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GENERALIZED HYERS–ULAM STABILITY OF A FUNCTIONAL EQUATION OF HOSSZÚ TYPE

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Abstract. The aim of the present paper is to give general solutions of the following functional equation of Hosszú type:

$$f(x - y + xy) + f(y) = f(x) + f(xy)$$

and its Pexiderized version

$$f(x - y + xy) + g(y) = h(x) + k(xy),$$

and prove the Hyers-Ulam stability of the above two functional equations in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [12] concerning the stability of group homomorphisms. The functional

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equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y-xy) + f(xy) = f(x) + f(y) \tag{1.1}$$

was considered first by Hosszú who has solved it under a differentiability assumption. It has also been treated by many authors ([1, 2, 8, 10, 11]).

In this paper, we solve the following functional equation of Hosszú type

$$f(x-y+xy) + f(y) = f(x) + f(xy) \tag{1.2}$$

and its Pexiderized version

$$f(x-y+xy) + g(y) = h(x) + k(xy), \tag{1.3}$$

and prove the Hyers-Ulam stability of the functional equations (1.2) and (1.3) in Banach spaces.

2. General solutions of (1.2) and (1.3) on $\mathbb{R}$

In this section, $\mathbb{X}$ denotes a linear space.

Theorem 2.1. *A mapping $f : \mathbb{R} \to \mathbb{X}$ satisfies* (1.2) *for all $x, y \in \mathbb{R}$ if and only if f has the form $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b \in \mathbb{R}$ is a constant.*

Proof. Let $f : \mathbb{R} \to \mathbb{X}$ satisfy (1.2). Setting $y = 1$ and $y = -1$, respectively, in (1.2), we get

$$f(2x-1) + f(1) = 2f(x) \tag{2.1}$$

and

$$f(x) + f(-x) = f(1) + f(-1). \tag{2.2}$$

Letting $x = 0$ in (2.1) and using (2.2), we get

$$f(x) + f(-x) = 2f(0), \quad (x \in \mathbb{R}). \tag{2.3}$$

Letting $x = 2$ and $x = -1$ in (2.1) and using (2.3), we get

$$f(3) + f(1) = 2f(2), \quad 3f(1) - f(3) = 2f(0), \quad (x \in \mathbb{R}).$$

Hence

$$f(2) + f(0) = 2f(1),\ (x \in \mathbb{R}). \tag{2.4}$$

Replacing y by $-y$ in (1.2), we obtain

$$f(x + y - xy) + f(-y) = f(x) + f(-xy),\ (x, y \in \mathbb{R}). \tag{2.5}$$

Adding (1.2) to (2.5) and using (2.2), we obtain

$$f(x + y - xy) + f(x - y + xy) = 2f(x),\ (x, y \in \mathbb{R}). \tag{2.6}$$

Letting $y = \frac{x}{x-1}$ in (2.6), we have

$$f(0) + f(2x) = 2f(x),\ (x \in \mathbb{R} \setminus \{1\}). \tag{2.7}$$

It follows from (2.4) that (2.7) holds for each $x \in \mathbb{R}$. By (2.6) and (2.7), we obtain that

$$f(x + y - xy) + f(x - y + xy) = f(2x) + f(0),\ (x, y \in \mathbb{R}). \tag{2.8}$$

Let $u, v \in \mathbb{R}$ with $u + v \neq 2$. We can find $x, y \in \mathbb{R}$ such that $x + y - xy = u$ and $x - y + xy = v$. Hence (2.8) implies that

$$f(u) + f(v) = f(u + v) + f(0). \tag{2.9}$$

We prove that (2.9) holds when $u + v = 2$. For this, letting $x = 2$ in (1.2) and using (2.7), we get

$$f(2 + y) - f(y) = f(2) - f(0),\ (y \in \mathbb{R}). \tag{2.10}$$

Replacing y by $-y$ in (2.10) and using (2.3), we obtain

$$f(2 - y) + f(y) = f(2) + f(0),\ (y \in \mathbb{R}).$$

Hence (2.9) holds for all $u, v \in \mathbb{R}$ and this shows $f - f(0)$ is additive. The converse is obvious. This completes the proof. □

