Nonlinear Functional Analysis and Applications
Vol. 23, No. 1 (2018), pp. 167-179
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME FORMULAS FOR BESSEL AND HYPER-BESSEL FUNCTIONS RELATED TO THE PROPER LORENTZ GROUP

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Abstract. In a representation space of the proper Lorentz group, we consider the so-called spherical and two parabolic bases and compute the matrix elements of restriction of the representation to matrix $\mathrm{diag}(1, 1, 1, -1)$ with respect to one of the above parabolic basis in the following three particular cases: matrix elements belong to 'zero row'; lie on the 'main diagonal'; lie on the 'anti-diagonal'. Taking the relations between above bases, we give a group theoretical treatment of one known formula and derive two new formulas for series involving modified hyper-Bessel functions of the first kind, which converge to products of (usual) cylinder functions. Some results here are pointed out to be able to be rewritten in terms of Bessel-Clifford functions.

1. Introduction and preliminaries

Various generalizations of certain special functions have been introduced and investigated in solving some problems associated with diverse research

[0]Received August 4, 2017. Revised January 26, 2018.
[0]2010 Mathematics Subject Classification: 33C10, 33C80, 33B15, 33C05.
[0]Keywords: Representation space, proper Lorentz group, spherical basis, parabolic bases, cylinder functions, modified hyper-Bessel functions of the first kind, Bessel-Clifford functions.
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subjects such as applied mathematics, nuclear physics, statistics and engineering. For example, we take the frequently used basic and modified Bessel functions. In particular, with a view to describing exact solution of problem of reflection and diffraction of atomic of Broglie waves, Witte [29] introduced the following so-called hyper-Bessel basic function of the first kind:

$$\begin{aligned} {}_0f_3(a,b,c;z) &:= \{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\}^{-1}\,{}_0F_3(a,b,c;z) \\ &= G^{1,0}_{0,4}\left(-z \,\middle|\, \begin{array}{c} \overline{} \\ 0,\,1-a,\,1-b,\,1-c \end{array}\right), \end{aligned}$$

where Γ is the familiar Gamma function (see, e.g., [26]), ${}_pF_q$ are the generalized hypergeometric functions (see, e.g., [21]), and $G^{m,n}_{p,q}$ is the Meijer G-function (see, e.g., [27]). Delerue [7] defined the modified hyper-Bessel functions of the first kind as follows: For $n \in \mathbb{N}$,

$$I_{\nu_1,\ldots,\nu_n}(z) := \frac{\left(\frac{z}{n+1}\right)^{\nu_1+\ldots+\nu_n}}{\prod\limits_{i=1}^{n}\Gamma(\nu_i+1)}\,{}_0F_n\left[-;\nu_1+1,\,\ldots,\,\nu_n+1;\,\left(\frac{z}{n+1}\right)^{n+1}\right],$$

which were re-derived, three decades later, by Klyuchantsev [16] and have found certain interesting applications in (for example) quantum mechanics [15], fractional calculus [14], and theory of Mittag-Leffler functions [11]. Also certain properties of the function $I_{\nu_1,\ldots,\nu_n}$ have been investigated (see [2, 18]). Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\}$.

Most of special functions involving in mathematical physics and their analogues arise in matrix elements of group representations and matrix elements of non-degenerate linear transformations of a representation space. For instance, let G be a proper Lorentz group. The left quasi-regular representation T of this group can be realised in the space $\mathfrak{D}$ of σ-homogeneous infinitely differentiable functions defined on the cone

$$\Lambda:\; x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0; \tag{1.1}$$

in this case T maps an element g to the linear operator which 'shifts' a function f according to the rule $f(x) \mapsto f(g^{-1}x)$ (see [17]). Shilin and Choi [22] expressed the matrix elements of the linear operator mapping the so-called spherical basis $B_1 := \{f_{p_1,q_1} \mid p_1 \in \mathbb{N}_0,\, q_1 \in \mathbb{Z},\, |q_1| \leq p_1\}$ of $\mathfrak{D}$, where

$$f_{p_1,q_1}(x) := x_1^{\sigma-|q_1|}C^{|q_1|+\frac{1}{2}}_{p_1-|q_1|}\left(\frac{x_4}{x_1}\right)(x_3 + \mathbf{i}x_2\,\mathrm{sign}\,q_1)^{|q_1|},$$

