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HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A NON-METRIC ϕ-SYMMETRIC CONNECTION

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Abstract. We study half lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection $\bar{\nabla}$ subject such that the tensor field ϕ is identical with the fundamental 2-form associated with the indefinite almost complex structure J of $\bar{M}$.

1. Introduction

A codimension 2 submanifold M of a semi-Riemannian manifold is called

(1) *half lightlike submanifold* if $rank\{Rad(TM)\} = 1$,
(2) *coisotropic submanifold* if $rank\{Rad(TM)\} = 2$,

where $Rad(TM)$ is the radical distribution given by $Rad(TM) = TM \cap TM^{\perp}$, TM and $TM^{\perp}$ are the tangent and normal bundle of M, respectively. Half lightlike submanifold was introduced by Duggal-Bejancu [2] and later, studied by Duggal-Jin [3]. Its geometry is more general than that of lightlike hypersurfaces or coisotropic submanifolds. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds. A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *non-metric*

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ϕ-symmetric connection if it and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y},\bar{Z}) = -\theta(\bar{Y})\phi(\bar{X},\bar{Z}) - \theta(\bar{Z})\phi(\bar{X},\bar{Y}), \tag{1.1}$$

$$\bar{T}(\bar{X},\bar{Y}) = \theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}, \tag{1.2}$$

where ϕ and J are tensor fields of types $(0,2)$ and $(1,1)$, respectively, and θ is a 1-form associated with a smooth unit spacelike vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X},\zeta)$. In the followings, denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$. The notion of non-metric ϕ-symmetric connection on indefinite almost complex or indefinite almost contact manifolds was defined by Jin [6, 7].

The subject of study in this paper is half lightlike submanifolds of an indefinite Kaehler manifold $(\bar{M},\bar{g},J)$ with a non-metric ϕ-symmetric connection, in which the tensor field J defined by (1.2) is identical with the indefinite almost complex structure J of $\bar{M}$ and the tensor field ϕ given by (1.1) is identical with the fundamental 2-form associated with J, that is,

$$\phi(\bar{X},\bar{Y}) = \bar{g}(J\bar{X},\bar{Y}). \tag{1.3}$$

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of an indefinite Kaehler manifold $(\bar{M},\bar{g},J)$. It is known [7] that a linear connection $\bar{\nabla}$ is a non-metric ϕ-symmetric connection if and only if $\bar{\nabla}$ satisfies

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})J\bar{X}. \tag{1.4}$$

2. Preliminaries

Let $\bar{M} = (\bar{M},\bar{g},J)$ be an indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an almost complex structure such that

$$J^2 = -I, \qquad \bar{g}(J\bar{X},J\bar{Y}) = \bar{g}(\bar{X},\bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the non-metric ϕ-symmetric connection $\bar{\nabla}$ given by (1.4), the third equation of (2.1) is reduced to

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \theta(\bar{Y})\bar{X} + \theta(J\bar{Y})J\bar{X}. \tag{2.2}$$

Let (M,g) be a half lightlike submanifold of $\bar{M}$. As $rank(Rad(TM)) = 1$, there exist two complementary non-degenerate vector bundles $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called *screen distribution* and *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use same notations for any others. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit spacelike vector field, without loss of generality. Consider the orthogonal complementary vector bundle $S(TM)^\perp$ to $S(TM)$ in $T\bar{M}$. Certainly, $Rad(TM)$ and $S(TM^\perp)$ are vector subbundles of $S(TM)^\perp$. As the co-screen distribution $S(TM^\perp)$ is non-degenerate, we have

$$S(TM)^\perp = S(TM^\perp) \oplus_{orth} S(TM^\perp)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. For any null section ξ of $Rad(TM)$, there exists a uniquely defined lightlike vector bundle $ltr(TM)$ and a null vector field N of $ltr(TM)$ satisfying

$$\bar{g}(\xi, N) = 1, \;\bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \;\forall\, X \in \Gamma(S(TM)).$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ the *null transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* with respect to $S(TM)$ respectively [3]. $T\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned}$$

Let P be the projection morphism of TM on $S(TM)$. Denote by X, Y and Z the smooth vector fields on M, unless otherwise specified. Then the local Gauss-Weingarten formular of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.3}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.4}$$
$$\bar{\nabla}_X L = -A_L X + \lambda(X)N + \mu(X)L; \tag{2.5}$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$
$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.7}$$

respectively, where the symbols ∇ and ∇^* are linear connections on M and $S(TM)$, respectively, B and D are the *local second fundamental forms* of M, C is the *local second fundamental form* on $S(TM)$. A_N, A_L and A^*_ξ are the *shape operators* and τ, ρ, λ, μ and σ are 1-forms on M.

