Nonlinear Functional Analysis and Applications
Vol. 15, No. 2 (2010), pp. 193-198

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SOLVABILITY FOR A CLASS OF SINGULAR NONLINEAR THIRD-ORDER BOUNDARY VALUE PROBLEMS WITH DERIVATIVE NONLINEARITY

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Abstract. By using Leray-Schauder nonlinear alternative, this paper presents the existence of solutions for a class of singular nonlinear third-order boundary value problem under some weaker conditions that the nonlinear term f contains first and second derivatives.

This paper deals with the existence of solution for the following nonlinear boundary value problem

$$\begin{cases} x'''(t) + h(t)f(t, x(t), x'(t), x''(t)) = 0, \quad 0 \le t \le 1, \\ x(0) = x'(0) = x(1), \end{cases} \tag{1}$$

where nonlinearity $h(t)$ may be singular at $t = 0$ and/or $t = 1$.

Boundary value problems have been considering widely, and there are some excellent results on the existence of solutions, (see [1], [2], [6]). At the same

[0]Received October 9, 2008. Revised January 5, 2009.
[0]2000 Mathematics Subject Classification: 34B15, 34B16, 34B18.
[0]Keywords: Existence, boundary value problems, Leray-Schauder nonlinear alternative.
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[0]The author is supported financially by the National Natural Science Foundation of China (10471075), the Natural Science Foundation of Shanghai Normal University (SK200703).

time, utilizing the method of upper and lower solutions or topological transversality, differential inequality, shooting argument, some authors established the existence of solutions for boundary value problems of third-order differential equations (see [5], [6]).

Motivated by the results in [5] and [6], the aim of this paper is to consider the existence of positive solutions for the BVP (1.1) using properties of Green's function and Leray-Schauder nonlinear alternative. The nonlinear term $h(t)$ may be singular at $t = 0$ and/or $t = 1$.

Definition 1. By a nonzero solution of the BVP (1.1), we mean a function $x(t) \in C^3(0,1) \cap C[0,1]$ satisfying the BVP (1.1) and with x not identically zero on $[0,1]$, $x(t)$ is called a positive solution of the BVP (1.1) if $x(t)$ is a solution of the BVP (1.1) and $x(t) > 0$ for any $t \in (0,1)$.

Let E be real Banach space and $K \subset E$ be a cone in E. For all $0 < r < R < +\infty$, let

$$K_r = \{x \in K : \|x\| < r\}, \quad \partial K_r = \{x \in K : \|x\| = r\},$$

$$\overline{K}_{r,R} = \{x \in K : r \leq \|x\| \leq R\}.$$

The proof of our main results uses the following fixed point theorem:

Lemma 1.([2]) *Let X be a Banach space, and Ω be a bounded open set in X, and $0 \in \Omega$, $T : \overline{\Omega} \longrightarrow X$ be a completely continuous operator satisfying $Tx \neq \lambda x$, $\lambda > 1$, $x \in \partial\Omega$. Then T has a fixed point in Ω.*

Lemma 2. *Let $G(t,s)$ be the Green's function for*

$$\begin{cases} x'''(t) = 0, \ 0 < t < 1, \\ x(0) = x'(0) = x(1) = 0, \end{cases}$$

that is

$$G(t,s) = \begin{cases} \dfrac{1}{2}s^2(1-t)^2 + s(1-t)(t-s), & 0 \leq s \leq t \leq 1, \\ \dfrac{1}{2}t^2(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2}$$

Lemma 3.([1]) $\frac{1}{2}a(t)b(s) \leq G(t,s) \leq b(s)$ *for any t, $s \in [0,1]$, where $b(s) = s(1-s)^2$, $a(t) = t^2(1-t)$.*

Lemma 4. $\max\limits_{0\leq t\leq 1} |G(t,s)| \leq 1$, $\max\limits_{0\leq t\leq 1} | \frac{\partial}{\partial t}G(t,s) | \leq 1$, $\max\limits_{0\leq t\leq 1} | \frac{\partial^2}{\partial t^2}G(t,s) | \leq 1$.

For convenience, we list the following assumptions:

($\mathbf{H}_1$) $h \in C((0,1),(0,+\infty))$ *and* $0 < \int_0^1 h(s)ds < +\infty$.