Theorem 2.2. *Mappings* $f, g, h, k : \mathbb{R} \to \mathbb{X}$ *satisfy* (1.3) *for all* $x, y \in \mathbb{R}$ *if and only if they have the form* $f(x) = A(x) + b_1, g(x) = A(x) + b_2, h(x) = A(x) + b_3, k(x) = A(x) + b_4$, *where* $A : \mathbb{R} \to \mathbb{X}$ *is additive and* $b_1, b_2, b_3, b_4 \in \mathbb{R}$ *are constants with* $b_1 + b_2 = b_3 + b_4$.

Proof. Let $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy (1.3). Setting $x = 0, y = 0$ and $x = 1$, respectively, in (1.3), we get

$$f(-y) + g(y) = h(0) + k(0), \tag{2.11}$$

$$f(x) + g(0) = h(x) + k(0), \tag{2.12}$$

$$f(1) + g(y) = h(1) + k(y). \tag{2.13}$$

Using (2.11), (2.12), (2.13) and (1.3), we have

$$\begin{aligned} &f(x - y + xy) - f(-y) + h(0) + k(0) \\ &= f(x) + g(xy) + f(1) + g(0) - h(1) - k(0),\ (x, y \in \mathbb{R}). \end{aligned} \tag{2.14}$$

Using (2.11) and (2.14), we get

$$\begin{aligned} &f(x-y+xy)-f(-y)+h(1)+k(0)\\ &=f(x)-f(-xy)+f(1)+g(0) \end{aligned} \tag{2.15}$$

for all $x, y \in \mathbb{R}$. It follows from (2.13) that $f(1)+g(0) = h(1)+k(0)$. Therefore, (2.15) implies that

$$f(x-y+xy)-f(-y)=f(x)-f(-xy),\ (x,y\in\mathbb{R}). \tag{2.16}$$

Replacing y by $-y$ in (2.16), we have

$$f(x+y-xy)+f(xy)=f(x)+f(y),\ (x,y\in\mathbb{R}), \tag{2.17}$$

which is the Hosszú's functional equation. Hence $f - f(0)$ is additive (see [2, 3]). Now using (2.11), (2.12) and (2.13), we also get the assertion for g, h and k. The converse is obvious. This completes the proof. □

Theorem 2.3. *A mapping* $f : \mathbb{R} \to \mathbb{X}$ *satisfies* (1.2) *if and only if* f *satisfies* (1.1).

Proof. Let f satisfy (1.2). By Theorem 2.1 f has the form $f(x) = A(x) + b$ where $A : \mathbb{R} \to \mathbb{X}$ is additive and $b \in \mathbb{R}$ is a constant. Therefore f satisfies (1.1). If f satisfy (1.1), $f-f(0)$ is additive (see [2, 3]). So f satisfies (1.2). □

Remark 2.4. Using the proofs of Theorems 2.1 and 2.2, the results of Theorems 2.1 and 2.2 can be extended to mappings $f, g, h, k : \mathbb{K} \to G$, where $\mathbb{K}$ is a commutative field of characteristic different from 2, and G is an abelian group.

3. Generalized Hyers-Ulam stability

In this section, we examine generalized Hyers-Ulam stability of functional equations (1.2) and (1.3). Let $\mathbb{X}$ be a Banach space.