For a half lightlike submanifold M of $\bar{M}$, it is known [4] that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$ with mutually trivial intersections, of rank 1. Thus there exist two non-degenerate almost complex distributions H_o and H on M with respect to J such that

$$\begin{aligned} S(TM) &= J(Rad(TM)) \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp) \oplus_{orth} H_o, \\ H &= \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} H_o. \end{aligned}$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{2.8}$$

Let a, b and e be the smooth functions defined by

$$a = \theta(N), \qquad b = \theta(\xi), \qquad e = \theta(L).$$

Consider two null vector fields $\{U, V\}$ and one spacelike vector field W on the screen distribution $S(TM)$ and their 1-forms u, v and w such that

$$U = -JN, \qquad V = -J\xi, \qquad W = -JL, \tag{2.9}$$

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W). \tag{2.10}$$

Denote by S the projection morphism of TM on H and F the tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Then

$$JX = FX + u(X)N + w(X)L. \tag{2.11}$$

Applying J to (2.11) and using (2.1) and (2.9), we have

$$F^2X = -X + u(X)U + w(X)W. \tag{2.12}$$

Substituting (2.11) into $(1.2)_2$ and using (2.11) and (2.10), we have

$$g(FX, FY) = g(X, Y) - u(X)v(Y) - u(Y)v(X) - w(X)w(Y). \tag{2.13}$$

In the sequel, we say that F is the *structure tensor field* of M.

3. Non-metric ϕ-symmetric connections

Let T be the torsion tensor on M with respect to ∇ and η the 1-form given by $\eta(X) = \bar{g}(X, N)$. Using (1.1), (1.2), (1.3), (2.3) and (2.11), we get

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) - \theta(Y)\phi(X, Z) - \theta(Z)\phi(X, Y), \tag{3.1}$$

$$T(X, Y) = \theta(Y)FX - \theta(X)FY, \tag{3.2}$$

$$B(X, Y) - B(Y, X) = \theta(Y)u(X) - \theta(X)u(Y), \tag{3.3}$$

$$D(X, Y) - D(Y, X) = \theta(Y)w(X) - \theta(X)w(Y), \tag{3.4}$$

$$\phi(X, Y) = g(FX, Y) + u(X)\eta(Y), \tag{3.5}$$

$$\phi(X, \xi) = u(X), \quad \phi(X, N) = v(X), \quad \phi(X, L) = w(X), \tag{3.6}$$

$$\phi(X, V) = 0, \quad \phi(X, U) = -\eta(X), \quad \phi(X, W) = 0.$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\xi, \xi) = 0$, $\bar{g}(\xi, L) = 0$, $\bar{g}(N, N) = 0$, $\bar{g}(N, L) = 0$, $\bar{g}(L, L) = 1$ and $\bar{g}(\xi, N) = 1$ by turns and using (1.1) and (3.6), we obtain

$$B(X, \xi) = bu(X), \qquad D(X, \xi) = -\lambda(X) + eu(X) + bw(X), \tag{3.7}$$

$$\bar{g}(A_N X, N) = -av(X), \quad \bar{g}(A_L X, N) = \rho(X) - ev(X) - aw(X), \tag{3.8}$$

$$\mu(X) = ew(X), \qquad \sigma(X) = \tau(X) - au(X) - bv(X), \tag{3.9}$$

respectively. From (3.3), (3.4) and (3.7), we see that

$$B(\xi, X) = 0, \qquad D(\xi, X) = -\lambda(X) + eu(X). \tag{3.10}$$

The local second fundamental forms are related to their shape operators by

$$\begin{aligned} B(X,Y) &= g(A^*_\xi X, Y) + bg(FX, Y) + u(X)\theta(Y), && (3.11)\\ D(X,Y) &= g(A_{_L} X, Y) + eg(FX, Y) + w(X)\theta(Y) && (3.12)\\ &\quad - \{\lambda(X) - eu(X)\}\eta(Y), \\ C(X, PY) &= g(A_{_N} X, PY) + ag(FX, PY) + v(X)\theta(PY). && (3.13) \end{aligned}$$