into the hyperbolic basis in terms of ${}_4F_3$-hypergeometric functions and evaluated some integrals involving Legendre functions in terms of ${}_4F_3$. For the three-dimensional analogue of G, they [4, 23] have considered restrictions of the matrix elements of T to some diagonal and block-diagonal matrices and obtained exact formulas for them with respect to the so-called parabolic basis. From these formulas, they [4, 23] have evaluated certain integrals involving products of cylinder and Whittaker functions and some linear combinations of cylinder and Struve functions. They [24] showed that, for $a := \operatorname{diag}(1, 1, 1 -1)$, the matrix elements $t_{p_2,q_2,\hat{p}_2,\hat{q}_2}(a)$ of the operator $T(a)$ with respect to the *parabolic* basis $B_2 = \{f_{p_2,q_2} \mid p_2 \geq 0,\ q_2 \in \mathbb{Z}\}$, where

$$f^*_{p_2,q_2}(x) := (x_1 + x_4)^\sigma (x_2^2 + x_3^2)^{-\frac{|q_2|}{2}} J_{|q_2|}\left(\frac{p_2\sqrt{x_2^2 + x_3^2}}{x_1 + x_4}\right)(x_2 + \mathbf{i}x_2 \operatorname{sign} q_2)^{|q_2|},$$

can be written in terms of the modified hyper-Bessel functions:

$$\begin{aligned} t_{p_2,q_2,\hat{p}_2,\hat{q}_2}(a) = {} & \frac{\pi}{2}\frac{\delta_{q_2,\hat{q}_2}\,\hat{p}_2}{\sin(\pi\sigma)}\left(\frac{p_2}{\hat{p}_2}\right)^{\sigma+2} \\ & \times \left[I_{|q_2|,\sigma+2,|q_2|+\sigma+2}\left(2\sqrt{p_2\hat{p}_2}\right) - I_{|q_2|,-\sigma-2,|q_2|-\sigma-2}\left(2\sqrt{p_2\hat{p}_2}\right)\right]. \end{aligned} \tag{1.2}$$

In addition to $\mathfrak{D}$ and B_2, in the following sections, we aim to consider the space $\mathfrak{D}^\bullet$, where $\sigma^\bullet = -\sigma - 2$ and the *second parabolic basis*, which will be introduced.

2. Some matrix elements of the operator $T^\bullet(a)$ computed with respect to the second parabolic basis

Consider the parabolic section $\gamma:\ x_1 + x_4 = 1$ on the cone Λ in (1.1), which can be parameterized as follows:

$$\gamma = \left\{\left(\frac{1 + \alpha_1^2 - \beta_1^2}{2},\ \alpha_1, \beta_1,\ \frac{1 - \alpha_1^2 + \beta_1^2}{2}\right)\ \Big|\ \alpha_1, \beta_1 \in \mathbb{R}\right\}.$$

For arbitrary s, $t \in \mathbb{R}$, we define the matrix

$$g(s,t) := \begin{pmatrix} 1 + \frac{s^2-t^2}{2} & s & t & \frac{s^2-t^2}{2} \\ s & 1 & 0 & s \\ t & 0 & 1 & t \\ \frac{t^2-s^2}{2} & -s & -t & 1 + \frac{t^2-s^2}{2} \end{pmatrix}.$$

It is easy to show that $g(s,t) \in G$, the subset H of matrices $g(s,t)$ in G is a subgroup with $g^{-1}(s,t) = g(-s,-t)$. Since γ is a H-orbit of the point $\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, this section is a homogeneous space of H, that is, for any $x(p_1, q_1) \in \gamma$ and $\hat{x}(\hat{p}_1, \hat{q}_1) \in \gamma$, we have $g(\hat{p}_1 - p_1, \hat{q}_1 - q_1)(x) = \hat{x}$. The functions $F(p_3, q_3) := \exp\frac{(p_3x_2 + q_3x_3)\mathbf{i}}{x_1 + x_4}$, defined on γ, are eigenfunctions of the operator $T(g(s,t))$ with respect to the eigenvalue $\exp(-[p_3 s + q_3 t]\mathbf{i})$. Since each point $y \in \Lambda$ can be represented in the form $y = rx$ for $x \in \gamma$, $r \in \mathbb{R}$ and functions belonging to $\mathfrak{D}$ are σ-homogeneous, it is possible for the function F to extend continuously to Λ via this homogeneity. Thus, the functions $f^{**}_{p_3,q_3} := (x_2 + x_4)^\sigma F(p_3, q_3)$ form the basis $B_3 = \{f^{**}_{p_3,q_3} \mid p_3, q_3 \in \mathbb{R}\}$ in $\mathfrak{D}$.

Further we use the bilinear functional

$$\mathsf{F}: \ \mathfrak{D} \times \mathfrak{D}^\bullet \longrightarrow \mathbb{C}, \ (f, g) \longmapsto \int\limits_\gamma f(x)g(x)\,\mathrm{d}x,$$

where $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_j$ ($j = 2, 3, 4$) means the invariant measure with respect to H on γ.