($\mathbf{H}_2$) *$f : [0,1] \times R \times R \times R \longrightarrow R$ is continuous and there exist nonnegative function a, b, c, $r \in C[0,1]$ such that*

$$| f(t,u,v,w) | \leq a(t)|u| + b(t)|v| + c(t)|w| + r(t), \qquad 0 \leq t \leq 1$$

and $\int_0^1 h(s)[a(s)+b(s)+c(s)]ds < 1$.

Let

$$K = \{x \in X : x(t) \geq 0,\ t \in [0,1],\ x(t) \geq \frac{1}{2}a(t)\|x\|,\ t \in [0,1]\},$$

where $a(t)$ is defined as Lemma 3.

Denote $\|x\|_0 = \max\limits_{0 \leq t \leq 1} |x(t)|$, $x(t) \in C[0,1]$, $\|x\| = \max\{\|x\|_0,\ \|x'\|_0,\ \|x''\|_0\}$. It is easy to see that K is a cone of X. Define an operator $A : K \longrightarrow K$ by

$$Ax(t) = \int_0^1 G(t,s)h(s)f(s,x(s),x'(s),\ x''(s))ds, \quad t \in [0,1].$$

Obviously, the existence of positive solutins for the BVP (1.1) is equivalent to the existence of fixed points of the operator equation $Ax = x,\ x \in C[0,1]$.

Lemma 5. *Assume that (H_1) and (H_2) hold, then $AK \subset K$ and $A : K \longrightarrow K$ is completely continuous.*

Proof. It follows from Lemma 3. that for all $t \in [0,1]$, we get

$$\begin{aligned}(Ax)(t) &= \int_0^1 G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds \\ &\leq \int_0^1 b(s)h(s)f(s,x(s),x'(s),x''(s))ds.\end{aligned}$$

Therefore,

$$\|Ax\| \leq \int_0^1 b(s)h(s)f(s,x(s),x'(s),x''(s))ds.$$

For any $x \in K$, we know by Lemma 3. that

$$\begin{aligned}(Ax)(t) &= \int_0^1 G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds \\ &\geq \frac{1}{2}a(t)\int_0^1 b(s)h(s)f(s,x(s),x'(s),x''(s))ds \\ &\geq \frac{1}{2}a(t)\|Ax\|, \quad 0 \leq t \leq 1.\end{aligned}$$

Therefore, we have $AK \subset K$.

Now, let us prove $A : K \longrightarrow K$ is completely continuous. For each $n \geq 1$ define the operator $A_n : K \longrightarrow K$ by

$$(A_n x)(t) = \int_{\frac{1}{n}}^{\frac{n-1}{n}} G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds,\ x \in K, \quad t \in [0,1]. \quad (3)$$

By (H_1), (H_2) and the Arzela-Ascoli theorem, we know that $A_n : K \longrightarrow K$ is completely continuous. By Lemma 3. we have

$$\begin{aligned}|(Ax)(t) - (A_n x)(t)| &= \Big| \int_0^{\frac{1}{n}} G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds \\ &\quad + \int_{\frac{n-1}{n}}^1 G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds \Big| \\ &\leq \int_0^{\frac{1}{n}} b(s)h(s)|f(s,x(s),x'(s),x''(s))|ds \\ &\quad + \int_{\frac{n-1}{n}}^1 b(s)h(s)|f(s,x(s),x'(s),x''(s))|ds,\end{aligned}$$

and so

$$\begin{aligned}|(Ax)(t) - (A_n x)(t)| \leq &\int_0^{\frac{1}{n}} b(s)h(s)|f(s,x(s),x'(s),x''(s))|ds \\ &+ \int_{\frac{n-1}{n}}^1 b(s)h(s)|f(s,x(s),x'(s),x''(s))|ds.\end{aligned}$$

Assumption (H_1), (H_2) and the absolute continuity of integral imply that

$$\lim_{n\to\infty} \|Ax - A_n x\| = 0,$$

then A is completely continuous. □

Theorem 6. *Assume that* (H_1) *and* (H_2) *hold, then the BVP* (1.1) *has at least one positive solution.*