Replacing X by ξ to (3.11) and using (2.10) and $(3.10)_1$, we obtain

$$A^*_\xi \xi = bV. \tag{3.14}$$

Applying $\bar\nabla_X$ to (2.9) and (2.11) and using (2.3)∼(2.5), (2.9), (2.11), (2.2) and (3.11)∼(3.13), we have

$$\begin{aligned} B(X,U) &= u(A_{_N}X) + \theta(U)u(X) \\ &= C(X,V) + \theta(U)u(X) - \theta(V)v(X), \\ D(X,U) &= w(A_{_N}X) + \theta(U)w(X) && (3.15)\\ &= C(X,W) - \theta(W)v(X) + \theta(U)w(X), \\ D(X,V) &= B(X,W) - \theta(W)u(X) + \theta(V)w(X), \\ \nabla_X U &= F(A_{_N}X) + \tau(X)U + \rho(X)W - aX + \theta(U)FX, && (3.16)\\ \nabla_X V &= F(A^*_\xi X) - \sigma(X)V + bu(X)U + D(X,\xi)W && (3.17)\\ &\quad - bX + \theta(V)FX, \\ \nabla_X W &= F(A_{_L}X) + \lambda(X)U + \mu(X)W - eX + \theta(W)FX, && (3.18)\\ (\nabla_X F)Y &= u(Y)A_{_N}X + w(Y)A_{_L}X - B(X,Y)U && (3.19)\\ &\quad - D(X,Y)W + \theta(Y)X + \theta(JY)FX. \end{aligned}$$

Definition 3.1. ([9]) A half lightlike submanifold M of a semi-Riemannian manifold is called *irrotational* if $\bar\nabla_X \xi \in \Gamma(TM)$, *i.e.*, $B(X,\xi) = D(X,\xi) = 0$.

Note that, from (3.7), we see that M is irrotational if and only if

$$b = 0, \qquad \lambda(X) = eu(X). \tag{3.20}$$

4. Recurrent and Lie recurrent submanifolds

Definition 4.1. ([5]) The structure tensor field F of M is said to be *recurrent* if there exists a smooth 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

Theorem 4.2. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. If F is recurrent, then the following six statements are satisfied:*

(1) *F is parallel with respect to the induced connection ∇ on M.*
(2) *M is irrotational.*
(3) *The 1-form θ vanishes, i.e., $\theta = 0$, on M.*
(4) *W is parallel vector field with respect to the connection ∇.*
(5) *H, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions on M.*
(6) *M is locally a product manifold $M = \mathcal{C}_{U} \times \mathcal{C}_{W} \times M^{\sharp}$, where $\mathcal{C}_{U}$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_{W}$ is a spacelike curve tangent to $J(S(TM^{\perp}))$, and $M^{\sharp}$ is a leaf of the distribution H.*

Proof. (1) From the above definition and (3.19), we get

$$\begin{aligned} \varpi(X)FY = u(Y)A_{N}X + w(Y)A_{L}X - B(X,Y)U \\ - D(X,Y)W + \theta(Y)X + \theta(JY)FX. \end{aligned} \tag{4.1}$$

Replacing Y by ξ and using (2.9), (2.10) and the fact: $F\xi = -V$, we get

$$\varpi(X)V = B(X,\xi)U + D(X,\xi)W - bX + \theta(V)FX. \tag{4.2}$$

Taking the scalar product with N to (4.2), we obtain $b\eta(X) - \theta(V)v(X) = 0$. Taking $X = \xi$ and $X = V$ to this result by turns, we have

$$b = 0, \qquad \theta(V) = 0. \tag{4.3}$$

Taking the scalar product with U to (4.2), we get $\varpi = 0$. It follows that $\nabla_X F = 0$. Therefore, F is parallel with respect to the connection ∇.

(2) Taking the scalar product with V and W to (4.2) by turns, we get

$$B(X,\xi) = 0, \qquad D(X,\xi) = 0. \tag{4.4}$$

It is equivalent to $\bar{\nabla}_X\xi \in \Gamma(TM)$. Therefore, M is irrotational.