Let $g \in G$. We denote the elements of the matrix of the operator $T^\bullet(g)$ with respect to the basis $B_3^\bullet$ by $t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g)$, i.e.,

$$T^\bullet(g)[f^{\bullet**}_{p_3,q_3}](x) = \iint\limits_{\mathbb{R}^2} t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g)\, f^{\bullet**}_{\hat{p}_3,\hat{q}_3}(x)\,\mathrm{d}\hat{p}_3\,\mathrm{d}\hat{q}_3. \tag{2.1}$$

Since $\mathrm{d}x = \mathrm{d}\alpha_1\,\mathrm{d}\beta_1$, from this equality we have

$$\begin{aligned}\mathsf{F}\left(T^\bullet(g)[f^{\bullet**}_{p_3,q_3}], f^{**}_{\tilde{p}_3,\tilde{q}_3}\right) &= \iint\limits_{\mathbb{R}^2} t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g)\,\mathsf{F}\left(f^{\bullet**}_{\hat{p}_3,\hat{q}_3}, f^{**}_{\tilde{p}_3,\tilde{q}_3}\right)\,\mathrm{d}\hat{p}_3\,\mathrm{d}\hat{q}_3 \\ &= 4\pi^2 \iint\limits_{\mathbb{R}^2} t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g)\,\delta(\hat{p}_3 + \tilde{p}_3)\,\delta(\hat{q}_3 + \tilde{q}_3)\,\mathrm{d}\hat{p}_3\,\mathrm{d}\hat{q}_3 \\ &= 4\pi^2 t^{\bullet**}_{p_3,q_3,-\hat{p}_3,-\hat{q}_3}(g),\end{aligned}$$

where δ is the Dirac delta function, and therefore,

$$t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g) = \frac{1}{4\pi^2}\,\mathsf{F}\left(T^\bullet(g)[f^{\bullet**}_{p_3,q_3}], f^{**}_{-\hat{p}_3,-\hat{q}_3}\right).$$

In this section, we compute the matrix elements $t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(a)$ in the following three particular cases: (i) the matrix element belongs to *zero row*; (ii) the matrix element belongs to *main diagonal*; (iii) the matrix element belongs to, that is, its corresponding indices differ by signum. We use the terminology *anti-diagonal* because the multi-indexes $I := (p_3, q_3)$ and $-I = (-p_3, -q_3)$ are mutually opposite in the additive group $\mathbb{R}^2$ as well as the indexes k and $n+1-k$ of the matrix element $a_{k,n+1-k}$, belonging to the $n \times n$ matrix (a_{ij}) and lying on its anti-diagonal, are mutually opposite in the additive group $\mathbb{Z}_{n+1}$ of residues modulo $n+1$. The results are given in Theorems 2.1 and 2.2.

Theorem 2.1. *For* $\Re(\sigma) \in \left(-\frac{3}{2}, -1\right)$ *and* p_3, $q_3 \in \mathbb{R} \setminus \{0\}$, *we have*

$$t^{\bullet**}_{0,0,\hat{p}_3,\hat{q}_3}(a) = \frac{\Gamma\left(\sigma + \frac{3}{2}\right)\,\Gamma(-2\sigma - 2)}{2\,\pi^{\frac{3}{2}}}\,\Omega^{2\sigma+2},$$

where

$$\Omega := \sqrt{p_3^2 + q_3^2} \quad (p_3,\, q_3 \in \mathbb{R})\,. \tag{2.2}$$

Proof. Consider the following (α_2, β_2)-parametrization on γ:

$$\gamma = \left\{ \left(\frac{1+\alpha_2^2}{2},\, \alpha_2 \cos\beta_2,\, \alpha_2 \sin\beta_2,\, \frac{1-\alpha_2^2}{2} \right) \;\middle|\; \alpha_2 \in \mathbb{R}_0^+,\, \beta_2 \in [-\pi, \pi) \right\},$$

we have $\mathrm{d}x = \alpha_2\, \mathrm{d}\alpha_2\, \mathrm{d}\beta_2$. The inner integral (with respect to α_2) in

$$\begin{aligned} t^{\bullet **}_{0,0,\hat{p}_3,\hat{q}_3}(a) &= \frac{1}{4\pi^2}\, \mathsf{F}\left(T^{\bullet}(a)[f^{\bullet **}_{0,0}], f^{**}_{-\hat{p}_3,-\hat{q}_3}\right) \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{0}^{+\infty} \alpha_2^{-2\sigma-3}\, \exp(-\mathbf{i}\alpha_2\, \Omega\, \sin\beta_2)\, \mathrm{d}\alpha_2\, \mathrm{d}\beta_2 \end{aligned}$$