Proof. Obviously, if $f(t,0,0,0) \equiv 0,\ 0 \leq t \leq 1$, then BVP (1.1) has only trivial solution. Therefore, we may assume that $(t, x_0, y_0, z_0) \in (0,1) \times R \times R \times R$ such that $f(t, x_0, y_0, z_0) = \alpha_0 > 0$. It follows from (H_2) that

$$\int_0^1 h(s)[a(s) + b(s) + c(s)]ds > 0$$

and

$$\int_0^1 h(s)r(s)ds > 0.$$

Now we assume that

$$M = \int_0^1 h(s)[a(s) + b(s) + c(s)]ds,$$

and

$$M^* = \int_0^1 h(s)r(s)ds.$$

Then $0 < M < 1$. Let $R = M^*(1 - M)^{-1}$. Then $R > 0$. Let $\Omega = \{x \in K : \|x\| < R\}$.

We claim that $Tx \neq \lambda x$, for all $\lambda > 1$.

In fact, if not, there exists $x_0 \in \partial\Omega$, $\lambda_0 > 1$ such that $Tx_0 = \lambda_0 x_0$. Since $\|x\| = R$, then $\|x\|_0 \leq R$, $\|x'\| \leq R$, $\|x''\| \leq R$. Thus

$$\begin{aligned}
\|Tx(t)\|_0 &= \max_{0\leq t\leq 1} \left| \int_0^1 G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds \right| \\
&\leq \int_0^1 b(s)h(s)|f(s,x(s),x'(s),x''(s))|ds \\
&\leq \int_0^1 h(s)[a(s)|x(s)| + b(s)|x'(s)| + c(s)|x''(s)| + r(s)]ds \\
&\leq \int_0^1 h(s)([a(s) + b(s) + c(s)]R + r(s))ds \\
&\leq R\int_0^1 h(s)[a(s) + b(s) + c(s)]ds + \int_0^1 h(s)r(s)ds \\
&\leq RM + M^*,
\end{aligned}$$

$$\begin{aligned}
\|(Tx)'(t)\|_0 &= \max_{0\leq t\leq 1} \left| \int_0^1 \frac{\partial}{\partial t}G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds \right| \\
&\leq \int_0^1 \max_{0\leq t\leq 1} \left| \frac{\partial}{\partial t}G(t,s) \right| h(s)|f(s,x(s),x'(s),x''(s))|ds \\
&\leq \int_0^1 h(s)[a(s)|x(s)| + b(s)|x'(s)| + c(s)|x''(s)| + r(s)]ds \\
&\leq \int_0^1 h(s)([a(s) + b(s) + c(s)]R + r(s))ds \\
&\leq R\int_0^1 h(s)([a(s) + b(s) + c(s)]ds + \int_0^1 h(s)r(s)ds \\
&\leq RM + M^*
\end{aligned}$$

and

$$\begin{aligned}
\|(Tx)''(t)\|_0 &= \max_{0\le t\le 1}\left|\int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)h(s)f(s,x(s),x'(s),x''(s))ds\right| \\
&\le \int_0^1 \max_{0\le t\le 1}\left|\frac{\partial^2}{\partial t^2}G(t,s)\right| h(s)|f(s,x(s),x'(s),x''(s))|ds \\
&\le \int_0^1 h(s)[a(s)|x(s)|+b(s)|x'(s)|+c(s)|x''(s)|+r(s)]ds \\
&\le \int_0^1 h(s)([a(s)+b(s)+c(s)]R+r(s))ds \\
&\le R\int_0^1 h(s)([a(s)+b(s)+c(s)]ds+\int_0^1 h(s)r(s)ds \\
&\le RM+M^*.
\end{aligned}$$

Therefore $\|Tx\|\le RM+M^*$. Hence

$$\begin{aligned}
\lambda_0 R = \lambda_0\|x\| = \|Tx\| &\le RM+M^* = M^*(1-M)^{-1}M+M^* \\
&= \frac{M^*M}{1-M}+M^* = \frac{M^*M+M^*-MM^*}{1-M} \\
&= M^*(1-M)^{-1} = R.
\end{aligned}$$

Since $R>0$, then $\lambda_0\le 1$, which is contradict with $\lambda_0>1$.

It follows from Lemma 3. that T has at least one fixed point $x^*\in\Omega$. So the BVP (1.1) has at least one solution $x^*\in E$. This completes the proof. □

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