(3) Replacing Y by V to (4.1) and using (4.3), we have

$$B(X,V) = 0, \qquad D(X,V) = 0. \tag{4.5}$$

Taking $Y = V$ to (3.3) and (3.4) by turns and using $(4.3)_2$ and (4.5), we get

$$B(V,X) = 0, \qquad D(V,X) = 0. \tag{4.6}$$

Taking $Y = U$ and $Y = W$ to (4.1) such that $\varpi = 0$ by turns, we have

$$A_{N}X = B(X,U)U + D(X,U)W - \theta(U)X - aFX, \tag{4.7}$$

$$A_{L}X = B(X,W)U + D(X,W)W - \theta(W)X - eFX. \tag{4.8}$$

Taking the scalar product with N and then, with U to (4.7) and (4.8) by turns and using (3.8), (3.12) and (3.13), we obtain

$$\theta(U) = 0, \qquad C(X, U) = 0, \tag{4.9}$$

$$\rho(X) - aw(X) = -\theta(W)\eta(X), \quad D(X, U) = -\theta(W)v(X). \tag{4.10}$$

Replacing X by V to $(4.10)_2$ and using $(4.6)_2$, we get $\theta(W) = 0$. Thus

$$\theta(W) = 0, \qquad \rho(X) = aw(X), \qquad D(X, U) = 0. \tag{4.11}$$

Taking the product with N to (4.1) and using (3.8) and $(4.11)_2$, we have

$$\theta(Y)\eta(X) + \{\theta(JY) - au(Y) - ew(Y)\}v(X) = 0.$$

Replacing X by ξ and V to this equation by turns, we obtain

$$\theta(X) = 0, \qquad \theta(JX) = au(X) + ew(X), \quad \forall\, X \in \Gamma(TM). \tag{4.12}$$

(4) Applying F to (4.7) and (4.8) and using (2.12), we obtain

$$F(A_N X) - aX = -au(X)U - aw(X)W,$$
$$F(A_L X) - eX = -eu(X)U - ew(X)W.$$

Using these, (3.9), $(3.20)_2$ and $(4.11)_2$, Eqs: (3.16) and (3.18) reduce

$$\nabla_X U = \sigma(X)U, \qquad \nabla_X W = 0. \tag{4.13}$$

From $(4.13)_2$, we see that W is parallel vector field with respect to ∇.

(5) It follows from (4.13) that $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions on M with respect to ∇, that is,

$$\nabla_X U \in \Gamma(J(ltr(TM))), \qquad \nabla_X W \in \Gamma(J(S(TM^{\perp}))).$$

On the other hand, using $(4.3)_2$, $(4.5)_2$ and $(4.11)_1$, from $(3.15)_3$ we get

$$B(X, W) = 0. \tag{4.14}$$

Taking $Y = FZ$ to (4.1) and using (4.12) and $u(FZ) = w(FZ) = 0$, we get

$$B(X, FZ) = 0, \qquad D(X, FZ) = 0. \tag{4.15}$$

For any $X \in \Gamma(TM)$ and $Z \in \Gamma(H_o)$, by using (2.2), (2.7), (2.13), $(3.6)_{1,4}$, (3.11), (3.12), (3.17), (4.3)$\sim$(4.5), (4.12), (4.14) and (4.15), we derive

$$g(\nabla_X \xi, V) = -B(X, V) + \theta(V)u(X) = 0,$$
$$g(\nabla_X \xi, W) = -B(X, W) + \theta(W)u(X) = 0,$$
$$g(\nabla_X V, V) = 0,$$
$$g(\nabla_X V, W) = D(X, \xi) - bw(X) = 0,$$
$$g(\nabla_X Z, V) = B(X, FZ) - \theta(FZ)u(X) = 0,$$
$$g(\nabla_X Z, W) = D(X, FZ) - \theta(FZ)w(X) = 0.$$

It follows that H is also a parallel distribution on M, that is,

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

(6) As $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H are parallel distributions and satisfied (2.8), by the decomposition theorem [1], M is locally a product manifold $\mathcal{C}_U \times \mathcal{C}_W \times M^{\sharp}$, where $\mathcal{C}_U$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_W$ is a spacelike curve tangent to $J(S(TM^{\perp}))$, and $M^{\sharp}$ is a leaf of H. □

Definition 4.3. ([5]) The structure tensor field F of M is said to be *Lie recurrent* if there exists a smooth 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X. The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$.