can be evaluated by using a known integral formula (see, e.g., [19]). We therefore have

$$t^{\bullet **}_{0,0,\hat{p}_3,\hat{q}_3}(a) = \pi^{-2}\, \Omega^{2\sigma+2}\, \Gamma(-2\sigma-2) \int_{0}^{\frac{\pi}{2}} \sin^{2\sigma+2} \beta_2\, \mathrm{d}\beta_2,$$

which can be evaluated with the help of a known formula (see, e.g., [19])

$$\int_{0}^{\frac{\pi}{2}} \sin^{\mu} t\, \mathrm{d}t = \frac{1}{2}\sqrt{\pi}\, \Gamma\left(\frac{\mu+1}{2}\right).$$

Hence we have the desired result. □

Theorem 2.2. *For* $\Re(\sigma) \in \left(-\frac{3}{2}, -\frac{1}{2}\right)$, *we have*

$$t^{\bullet **}_{p_3,q_3,p_3,q_3}(a) = \frac{1}{\pi}\left[I_{\sigma+1}(\Omega) + I_{-\sigma-1}(\Omega)\right] K_{\sigma+1}(\Omega) \tag{2.3}$$

and

$$t^{\bullet **}_{p_3,q_3,-p_3,-q_3}(a) = \frac{1}{8}\left[J^2_{-\sigma-1}(\Omega) - J^2_{\sigma+1}(\Omega)\right], \tag{2.4}$$

where Ω *is the same as given in* (2.2).

Proof. For (2.3), by changing the order of integrals in

$$\begin{aligned} t^{\bullet **}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(a) &= \frac{1}{4\pi^2}\, \mathsf{F}(T^{\bullet}(a)[f^{\bullet **}_{p_3,q_3}], f^{**}_{-\hat{p}_3,-\hat{q}_3}) \\ &= \frac{1}{2\pi^2} \int_{0}^{+\infty} \alpha_2^{-2\sigma-3}\, \mathrm{d}\alpha_2 \int_{0}^{\pi} \cos\left(\Omega\left(\alpha_2 - \frac{1}{\alpha_2}\right)\sin\beta_2\right) \mathrm{d}\beta_2 \end{aligned}$$

and using a known formula (see, e.g., [19])

$$\int_0^{+\infty} x^{\alpha-1}\cos\left(ax-\frac{b}{x}\right)\mathrm{d}x = 2\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{2}\right)K_\alpha\left(2\sqrt{ab}\right)$$

$$\left(a,\, b\in\mathbb{R}_0^+,\ |\Re(\alpha)|<1\right),$$

we obtain

$$t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(a) = \frac{1}{\pi^2}\cos([\sigma+1]\pi)\int_0^\pi K_{2\sigma+2}(2\Omega\sin\beta_2)\,\mathrm{d}\beta_2.$$

Then we complete the proof by using the following known formula (see, e.g., [20])

$$\int_0^a \frac{1}{\sqrt{a^2-x^2}}K_\nu(cx)\,\mathrm{d}x = \frac{\pi}{2}\sec\frac{\nu\pi}{2}\left[I_{\frac{\nu}{2}}\left(\frac{ac}{2}\right)+I_{-\frac{\nu}{2}}\left(\frac{ac}{2}\right)\right]K_{\frac{\nu}{2}}\left(\frac{ac}{2}\right)$$

$$\left(a\in\mathbb{R}^+,\ |\Re(\nu)|<1\right).$$

For (2.4), some known formulas [19] and [20] may be used. We omit the details. □

Remark 2.3. The relation (2.3) can be rewritten in the form

$$t^{\bullet**}_{p_3,q_3,p_3,q_3}(a) = \frac{1}{\pi}\left[\mathcal{C}_{\sigma+1}\left(\frac{\Omega}{4}\right)+\mathcal{C}_{-\sigma-1}\left(\frac{\Omega}{4}\right)\right]\mathcal{K}_{\sigma+1}\left(\frac{\Omega}{4}\right).$$

Here

$$\mathcal{C}_\nu(z) = z^{-\frac{\nu}{2}}\,I_\nu\left(2\sqrt{z}\right) \tag{2.5}$$

and

$$\mathcal{K}_\nu(z) = z^{-\frac{\nu}{2}}\,K_\nu\left(2\sqrt{z}\right), \tag{2.6}$$

which are known as the modified Bessel-Clifford functions of the first and second order, respectively. Clifford [5] introduced (2.5) and (2.6) which had been used, in particular, in describing solutions of the Coloumb wave equation [1] and asymptotic expressions for Dirac-delta function [12]. Indeed, $\mathcal{C}_\nu$ and $\mathcal{K}_\nu$ are particular cases of the Wright function playing an important role in the theory of special functions (see, e.g., [13]). On the other hand, the functions J_ν, I_ν, and K_ν in (2.3) and (2.4) can be considered as particular cases of the two-variable coefficients $\mathcal{D}_n^{(1,m)}(a,b)$ arising in the Laurent expansion

$$\exp\left(ax-\frac{y}{x^m}\right) = \sum_{n\in\mathbb{Z}}\mathcal{D}_n^{(1,m)}(a,b)\,x^n$$

of the function $\exp\left(ax-\frac{y}{x^m}\right)$ (see [6]) which are, as we see, related to the matrix elements of the operator $T(a)$ with respect to the parabolic bases.