Theorem 3.1. *Let* $\varphi : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ *be a function such that*

$$\begin{aligned} \sum_{n\in D_x}\frac{1}{2^n}\Big[\varphi\Big(2^nx,\frac{2^ny}{2^nx-1}\Big)+\varphi\Big(2^nx,-\frac{2^ny}{2^nx-1}\Big)+\varphi\Big(\frac{2^ny}{2^nx-1},-1\Big)\\ +\varphi\Big(\frac{4^nxy}{2^nx-1},-1\Big)\Big]<\infty, \end{aligned} \tag{3.1}$$

where $D_x := \{n \in \mathbb{N} \cup \{0\} : 2^n x \neq 1\}$. *If a mapping* $f : \mathbb{R} \to \mathbb{X}$ *satisfies the inequality*

$$\|f(x-y+xy)+f(y)-f(x)-f(xy)\|\leqslant\varphi(x,y) \tag{3.2}$$

for all $x, y \in \mathbb{R}$, then there exists a unique additive mapping $A : \mathbb{R} \to \mathbb{X}$ such that

$$\|f(x)-A(x)-f(0)\| \leqslant \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\psi(2^i x)}{2^{i+1}}, & \text{if } x \in \mathbb{R} \setminus \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}; \\[2ex] \displaystyle\sum_{\substack{i=0,\\ i\neq k}}^{\infty} \frac{\psi(2^i x)}{2^{i+1}} + \frac{\delta}{2^{k+1}}, & \text{if } 2^k x = 1, \end{cases} \tag{3.3}$$

where

$$\psi(x) := \varphi\Big(x, \frac{x}{x-1}\Big) + \varphi\Big(x, -\frac{x}{x-1}\Big) + \varphi\Big(\frac{x}{x-1}, -1\Big) + \varphi\Big(\frac{x^2}{x-1}, -1\Big),$$
$$\delta := \frac{1}{2}\Big[\varphi(2,1) + \varphi(0,1) + \varphi(-1,1) + \varphi(3,-1) + \varphi(1,-1)\Big].$$

Proof. Setting $y = 1$ and $y = -1$, respectively, in (3.2), we get

$$\|f(2x-1) + f(1) - 2f(x)\| \leqslant \varphi(x,1) \tag{3.4}$$

and

$$\|f(x) + f(-x) - f(1) - f(-1)\| \leqslant \varphi(x,-1). \tag{3.5}$$

Letting $x = 0$ in (3.4) and using (3.5), we get

$$\|f(x) + f(-x) - 2f(0)\| \leqslant \varphi(0,1) + \varphi(x,-1), \ (x \in \mathbb{R}). \tag{3.6}$$

Replacing y by $-y$ in (3.2), we obtain

$$\|f(x+y-xy) + f(-y) - f(x) - f(-xy)\| \leqslant \varphi(x,-y), \ (x,y \in \mathbb{R}). \tag{3.7}$$

Adding (3.2) to (3.7) and using (3.5), we obtain

$$\|f(x+y-xy) + f(x-y+xy) - 2f(x)\| \leqslant \Phi(x,y), \ (x,y \in \mathbb{R}), \tag{3.8}$$

where $\Phi(x,y) := \varphi(x,y) + \varphi(x,-y) + \varphi(y,-1) + \varphi(xy,-1)$. Letting $y = \frac{x}{x-1}$ in (3.8), we have

$$\|f(2x) - 2f(x) + f(0)\| \leqslant \psi(x), \ (x \in \mathbb{R} \setminus \{1\}). \tag{3.9}$$

Letting $x = 2$ and $x = -1$ in (3.4) and using (3.5) with (3.6), we get

$$\begin{aligned} &\|f(2) - 2f(1) + f(0)\| \\ &\leqslant \frac{1}{2}\Big[\varphi(2,1) + \varphi(0,1) + \varphi(-1,1) + \varphi(3,-1) + \varphi(1,-1)\Big]. \end{aligned} \tag{3.10}$$

Let $x \in \mathbb{R}$. There exists $N \in \mathbb{N}$ such that $2^n x \neq 1$ for all $n \geqslant N$. Replacing x by $2^n x$ in (3.9) and dividing by 2^{n+1}, we have

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n} + \frac{f(0)}{2^{n+1}}\Big\| \leqslant \frac{\psi(2^n x)}{2^{n+1}}, \ (n \geqslant N).$$