Theorem 4.4. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. If F is Lie recurrent, then the following three statements are satisfied:*

1. *F is Lie paralle.*
2. *τ and σ satisfy $\tau(X) = au(X)$ and $\sigma(X) = -bv(X)$.*
3. *The shape operator A^*_ξ satisfies $A^*_\xi U = A^*_\xi V = 0$.*

Proof. (1) Using (3.2) and (3.19), we have

$$\begin{aligned}\vartheta(X)FY = &-\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X + w(Y)A_L X \\ &- \{B(X,Y) - \theta(Y)u(X)\}U - \{D(X,Y) - \theta(Y)w(X)\}W \\ &+ \{au(Y) + ew(Y)\}FX.\end{aligned} \tag{4.16}$$

Taking $Y = \xi$ to (4.16) and using $(3.7)_{1,2}$ and the fact: $F\xi = -V$, we get

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \{\lambda(X) - eu(X)\}W. \tag{4.17}$$

Taking the scalar product with V and W to (4.17) by turns, we have

$$u(\nabla_V X) = 0, \qquad w(\nabla_V X) = -\lambda(X) + eu(X). \tag{4.18}$$

Replacing Y by V to (4.16) and using the fact: $FV = \xi$, we have

$$\begin{aligned}\vartheta(X)\xi = &-\nabla_\xi X + F\nabla_V X \\ &- \{B(X,V) - \theta(V)u(X)\}U - \{D(X,V) - \theta(V)w(X)\}W.\end{aligned} \tag{4.19}$$

Applying F to this equation and using (2.12) and (4.18), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \{\lambda(X) - eu(X)\}W.$$

Comparing this equation with (4.17), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N to (4.16) and using (3.8), we get

$$-\bar{g}(\nabla_{FY}X, N) + g(\nabla_Y X, U) + \rho(X)w(Y) - aw(X)w(Y) = 0. \tag{4.20}$$

Replacing X by ξ to (4.20) and using (2.7), $(3.9)_2$ and (3.11), we have

$$B(Y,U) - \rho(\xi)w(Y) - \theta(U)u(Y) = \tau(FY). \tag{4.21}$$

Taking $Y = U$ and $Y = W$ by turns and using (3.3) and $(3.15)_{1,3}$, we get

$$\begin{aligned} C(U,V) &= B(U,U) - \theta(U) = 0, \\ D(U,V) &= B(U,W) - \theta(W) = \rho(\xi). \end{aligned} \tag{4.22}$$

On the other hand, taking the scalar product with V to (4.16) and then, replacing X by U and using (3.3), (3.12), (3.13) and (3.16), we have

$$-\tau(FY) - B(Y,U) + \theta(U)u(Y) + u(Y)C(U,V) + w(Y)D(U,V) = 0. \tag{4.23}$$

From the last two equations (4.22) and (4.23), we see that

$$B(Y,U) - \rho(\xi)w(Y) - \theta(U)u(Y) = -\tau(FY).$$

Comparing this equation with (4.21), we obtain $\tau(FX) = 0$.

Replacing X by W to (4.16) and using (2.12), (3.3), (3.4), $(3.9)_1$, (3.12), $(3.15)_3$ and (3.18), we obtain

$$u(Y)A_N W + w(Y)A_L W - A_L Y - F(A_L FY) - \lambda(FY)U = 0.$$

Taking the scalar product with N and using (2.12), (3.8) and (3.12), we get

$$D(FY,U) = w(Y)\rho(W) - \rho(Y).$$

Replacing Y by U and V by turns and using $(3.10)_2$, we have $\rho(U) = 0$ and

$$-\rho(V) = D(\xi, U) = -\lambda(U) + e.$$

On the other hand, replacing X by U to $(4.18)_2$ and using (3.16), we get

$$\rho(V) = -\lambda(U) + e.$$

Comparing the last two equations, we get $\rho(V) = 0$ and $\lambda(U) = e$. Replacing X by W to $(4.18)_1$ and using (3.18), we obtain $\lambda(V) = 0$. Thus

$$\rho(U) = 0, \qquad \rho(V) = 0, \qquad \lambda(U) = e, \qquad \lambda(V) = 0. \tag{4.24}$$

Replacing X by V to (4.20) and using (2.12), (3.17) and $(4.24)_2$, we have

$$g(A^*_\xi FY, U) + \sigma(Y) = 0.$$

Using this equation, $(3.9)_2$ and (3.11), we obtain

$$B(FY,U) = -\tau(Y) + au(Y).$$

Taking $Y = U$ and $Y = W$ by turns and using $FU = FW = 0$, we obtain

$$\tau(U) = a, \qquad \tau(W) = 0. \tag{4.25}$$

Replacing X by FY to $\tau(FX)=0$ and using (2.12) and (4.25), we see that $\tau(X)=au(X)$. From this result and $(3.9)_2$, we obtain $\sigma(X)=-bv(X)$.