3. A group theoretical interpretation of one known formula

Let ω be the intersection of Λ with the plane $x_1 = 1$. We parameterize ω in the following manner:

$$\omega = \{(1, \sin\alpha_3 \sin\beta_3, \sin\alpha_3 \cos\beta_3, \cos\alpha_3) \mid \alpha_3 \in [0, 2\pi),\ \beta_3 \in [0, \pi)\}$$

and express the function $f^{\bullet**}_{p_3,q_3}$ as a linear combination of functions belonging to the basis $B^{\bullet}_1$:

$$f^{\bullet**}_{p_3,q_3}(x) = \sum_{p_1=0}^{\infty} \sum_{q_1=-|p_1|}^{|p_1|} c_{p_3,q_3,p_1,q_1}\, f^{\bullet}_{p_1,q_1}(x). \tag{3.1}$$

Introducing the bilinear functional $\Phi : (\mathfrak{D}, \mathfrak{D}^{\bullet}) \longrightarrow \mathbb{C}$ defined by

$$(f, g) \longmapsto \int_{-\pi}^{\pi} \sin\alpha_3\, \mathrm{d}\alpha_3 \int_{0}^{\pi} f(\alpha_3, \beta_3)\, g(\alpha_3, \beta_3)\, \mathrm{d}\beta_3,$$

and using the orthogonality relations

$$\int_0^{2\pi} \exp(\mathbf{i}(p+q)z)\, \mathrm{d}z = 2\pi\delta_{p,-q}$$

and (see, e.g., [21, p. 281, Eqs. (27) and (28)]; see also [28, p. 462, Eqs. (4) and (5)])

$$\int_{-1}^{1} C^q_p(z) C^q_{\tilde{p}}(z)(1-z^2)^{q-\frac{1}{2}}\, \mathrm{d}z = \frac{\pi\Gamma(2q+p)\delta_{p,\tilde{p}}}{2^{2q-1}p!(p+q)[\Gamma(q)]^2},$$

where $\delta_{i,j}$ is the Kronecker symbol, we find that

$$c_{p_3,q_3,p_1,q_1} = \frac{2^{2q_1-1}(p_1-q_1)!\left(p_1+\frac{1}{2}\right)\left[\Gamma\left(q_1+\frac{1}{2}\right)\right]^2}{\pi^2\,\Gamma(2q_1+1)} \Phi(f^{\bullet**}_{p_3,q_3}, f_{p_1,-q_1}).$$

Theorem 3.1. *For* $p_1 = q_1$ and $\Re(\sigma) < 0$, *we have*

(1) $\Omega \neq 0$,

$$\begin{aligned} c_{p_3,q_3,p_1,q_1} = &\frac{2^{\sigma+|q_1|}\ \left(p_1+\frac{1}{2}\right)\ \left[\Gamma\left(|q_1|+\frac{1}{2}\right)\right]^2\ \Omega^{|q_1|-\sigma-\frac{1}{2}}}{\pi^2\,\Gamma(2|q_1|+1)\,\Gamma(|q_1|-\sigma)} \\ &\times \exp\left(\mathbf{i}|q_1|\left[\frac{\pi}{2}\operatorname{sign} p_3 - \frac{\pi}{2} - \arctan\frac{q_3}{p_3}\right]\right) K_{\sigma+1}(\Omega); \end{aligned} \tag{3.2}$$

(2) $q_1 \neq 0$,

$$c_{0,0,p_1,q_1} = \frac{2^{2q_1}\,\left(p_1+\frac{1}{2}\right)\,\left[\Gamma\left(|q_1|+\frac{1}{2}\right)\right]^2\,\sin(\pi q_1)}{\pi^2\, q_1\,\Gamma(2q_1+1)} \times \mathrm{B}\left(\frac{|q_1|+1}{2}, \frac{|q_1|-1}{2}-\sigma\right), \tag{3.3}$$

where Ω *is the same as given in* (2.2) *and* $\mathrm{B}(\cdot,\cdot)$ *is the familiar beta function* (see, e.g., [26]).