Therefore

$$\begin{aligned}&\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^m x)}{2^m}+\sum_{i=m}^{n}\frac{f(0)}{2^{i+1}}\Big\|\\&=\Big\|\sum_{i=m}^{n}\Big[\frac{f(2^{i+1}x)}{2^{i+1}}-\frac{f(2^i x)}{2^i}+\frac{f(0)}{2^{i+1}}\Big]\Big\|\\&\leqslant\sum_{i=m}^{n}\frac{\psi(2^i x)}{2^{i+1}},\ (m,n\geqslant N).\end{aligned}\tag{3.11}$$

This implies that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy. Let us define $A:\mathbb{R}\to\mathbb{X}$ by

$$A(x):=\lim_{n\to\infty}\frac{f(2^n x)}{2^n},\ (x\in\mathbb{R}).$$

If $x\in\mathbb{R}\setminus\{\frac{1}{2^n}:n\in\mathbb{N}\cup\{0\}\}$, then (3.11) holds for all $n\geqslant m\geqslant 0$. Letting $m=0$ and allowing $n\to\infty$ in (3.11), we get (3.3). If $2^k x=1$ for some $k\in\mathbb{N}\cup\{0\}$, then (3.9) implies

$$\begin{aligned}\Big\|\frac{f(2^k x)}{2^k}-f(x)+\sum_{i=0}^{k-1}\frac{f(0)}{2^{i+1}}\Big\|&=\Big\|\sum_{i=0}^{k-1}\Big[\frac{f(2^{i+1}x)}{2^{i+1}}-\frac{f(2^i x)}{2^i}+\frac{f(0)}{2^{i+1}}\Big]\Big\|\\&\leqslant\sum_{i=0}^{k-1}\frac{\psi(2^i x)}{2^{i+1}},\end{aligned}\tag{3.12}$$

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^{k+1}x)}{2^{k+1}}+\sum_{i=k+1}^{n}\frac{f(0)}{2^{i+1}}\Big\|\leqslant\sum_{i=k+1}^{n}\frac{\psi(2^i x)}{2^{i+1}}.$$

It follows from (3.10) that

$$\Big\|\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}+\frac{f(0)}{2^{k+1}}\Big\|\leqslant\frac{\delta}{2^{k+1}},\tag{3.13}$$

where $\delta:=\frac{1}{2}\Big[\varphi(2,1)+\varphi(0,1)+\varphi(-1,1)+\varphi(3,-1)+\varphi(1,-1)\Big]$. Hence we have from (3.12) and (3.13),

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-f(x)+\sum_{i=0}^{n}\frac{f(0)}{2^{i+1}}\Big\|\leqslant\sum_{i=0,i\neq k}^{n}\frac{\psi(2^i x)}{2^{i+1}}+\frac{\delta}{2^{k+1}}.\tag{3.14}$$

Allowing $n\to\infty$ in (3.14), we get (3.3). We now show that A is additive. Let $u,v\in\mathbb{R}$ with $u+v\neq 1$. We can find $x,y\in\mathbb{R}$ such that $x+y-xy=2u$ and $x-y+xy=2v$. Hence (3.8) implies that

$$\|f(2u)+f(2v)-2f(u+v)\|\leqslant\Phi\Big(u+v,\frac{u-v}{1-u-v}\Big).\tag{3.15}$$

Let $x, y \in \mathbb{R}$. There exists $N \in \mathbb{N}$ such that $2^n(x+y) \neq 1$ for all $n \geqslant N$. Hence (3.15) implies

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}+\frac{f(2^{n+1}y)}{2^{n+1}}-\frac{f(2^n(x+y))}{2^n}\Big\| \leqslant \frac{1}{2^{n+1}}\Phi\Big(2^nx+2^ny, \frac{2^nx-2^ny}{1-2^nx-2^ny}\Big),$$

for all $n \geqslant N$. Allowing $n \to \infty$, we see that $A(x)+A(y) = A(x+y)$. Therefore A is additive. The uniqueness of A follows from (3.3). □