(3) Taking the scalar product with W to (4.19), we have

$$D(X,V)-\theta(V)w(X)=-g(\nabla_\xi X,W).$$

Replacing X by U to this and using (3.16), we obtain

$$D(U,V)=-\rho(\xi).$$

From this result and $(4.22)_2$, we obtain $\rho(\xi)=0$.

Taking the scalar product with V to (4.19), we obtain

$$B(X,V)-\theta(V)u(X)+g(\nabla_\xi X,V)=0.$$

Replacing X by W to this equation and using (3.3) and (3.18), we have

$$B(V,W)=-\lambda(\xi).$$

Replacing X by ξ to $(4.18)_2$ and using (2.7) and (3.11), we get

$$B(V,W)=\lambda(\xi).$$

Comparing the last two equations, we obtain $\lambda(\xi)=0$. Therefore,

$$\rho(\xi)=0,\quad \lambda(\xi)=0,\quad D(U,V)=0,\quad B(X,U)=\theta(U)u(X). \tag{4.26}$$

Taking $Y=U$ to (3.3) and using $(4.26)_4$, we get $B(U,X)=\theta(X)$. Taking $X=U$ to (3.11), we have $g(A^*_\xi U,X)=0$. As $S(TM)$ is non-degenerate, this result implies $A^*_\xi U=0$. Replacing X by ξ to (4.17) and using (2.7), (3.14) and the facts that $\lambda(\xi)=0$ and $\sigma(X)=-bv(X)$, we obtain $A^*_\xi V=0$. □

5. Indefinite complex space forms

Definition 5.1. An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c;

$$\begin{aligned}\widetilde{R}(\bar{X},\bar{Y})\bar{Z}=\frac{c}{4}\{&\bar{g}(\bar{Y},\bar{Z})\bar{X}-\bar{g}(\bar{X},\bar{Z})\bar{Y}\\ &+\bar{g}(J\bar{Y},\bar{Z})J\bar{X}-\bar{g}(J\bar{X},\bar{Z})J\bar{Y}+2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\},\end{aligned} \tag{5.1}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Let $\bar{R}$ be the curvature tensor of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.2) and (1.4), we see that

$$\bar{R}(\bar{X},\bar{Y})\bar{Z}=\widetilde{R}(\bar{X},\bar{Y})\bar{Z}+(\bar{\nabla}_X\theta)(Z)JY-(\bar{\nabla}_Y\theta)(Z)JX. \tag{5.2}$$

Denote by R and R^* the curvature tensors of the induced connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formular, we have two Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}\bar{R}(X,Y)Z \;=\;& R(X,Y)Z + B(X,Z)A_{N}Y - B(Y,Z)A_{N}X \\ &+ D(X,Z)A_{L}Y - D(Y,Z)A_{L}X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &+ \lambda(X)D(Y,Z) - \lambda(Y)D(X,Z) \\ &+ \theta(Y)B(FX,Z) - \theta(X)B(FY,Z)\}N \\ &+ \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) \\ &+ \rho(X)B(Y,Z) - \rho(Y)B(X,Z) \\ &+ \mu(X)D(Y,Z) - \mu(Y)D(X,Z) \\ &+ \theta(Y)D(FX,Z) - \theta(X)D(FX,Z)\}L,\end{aligned} \tag{5.3}$$

$$\begin{aligned}R(X,Y)PZ \;=\;& R^*(X,Y)PZ + C(X,PZ)A^*_{\xi}Y - C(Y,PZ)A^*_{\xi}X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &- \sigma(X)C(Y,PZ) + \sigma(Y)C(X,PZ) \\ &+ \theta(Y)C(FX,PZ) - \theta(X)C(FY,PZ)\}\xi.\end{aligned} \tag{5.4}$$