Proof. We first prove (1). Since $C_0^\lambda(z) = 1$, we have

$$\mathsf{F}_2(f^{\bullet**}_{p_3,q_3}, f_{p_1,-q_1}) = 2^{|q_1|-\sigma} \int_0^{+\infty} \alpha_2^{|q_1|+1}\,(\alpha_2^2+1)^{\sigma-|q_1|}\,\mathrm{d}\alpha_2 \times \int_{-\pi}^{\pi} \exp\left(\mathbf{i}\left[|q_1|\beta_2 - r\alpha_2 \sin\left(\beta_2 + \arctan\frac{q_3}{p_3}\right)\right]\right)\,\mathrm{d}\beta_2.$$

Here, in order to evaluate the inner integral, we use the formula (see, e.g., [19])

$$\int_a^{a+2\pi} \exp(\mathbf{i}[nx - z\sin x])\,\mathrm{d}x = 2\pi\, J_n(z) \quad (n\in\mathbb{Z},\ |\arg z|<\pi). \tag{3.4}$$

The external integral

$$\int_0^{+\infty} \alpha_2^{|q_1|+1}\,(\alpha_2^2+1)^{\sigma-|q_1|}\,J_{|q_1|}\,(\Omega\alpha_2)\,\mathrm{d}\alpha_2,$$

which is a Hankel integral transform, can be evaluated by using a known formula (see [10]).
To prove (2), we use the Fourier cosine transform [9] and the Mellin transform [19]. □

From (2.1) and (3.1), we obtain

$$T^\bullet(g)[f^{\bullet**}_{p_3,q_3}](x) = \sum_{p_1=0}^{\infty}\sum_{q_1=-|p_1|}^{|p_1|}\left(\iint_{\mathbb{R}^2} t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g)\,c_{\hat{p}_3,\hat{q}_3,p_1,q_1}\,\mathrm{d}\hat{p}_3\,\mathrm{d}\hat{q}_3\right) f^\bullet_{p_1,q_1}(x). \tag{3.5}$$

On the other hand,

$$T^{\bullet}(g)[f^{\bullet**}_{p_3,q_3}](x) = \sum_{p_1=0}^{\infty} \sum_{q_1=-|p_1|}^{|p_1|} c_{p_3,q_3,p_1,q_1}\, T^{\bullet}(g)[f^{\bullet}_{p_1,q_1}](x). \tag{3.6}$$

Since, in case of $p_1 = q_1$, the function $f^{\bullet}_{p_1,q_1}$ is a fixed point of the linear operator $T^{\bullet}(a)$ (that is, $f^{\bullet}_{p_1,q_1}$ is an eigenfunction with respect to eigenvalue 1), we derive from (3.5) and (3.6) that, for $p_1 = q_1$,

$$c_{p_3,q_3,p_1,q_1} = \iint\limits_{\mathbb{R}^2} t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(a)\, c_{\hat{p}_3,\hat{q}_3,p_1,q_1}\, \mathrm{d}\hat{p}_3\, \mathrm{d}\hat{q}_3. \tag{3.7}$$

Putting $p_3 = q_3 = 0$ in (3.7) (i.e., considering the matrix elements of the operators $T^{\bullet}(a)$ and $B^{\bullet}_1 \mapsto B^{\bullet}_3$ belonging to 'zero row'), taking the results from Theorems 2.1 and 3.1 and choosing the polar coordinate system for evaluating the double integral in the right side of (3.7), we obtain a particular case of a known formula (see, e.g., [20])

$$\int\limits_0^{+\infty} x^{\alpha-1} K_{\nu}(cx)\,\mathrm{d}x = \frac{2^{\alpha-2}}{c^{\alpha}}\,\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right) (\Re(\alpha) > |\Re(\nu)|, \Re(c) > 0).$$

Thus, in the considered particular case, this formula expresses the relationship between the matrix elements of the operator $T^{\bullet}(a)$ computed with respect to bases $B^{\bullet}_3$ and $B^{\bullet}_1$ and lying in the 'zero row' of the corresponding matrices.

For general case, this formula, as a K-transform of the function $x^{\alpha-\frac{3}{2}}$, can be derived by using a known formula (see, e.g., [8]).