Corollary 3.2. *Let δ be a fixed positive real number. If a mapping $f : \mathbb{R} \to \mathbb{X}$ satisfies the inequality*

$$\|f(x-y+xy)+f(y)-f(x)-f(xy)\| \leqslant \delta \tag{3.16}$$

for all $x, y \in \mathbb{R}$, then there exists a unique additive mapping $A : \mathbb{R} \to \mathbb{X}$ such that

$$\|f(x)-A(x)-f(0)\| \leqslant 4\delta,\ x \in \mathbb{R}. \tag{3.17}$$

Proof. Setting $\varphi(x, y) = \delta$, It follows from (3.9) and (3.10) that

$$\|f(2x)-2f(x)+f(0)\| \leqslant 4\delta,\ (x \in \mathbb{R}). \tag{3.18}$$

Let $x \in \mathbb{R}$. Replacing x by 2^nx in (3.18) and dividing by 2^{n+1}, we have

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n}+\frac{f(0)}{2^{n+1}}\Big\| \leqslant \frac{2\delta}{2^n},\ (n \in \mathbb{N}\cup\{0\}).$$

Therefore

$$\begin{aligned}
&\Big\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^mx)}{2^m}+\sum_{i=m}^{n}\frac{f(0)}{2^{i+1}}\Big\| \\
&=\Big\|\sum_{i=m}^{n}\Big[\frac{f(2^{i+1}x)}{2^{i+1}}-\frac{f(2^ix)}{2^i}+\frac{f(0)}{2^{i+1}}\Big]\Big\| \\
&\leqslant \sum_{i=m}^{n}\frac{2\delta}{2^i},\ (m, n \geqslant N).
\end{aligned} \tag{3.19}$$

This implies that the sequence $\{\frac{f(2^nx)}{2^n}\}$ is Cauchy. Let us define $A : \mathbb{R} \to \mathbb{X}$ by

$$A(x) := \lim_{n\to\infty}\frac{f(2^nx)}{2^n},\ (x \in \mathbb{R}).$$

Letting $m = 0$ and allowing $n \to \infty$ in (3.19), we get (3.17). The proof of additivity and uniqueness of A is similar to the proof in Theorem 3.1. □

Theorem 3.3. *Let δ be a fixed positive real number. If mappings $f, g, h, k : \mathbb{R} \to \mathbb{X}$ satisfy*

$$\|f(x-y+xy)+g(y)-h(x)-k(xy)\| \leqslant \delta \tag{3.20}$$

for all $x, y \in \mathbb{R}$*, then there exists a unique additive mapping* $A : \mathbb{R} \to \mathbb{X}$ *such that*

$$\|f(x) - A(x) + A(1) - f(1)\| \leqslant 24\delta, \tag{3.21}$$

$$\|g(x) - A(x) - A(1) - g(-1)\| \leqslant 26\delta, \tag{3.22}$$

$$\|h(x) - A(x) + A(1) - h(1)\| \leqslant 26\delta, \tag{3.23}$$

$$\|k(x) - A(x) - A(1) - k(-1)\| \leqslant 28\delta \tag{3.24}$$

for all $x \in \mathbb{R}$*.*

Proof. Setting $x = 0, y = 0$ and $x = 1$, respectively, in (3.20), we get

$$\|f(-y) + g(y) - h(0) - k(0)\| \leqslant \delta, \tag{3.25}$$

$$\|f(x) - h(x) + g(0) - k(0)\| \leqslant \delta, \tag{3.26}$$

$$\|g(y) - k(y) + f(1) - h(1)\| \leqslant \delta. \tag{3.27}$$

Using (3.25), (3.26), (3.27) and (3.20), we have

$$\begin{aligned} &\|f(x - y + xy) - f(x) - f(-y) - g(xy) \\ &\quad - f(1) - g(0) + h(0) + h(1) + 2k(0)\| \leqslant 4\delta, \ (x, y \in \mathbb{R}). \end{aligned} \tag{3.28}$$