Comparing the tangential components of (5.2) and (5.3), we obtain

$$\begin{aligned}R(X,Y)Z \;=\;& B(Y,Z)A_{N}X - B(X,Z)A_{N}Y \\ &+ D(Y,Z)A_{L}X - D(X,Z)A_{L}Y \\ &+ (\bar{\nabla}_X\theta)(Z)FY - (\bar{\nabla}_Y\theta)(Z)FX \\ &+ \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \bar{g}(JY,Z)FX \\ &- \bar{g}(JX,Z)FY + \; 2\bar{g}(X,JY)FZ\}.\end{aligned} \tag{5.5}$$

Taking the scalar product with N to (5.4) and then, substituting (5.5) into the resulting equation and using (3.2) and (3.8), we obtain

$$\begin{aligned}&(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad - \sigma(X)C(Y,PZ) + \sigma(Y)C(X,PZ) \\ &\quad - \theta(X)C(FY,PZ) + \theta(Y)C(FX,PZ) \\ &\quad + a\{v(X)B(Y,PZ) - v(Y)B(X,PZ)\} \\ &\quad - \{\rho(X) - ev(X) - aw(X)\}D(Y,PZ) \\ &\quad + \{\rho(Y) - ev(Y) - aw(Y)\}D(X,PZ) \\ &\quad - (\bar{\nabla}_X\theta)(PZ)v(Y) + (\bar{\nabla}_Y\theta)(PZ)v(X)\end{aligned}$$

$$= \frac{c}{4}\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ) + v(X)g(FY,PZ) - v(Y)g(FX,PZ) + 2v(PZ)\bar{g}(X,JY)\}. \tag{5.6}$$

Theorem 5.2. *Let M be a half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-symmetric connection. If one of the following four statements is satisfied;*

(1) *F is recurrent,*
(2) *F is Lie recurrent,*
(3) *U is parallel with respect to the connection ∇,*
(4) *V is parallel with respect to the connection ∇,*

then $\bar{M}(c)$ is flat, i.e., $c = 0$. In case (1), σ satisfies $d\sigma = 0$.

Proof. (1) Applying $\bar{\nabla}_X$ to $(4.9)_1$: $\theta(U) = 0$ and using (2.3), $(4.11)_3$, $(4.12)_1$ and the fact that $\theta(N) = a$, we obtain

$$(\bar{\nabla}_X\theta)(U) = -aB(X,U). \tag{5.7}$$

Applying ∇_X to $(4.9)_2$: $C(Y,U) = 0$ and using $(4.13)_1$, we obtain

$$(\nabla_X C)(Y,U) = 0.$$

Taking $Z = U$ to (5.6) and using $(4.11)_3$ and the last two equations, we get

$$\frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this, we have $c = 0$. Thus $\bar{M}(c)$ is flat.

By directed calculation from $(4.13)_1$: $\nabla_X U = \sigma(X)U$, we obtain

$$R(X,Y)U = 2d\sigma(X,Y)U.$$

On the other hand, by using $(4.9)_1$ and $(4.11)_3$, Eq.(4.7) reduces

$$A_N X = B(X,U)U - aFX.$$

Replacing Z by U to (5.5) and using $(4.11)_3$, (5.7) and the last equation, we get $R(X,Y)U = 0$. Therefore, we obtain $d\sigma = 0$.

(2) Using the Gauss-Weingarten formulae (2.6) and (2.7) for the screen distribution $S(TM)$, we have the following Codazzi equation for $S(TM)$:

$$\begin{aligned} R(X,Y)\xi = &-\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] \\ &- \sigma(X)A^*_\xi Y + \sigma(Y)A^*_\xi X \\ &+ \{C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\sigma(X,Y)\}\xi. \end{aligned} \tag{5.8}$$

Applying $\bar{\nabla}_X$ to $\theta(\xi) = b$ and using (2.7), (3.7) and $\sigma = -bv$, we get

$$\begin{aligned} (\bar{\nabla}_X\theta)(\xi) = &Xb + \theta(A^*_\xi X) - b^2v(X) - abu(X) \\ &+ e\{\lambda(X) - eu(X) - bw(X)\}. \end{aligned} \tag{5.9}$$