4. Two series involving the modified hyper Bessel functions of the second kind

Yasar and Özarslan [30] dealt with the idea concerning a unification of (usual) Bessel, modified Bessel, spherical Bessel and Bessel-Clifford functions by means of the generalized Pochhammer symbol (see [25])

$$(\lambda;\rho)_{\nu} = \begin{cases} \Gamma_{\rho}(\lambda+\nu) & (\Re(\rho) > 0,\ \lambda,\, \nu \in \mathbb{C}) \\ (\lambda)_{\nu} & (\rho = 0,\ \lambda,\, \nu \in \mathbb{C}), \end{cases}$$

where $(\lambda)_{\nu}$ is the Pochhammer symbol (see, e.g., [26]) and $\Gamma_{\rho}(x)$ is the extended gamma function defined by (see [3])

$$\Gamma_{\rho}(x) = \int\limits_0^{+\infty} t^{x-1} \exp\left(-t - \frac{\rho}{t}\right) \mathrm{d}t \quad (\Re(\rho) > 0).$$

From this unification, they [30] derived the integral representation, for $\Re(\nu) > -1$,

$$\begin{aligned} & I_\nu(z) \\ & = \left(\frac{z}{2}\right)^\nu [\Gamma(\nu+1)]^2 \int_0^{+\infty} t^\nu \exp(-t)\, {}_0F_3\left(-;\, \nu+1,\, \frac{\nu+1}{2},\, \frac{\nu+2}{2};\, \frac{z^2t^2}{2}\right) \mathrm{d}t, \end{aligned}$$

whose integrand can be rewritten in terms of $I_{\nu,\frac{\nu-1}{2},\frac{\nu}{2}}\left(2\sqrt{zt}\right)$.

In this section, we aim to establish two series converging to products of cylinder functions. Consider the matrix elements c_{p_2,q_2,p_3,q_3} of the linear operator $B_2^\bullet \mapsto B_3^\bullet$ such that

$$f^{\bullet *}_{p_2,q_2}(x) = \iint_{\mathbb{R}^2} c_{p_2,q_2,p_3,q_3}\, f^{\bullet **}_{p_3,q_3}(x)\, \mathrm{d}p_3\, \mathrm{d}q_3. \tag{4.1}$$

Theorem 4.1. *The matrix elements c_{p_2,q_2,p_3,q_3} can be expressed in the form*

$$c_{p_2,q_2,p_3,q_3} = \frac{(\operatorname{sign} q_2)^{q_2}}{2\pi\Omega} \exp\left(\mathbf{i}q_2\left[\frac{\pi}{2}\operatorname{sign} p_3 - \frac{\pi}{2} - \arctan\frac{q_3}{p_3}\right]\right) \delta(\Omega - p_2),$$

where Ω is the same as given in (2.2).

Proof. As for $t^{\bullet **}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g)$, it is easy to show that

$$c_{p_2,q_2,p_3,q_3} = \frac{1}{4\pi^2}\mathsf{F}(f^{\bullet *}_{p_2,q_2}, f^{**}_{-p_3,-q_3}).$$

We have

$$\begin{aligned} & c_{p_2,q_2,p_3,q_3} \\ & = \frac{1}{4\pi^2}\mathsf{F}(f^{\bullet *}_{p_2,q_2}, f^{**}_{-p_3,-q_3}) \\ & = \int_0^{+\infty}\int_{-\pi}^{\pi} \alpha_2\, J_{|q_2|}(p_2\alpha_2)\, \exp(\mathbf{i}[q_2\beta_2 - p_3\alpha_2\sin\beta_2 - q_3\alpha_2\cos\beta_2])\, \mathrm{d}\alpha_2\, \mathrm{d}\beta_2. \end{aligned} \tag{4.2}$$

Interchanging the order of integrations in (4.2) and using the formula (3.4), we obtain

$$\begin{aligned} & c_{p_2,q_2,p_3,q_3} \\ & = \frac{1}{2\pi}\exp\left(\mathbf{i}q_2\left[\frac{\pi}{2}\operatorname{sign} p_3 - \frac{\pi}{2} - \arctan\frac{q_3}{p_3}\right]\right) \int_0^{+\infty} \alpha_2\, J_{|q_2|}(p_2\alpha_2)\, J_{q_2}(\Omega\alpha_2)\, \mathrm{d}\alpha_2, \end{aligned}$$

which, upon using the symmetry relation $J_{-q_2}(x) = (-1)^{q_2} J_{q_2}(x)$ and the orthogonality property for the functions J_n, yields the desired result. □