Using (3.25) and (3.28), we get

$$\begin{aligned} &\|f(x - y + xy) - f(x) - f(-y) + f(-xy) \\ &\quad - f(1) - g(0) + h(1) + k(0)\| \leqslant 5\delta, \end{aligned} \tag{3.29}$$

for all $x, y \in \mathbb{R}$. It follows from (3.27) that

$$\|f(1) + g(0) - h(1) - k(0)\| \leqslant \delta.$$

Therefore (3.29) implies that

$$\|f(x - y + xy) - f(x) - f(-y) + f(-xy)\| \leqslant 6\delta, \ (x, y \in \mathbb{R}). \tag{3.30}$$

Replacing y by $-y$ in (3.30), we have

$$\|f(x + y - xy) - f(x) - f(y) + f(xy)\| \leqslant 6\delta, \ (x, y \in \mathbb{R}).$$

Hence by a result of [13] (see also [6, 8]), there exists a unique additive mapping $A : \mathbb{R} \to \mathbb{X}$ such that

$$\|f(x) - A(x) + A(1) - f(1)\| \leqslant 24\delta, \ x \in \mathbb{R}. \tag{3.31}$$

Now using (3.25) and (3.31), we have

$$\|A(x) - g(x) + A(1) - f(1) + h(0) + k(0)\| \leqslant 25\delta, \ x \in \mathbb{R}.$$

Using again (3.25) (by letting $y = -1$), we conclude (3.22).

Similarly, using (3.21), (3.22), (3.26) and (3.27), we obtain (3.23) and (3.24). This completes the proof. □

Here we use the Gajda's example [4] to give the next result.

Example 3.4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{for } x \geqslant 1; \\ -1 & \text{for } x \leqslant -1. \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x).$$

Then f is continuous and satisfies

$$|f(x - y + xy) + f(y) - f(x) - f(xy)| \leqslant 16(|x| + |y|) \tag{3.32}$$

for all $x, y \in \mathbb{R}$, and the range of $|f(x) - A(x)|/|x|$ for $x \neq 0$ is unbounded for each additive function $A : \mathbb{R} \to \mathbb{R}$.

Proof. It is clear that f is continuous and bounded by 2 on $\mathbb{R}$. If $|x| + |y| = 0$ or $|x| + |y| \geqslant \frac{1}{2}$, then

$$|f(x - y + xy) + f(y) - f(x) - f(xy)| \leqslant 8 \leqslant 16(|x| + |y|).$$

Now suppose that $0 < |x| + |y| < \frac{1}{2}$. Then there exists an integer $k \geqslant 1$ such that

$$\frac{1}{2^{k+1}} \leqslant |x| + |y| < \frac{1}{2^k}. \tag{3.33}$$

Therefore,

$$2^m|x - y + xy|, \;\; 2^m|x|, \;\; 2^m|y|, \;\; 2^m|xy| < 1$$

for all $m = 0, 1, \cdots, k - 1$. From the definition of f and (3.33), we have

$$\begin{aligned} &|f(x - y + xy) + f(y) - f(x) - f(xy)| \\ &\leqslant \sum_{n=k}^{\infty} 2^{-n} \Big[|\phi(2^n(x - y + xy))| + |\phi(2^n y)| + |\phi(2^n x)| + |\phi(2^n xy)| \Big] \\ &\leqslant \frac{8}{2^k} \leqslant 16(|x| + |y|). \end{aligned}$$

Thus f satisfies (3.32).

Let $A : \mathbb{R} \to \mathbb{R}$ be an additive function such that

$$|f(x) - A(x)| \leqslant \beta|x|$$

for all $x \in \mathbb{R}$, where $\beta > 0$ is a constant. Since A is additive, there exists a constant $c \in \mathbb{R}$ such that $A(x) = cx$ for all rational numbers x. So we have

$$|f(x)| \leqslant (\beta + |c|)|x| \tag{3.34}$$

for all rational numbers x. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If x is a rational number in $(0, 2^{1-m})$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \cdots, m-1$. So

$$f(x) \geqslant \sum_{n=0}^{m-1} 2^{-n} \phi(2^n x) = mx > (\beta + |c|)x$$

which contradicts (3.34). □

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