_0 = f(t,\ x_0)$ for $t \geq t_0$,*

$$\begin{aligned} f(t,\ x_0 + z) = f(t,\ x_0) \quad &+ \quad Bz + g(t,z), \quad g(t,0) = 0, \\ ||g(t,z_1) - g(t,z_2)|| \quad &\leq \quad k(\rho)\,||z_1 - z_2||_a, \\ ||z_1||_a \leq \rho, \qquad ||z_2||_a \quad &\leq \quad \rho, \qquad \text{and} \quad k(\rho) \to 0, \quad \rho \to 0^+. \end{aligned}$$

If we set $L = \mathcal{A} - B$ and assume that $\sigma(L) \cap \{\mathbb{R}\,\lambda < 0\}$ is a nonempty spectral set, then we obtain that the equilibrium solution x_0 is unstable for the original equation in X^a (where X^a is as in Theorem 4.2)

Next, we will compute the eigenvalues of $\hat{A}$. Let $\bar{x_j} = [\phi_j,\ \psi_j] \in D(A)$. Eigenvalues of $\hat{A}$ satisfy the following relation

$$\hat{A}\,\bar{x_j} = \mu_j\,\bar{x_j},$$

or,

$$\begin{bmatrix} 0 & -f'(0) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \mu_j \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix}. \tag{4.4}$$

Therefore, in order to find the eigenvalues of $\hat{A}$, we compute the characteristic polynomial of $\hat{A}$, that is,

$$\det \begin{vmatrix} \mu_j & f'(0) \\ 1 & \mu_j \end{vmatrix} = 0 \quad \Leftrightarrow \quad \mu_j^2 - f'(0) = 0.$$

Then according to the sign of $f'(0)$, we have the following cases:

Case I. Let $f'(0) > 0$. Then the operator $\hat{A}$ admits the following two real eigenvalues of different sign:

$$\mu_{j^\pm} = \pm(f'(0))^1/2. \tag{4.5}$$

Also, since we have that $f'(0) > 0$ from the initial hypothesis, we may easily see that the continuous spectrum of $\hat{A}$, $\sigma_c(\hat{A}) = \emptyset$. So, by Theorem 4.3 we have that $\min Re_{n\in\sigma_c(\hat{A})}\, n < 0$, which implies that 0 is unstable for the initial Kirchhoff's system.

Case II. Let $f'(0) < 0$. Then we have that the eigenvalues $\mu_{1^\pm}$ are complex and

$$\min Re_{\mu_1\in\sigma(\hat{A})}\mu_{1^\pm} > f'(0).$$

Therefore, using Theorem 4.2, we have that the solution $u = 0$ is asymptotically stable for the initial Kirchhoff's system.

Case III. Let $f'(0) = 0$. Then we obtain that $\sigma_c(\hat{A}) = \{0\}$, thus we have that the solution $u = 0$ is stable for the initial problem, (according to Theorem 4.3).

References

[1] K.J. Brown and N.M. Stavrakakis, *Global bifurcation results for a semilinear elliptic equation on all of* $\mathbb{R}^N$, Duke Math. J., **85** (1996), 77–94.

[2] S.N. Chow and K. Lu, $\mathcal{C}^k$ *center unstable manifolds*, Proc. Roy. Soc. Edinburgh, **108** A, (1988), 303–320.

[3] S.N. Chow and K. Lu, *Invariant manifolds for flows in Banach spaces*, J. Diff. Eq., **74** (1988), 285–317.

[4] S. Dunford and D. Schwartz, *Linear operators, I*, Wiley-Interscience, New-York, 1958.

[5] Th. Gallay and G. Raugel, *Scaling variables and asymptotic expansions in dambed wave equations*, J. of Diff. Equ., **150** (1998), 42–97.

[6] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes in Math., Springer-Verlag, Berlin, 1981.

[7] N.I. Karahalios and N.M. Stavrakakis, *Existence of global attractors for semilinear dissipative wave equations on* $\mathbb{R}^N$, J. of Dif. Equ., **157** (1999), 183–205.

[8] N.I. Karahalios and N.M. Stavrakakis, *Asymptotic behavior of solutions of some nonlinearly damped wave equation on* $\mathbb{R}^N$, Topol. Methods Nonlinear Anal., **18** (2001), 73–87.

[9] N.I. Karahalios and N.M. Stavrakakis, *Global existence and blow-up results for some nonlinear wave equations on* $\mathbb{R}^N$, Adv. Diff. Equ., **6** (2001), 155–174.

[10] G. Kirchhoff , *Vorlesungen Über Mechanik*, Teubner, Leipzig, 1883.

[11] M.P. Matos and D.C. Pereira, *On a hyperbolic equation with strong damping*, Funkcialaj Ekvacioj, **34** (1991), 303–311.

[12] T. Mizumachi, *The asymptotic behavior of solutions to the Kirchhoff equation with a Viscous damping term*, J. of Dyna. and Diff. Equ., **9** (1997), 211–247.

[13] M. Nakao and K. Ono, *Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equation*, Math. Z., **214** (1993), 325–342.

[14] K. Nishihara, *Degenerate quasilinear hyperbolic equation with strong damping*, Funkcialaj Ekvacioj, **27** (1984), 125–145.

[15] K. Nishihara, *Decay properties of solutions of some quasilinear hyperbolic equations with stong damping*, Nonlinear Anal. TMA., **21** (1993), 17–21.

[16] K. Nishihara and K. Ono, *Asymptotic behaviors of solutions of some nonlinear oscillation equations with strong damping*, Adv. Math. Sci. Appl., **4** (1994), 285–295.

[17] K. Ono, *On global existence, asymptotic stability and blowing - up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation*, Math. Methods Appl. Sci., **20** (1997), 151–177.

[18] K. Ono, *Global existence and decay properties of solutions for some mildly degenerate nonlinear dissipative Kirchhoff strings*, Funkcialaj Ekvacioj, **40** (1997), 255–270.

[19] K. Ono, *Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings*, J. of Diff. Equ., **137** (1997), 273–301.

[20] P.G. Papadopoulos and N.M. Stavrakakis, *Global existence and blow-up results for an equation of Kirchhoff type on* $\mathbb{R}^N$, Topolo. Methods Nonlinear Anal., **17** (2001), 91–109.

[21] R.L. Pego, *Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability*, Arch. Rat. Anal., **97** (1987), 353–394.

[22] M. Tsutsumi, *Some nonlinear evolution equations of second order*, Proc. Japan. Acad., **47** (1971), 950–955.

[23] R. Temam, *Infinite- dimensional dynamical systems in mechanics and physics*, Appl. Math. Sc., 68, (2nd Edition), Springer-Verlag, 1997.