Theorem 4.2. *For $\Re(\sigma) \in \left(-1, -\frac{1}{2}\right)$, we have*

$$\sum_{q_2\in\mathbb{Z}} \exp\left(-\mathbf{i}2q_2 \arctan\frac{q_3}{p_3}\right)\left[I_{|q_2|,-\sigma,|q_2|-\sigma}(2\Omega) - I_{|q_2|,\sigma,|q_2|+\sigma}(2\Omega)\right] = 8\Omega\,\sin(\pi\sigma)\exp\left(2\mathbf{i}q_2\arctan\frac{p_3}{q_3}\right)\left[I_{\sigma+1}(\Omega) + I_{-\sigma-1}(\Omega)\right]K_{\sigma+1}(\Omega) \tag{4.3}$$

and

$$\sum_{q_2\in\mathbb{Z}} (-1)^{q_2}\exp\left(-\mathbf{i}2q_2 \arctan\frac{q_3}{p_3}\right)\left[I_{|q_2|,-\sigma,|q_2|-\sigma}(2\Omega) - I_{|q_2|,\sigma,|q_2|+\sigma}(2\Omega)\right] = \pi\Omega\,\sin(\pi\sigma)\exp\left(2\mathbf{i}q_2\arctan\frac{p_3}{q_3}\right)\left[J^2_{-\sigma-1}(\Omega) - J^2_{\sigma+1}(\Omega)\right], \tag{4.4}$$

where Ω is the same as given in (2.2).

Proof. We denote c_{p_3,q_3,p_2,q_2} the matrix elements of the linear operator which is inverse for $B_2^\bullet \mapsto B_3^\bullet$. In general case, for arbitrary bases B and B' in $\mathfrak{D}^\bullet$, the matrix elements of the bases transformations $B \mapsto B'$ and $B' \mapsto B$ differ (up to a multiplicative constant) from each other by σ: namely, in order to obtain the matrix elements of the inverse transformation we must change σ by the dual number $\sigma^\bullet$. However, it is easy to see that in case $B = B_2^\bullet$ and $B' = B_3^\bullet$ the integral $c_{p_3,q_3,p_2,q_2} = \frac{1}{4\pi^2}\mathsf{F}(f^{\bullet**}_{p_3,q_3}, f^*_{p_2,q_2})$ does not depend on σ, therefore, we have $c_{p_3,q_3,p_2,q_2} = c_{p_2,q_2,p_3,q_3}$.

Let us express the operand in $T^\bullet(g)[f^{\bullet**}_{p_3,q_3}]$ as a linear combination of functions belonging to $B_2^\bullet$:

$$T^\bullet(g)[f^{\bullet**}_{p_3,q_3}] = \sum_{q_2\in\mathbb{Z}}\int_0^{+\infty} c_{p_2,q_2,p_3,q_3}\,T^\bullet(g)[f^{\bullet*}_{p_2,q_2}]\,\mathrm{d}p_2.$$

Applying the same argument to the function $T^\bullet(g)[f^{\bullet*}_{p_2,q_2}]$ and considering the formula (4.1), we have

$$T^\bullet(g)[f^{\bullet**}_{p_3,q_3}] = \iint_{\mathbb{R}^2}\left(\sum_{q_2,\hat{q}_2\in\mathbb{Z}}\iint_{\mathbb{R}^2} c_{p_2,q_2,p_3,q_3}\,t^{\bullet*}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(g)c_{\hat{p}_2,\hat{q}_2,\hat{p}_3,\hat{q}_3}\mathrm{d}p_2\mathrm{d}\hat{p}_2\right)\mathrm{d}\hat{p}_3\mathrm{d}\hat{q}_3.$$

Comparing the coefficients in this expression and (2.1), we derive

$$t^{\bullet**}_{p_3,q_3,\hat{p}_3,\hat{q}_3}(g) = \frac{1}{4\pi^2\Omega^2}\sum_{q_2,\hat{q}_2\in\mathbb{Z}}\exp\left(-2\mathbf{i}q_2\,\arctan\frac{q_3}{p_3}\right)t^{\bullet*}_{\Omega,q_2,\Omega,\hat{q}_2}(g).$$

In this equality we substitute (1.2) and obtain the result in Theorem 4.1. To obtain (4.3) and (4.4), we substitute (2.3) and (2.4), respectively. □

Remark 4.3. The equality (4.3) can be rewritten in terms of $\mathcal{C}_\nu$ and $\mathcal{K}_\nu$ as in Remark 2.3. In addition, considering the symmetry relation $K_{-\nu}(z) = K_\nu(z)$, we can consider this equality as a series representation of the function

$$I_{\sigma+1}(\Omega)K_{\sigma+1}(\Omega) + I_{-\sigma-1}(\Omega)K_{-\sigma-1}(\Omega). \tag{4.5}$$

An integral representation of this function (4.5) can be given by using a known formula (see, e.g., [8])

$$I_\nu(x)K_\mu(x) + I_\nu(x)K_\mu(x) = 2\int_0^{+\infty} J_{\nu+\mu}(2x\,\sinh t)\,\cosh[(\mu-\nu)t]\,\mathrm{d}t. \tag{4.6}$$

Acknowledgments. The authors would like to express their deep-felt thanks for the reviewer's very helpful comments to improve this paper as it stands.

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