Taking the scaler product with N to (5.5) with $Z = \xi$ and then, comparing this result with the radical component of (5.8), we obtain

$$\begin{aligned} & C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\sigma(X, Y) \\ = & \frac{c}{4}\{u(Y)v(X) - u(X)v(Y)\} \\ & + \{\lambda(X) - eu(X) - bw(X)\}\{\rho(Y) - aw(Y)\} \\ & - \{\lambda(Y) - eu(Y) - bw(Y)\}\{\rho(X) - aw(X)\} \\ & + (Xb)v(Y) - (Yb)v(X) + \theta(A^*_\xi X)v(Y) - \theta(A^*_\xi Y)v(X), \end{aligned}$$

due to (3.7), (3.8) and (5.9). Taking $X = U$ and $Y = V$ to the last equation and using (4.24) and the item (3) in Theorem 4.2, we obtain

$$2d\sigma(U, V) = \frac{c}{4} - Ub. \tag{5.10}$$

On the other hand, in general, applying ∇_X to $v(Y) = g(Y, U)$ and using (2.11), (3.1), (3.5), $(3.6)_5$, $(3.8)_1$, $(3.15)_1$ and (3.16), we have

$$\begin{aligned} (\nabla_X v)Y \;=\; & v(Y)\tau(X) + w(Y)\rho(X) + \theta(Y)\eta(X) \\ & -g(A_N X, FY) - a\{g(X, Y) - u(Y)v(X)\}. \end{aligned}$$

By directed calculation from $\sigma(X) = -bv(X)$ and by using (3.2), we derive

$$\begin{aligned} 2d\sigma(X, Y) \;=\; & -(Xb)v(Y) + (Yb)v(X) + ab\{u(X)v(Y) - u(Y)v(X)\} \\ & +b\{v(X)\tau(Y) - v(Y)\tau(X) + w(X)\rho(Y) - w(Y)\rho(X) \\ & +g(A_N X, FY) - g(A_N Y, FX)\}. \end{aligned}$$

Taking $X = U$ and $Y = V$ to this equation and using $(4.25)_1$, we have

$$2d\sigma(U, V) = -Ub.$$

Comparing this result with (5.10), we obtain $c = 0$.

(3) Assume that $\nabla_X U = 0$. Taking the scalar product with U to (3.16) and using $(3.8)_1$, we obtain $\theta(U)\eta(X) = 0$. It follows that

$$\theta(U) = 0.$$

Applying $\bar{\nabla}_X$ to $\theta(U) = 0$ and using (2.3) and the fact $\nabla_X U = 0$, we get

$$(\bar{\nabla}_X \theta)(U) = -aB(X, U) - eD(X, U). \tag{5.11}$$

Taking the scalar product with W and N to (3.16) and using (3.13), we have

$$\rho(X) = aw(X), \quad C(X, U) = 0, \tag{5.12}$$

respectively. Applying ∇_Y to $(5.12)_2$ and the fact $\nabla_Y U = 0$, we obtain

$$(\nabla_X C)(Y, U) = 0. \tag{5.13}$$

Taking $PZ = U$ to (5.6) and using (5.11), $(5.12)_{1,2}$ and (5.13), we have

$$\frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this equation, we obtain $c = 0$.

(4) Assume that $\nabla_X V = 0$. Taking the scalar product with W and N to (3.17) by turns and using (3.4), (3.11) and $(3.15)_1$, we obtain

$$D(X, \xi) = bw(X), \quad D(\xi, X) = 0, \quad C(X, V) = 0. \tag{5.14}$$

Taking $X = U$ and $Y = W$ to (3.3), we obtain

$$B(U, W) - \theta(W) = B(W, U).$$

Replacing X by U to $(3.15)_3$ and using the last equation, $(3.15)_1$ and $(5.14)_3$, we see that $D(U, V) = B(U, W) - \theta(W) = B(W, U) = C(W, V) = 0$. Thus

$$D(U, V) = 0. \tag{5.15}$$

Applying ∇_X to $(5.14)_3$ and using the fact that $\nabla_X V = 0$, we have

$$(\nabla_X C)(Y, V) = 0.$$

Taking $PZ = V$ to (5.6) and using $(5.14)_3$ and the last equation, we get

$$\begin{aligned}
&a\{v(X)B(Y, V) - v(Y)B(X, V)\}\\
&- \{\rho(X) - ev(X) - aw(X)\}D(Y, V)\\
&+ \{\rho(Y) - ev(Y) - aw(Y)\}D(X, V)\\
&- (\bar{\nabla}_X\theta)(V)v(Y) + (\bar{\nabla}_Y\theta)(V)v(X)\\
&= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.
\end{aligned}$$

Taking $X = \xi$ and $Y = U$ and using $(5.14)_2$ and (5.15), we get $c = 0$. □

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