Nonlinear Functional Analysis and Applications
Vol. 23, No. 2 (2018), pp. 205-224
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa
Copyright © 2018 Kyungnam University Press

EXISTENCE OF SOLUTION TO BESSEL-TYPE BOUNDARY VALUE PROBLEM VIA $G-l$ CYCLIC F-CONTRACTIVE MAPPING WITH GRAPHICAL VERIFICATION

Deepak Singh[1], Vishal Joshi[2] and Jong Kyu Kim[3]

[1]Department of Applied Sciences, NITTTR,
Under Ministry of HRD, Govt. of India, Bhopal, 462002, India
e-mail: dk.singh1002@gmail.com

[2]Department of Applied Mathematics,
Jabalpur Engineering College, Jabalpur, India
e-mail: joshinvishal76@gmail.com

[3]Department of Mathematics Education,
Kyungnam University, Changwon, Gyeongnam, 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. The purpose of this paper is to establish the existence of solution to Bessel-type boundary value problem

$$\begin{cases} t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} = K(t, u(t)); \\ u(0) = u(1) = 0, \end{cases}$$

where $K : [0, 1] \times R^+ \to R$ is a continuous function, by highlighting role of the $G - l$ cyclic F-contractive mapping. In the sequel some fixed point theorems are proved. It is worth mentioning that our results cannot be concluded from the existing results in the milieu of associated metric spaces. Utilizing an innovative pictorial technique, an example is worked out substantiating the utility of hypothesis of our results. In an another paper, authors recently established some inventive applications of G-metric spaces in science and engineering.

---

[0]Received February 14, 2017. Revised July 5, 2017.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: $G - l$ cyclic F-contraction G-metric spaces, Cauchy sequence, fixed point theorem, Bessel-type boundary value problem.
[0]Corresponding author: J. K. Kim(jongkyuk@kyungnam.ac.kr).

1. Introduction

The instigation of fixed point theory on complete metric space is associated to Banach Contraction Principle due to Banach [6], published in 1922. Banach Contraction Principle pronounces that any contractive self-mappings on a complete metric space has a unique fixed point. This principle is one of a very supremacy test for the existence and uniqueness of the solution of substantial problems arising in mathematics. Because of its significance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions. Recently, one of the most interesting generalization of it was given by Wardowski [24]. He introduced a new contraction called F-contraction and established a fixed point result as a generalization of the Banach contraction principle in an dissimilar way than in the other recognized results from the literature. Recently, Secelean *et al.* [19] described a large class of functions by replacing condition $(F2')$ instead of the condition $(F2)$ in the definition of F-contraction presented by Wardowski [24]. Very recently, Piri *et al.* [15] improved the result of Secelean *et al.* [19] by replacing condition $(F3')$ instead of the condition $(F3)$.

The concept of G-metric space was introduced by Mustafa and Sims in [11], [12]. Physically, this notion is a measure of mutual distance between three elements taken together. A metric space is a special case of a generalized metric space. Analysis of the structure of this space was done in some detail in Mustafa and Sims [11]. Several studies relevant to metric spaces have been and are being extended to G-metric spaces as, for instances, fixed point results have been proved in [1, 2, 3, 4, 5, 7, 13, 14, 20, 22, 23]. Samet *et al.* [18] and Jleli *et al.* [8] raised the question that some theorems in the framework of a G- metric spaces in the literature can be obtain directly by some existing results in the setting of usual metric spaces. Further E. Karapinar *et al.* [9] answered the issues raised by [18] and [8] with the remark that techniques used in [18] and [8] are inapplicable unless the contraction condition in the statement of the theorem can be reduced into two variables. Thus a study in G-metric spaces is meaningful only when the problem cannot be transferred to a corresponding problem in a metric space.

The study of fixed points of mappings satisfying cyclic contractive conditions has been at the center of vigorous research activity in the last years. In 2003, Kirk *et al.* [10] generalized the Banach contraction principle by using two closed subsets of a complete metric space. Then, Petrusel [17] proved some results about periodic points of cyclic contraction maps. His results generalized the main result of Kirk *et al.* [10]. Later, Pacurar and Rus [16] proved some fixed point results for cyclic ϕ-contraction mappings on a metric space.

In this paper, we establish the existence of solution to Bessel-type boundary value problem by highlighting the role of $G-l$ cyclic F-contraction mapping and some fixed point theorems.

For the sake of completeness, we will comprise basic definitions and crucial concepts that we need in the rest of this note.

Definition 1.1. ([11]) Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G-1) $G(x, y, z) = 0$ if $x = y = z$;

(G-2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G-3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.

The function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. ([11]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that the sequence $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that

$$G(x, x_n, x_m) < \epsilon,$$

for all $m, n > N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim\limits_{n,m\to+\infty} x_n = x$.

Proposition 1.3. ([11]) *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x;*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$;*
(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$;*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.4. ([11]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that

$$G(x_n, x_m, x_l) < \epsilon,$$

for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.5. ([11]) *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

(1) *$\{x_n\}$ is G-Cauchy;*
(2) *For every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that $G(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

Definition 1.6. ([11]) A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Lemma 1.7. ([12]) *By the rectangle inequality* $(G5)$ *together with the symmetry* $(G4)$*, we have*

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2G(y, x, x).$$

The notion of cyclic mapping is introduced by Kirk *et al.* [10] in 2003.

Definition 1.8. Let X be a nonempty set, m a positive integer, and $T : X \to X$ a mapping. $X = \bigcup_{i=1}^{m} A_i$ is said to be a cyclic representation of X with respect to T if

(i) $A_i, i = 1, 2, \ldots, m$ are nonempty closed sets,

(ii) $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset T(A_m), T(A_m) \subset A_1$.

Wardowski [24] initiated and considered a new contraction which is called F-contraction to prove a fixed point result as a generalization of the Banach contraction principle.

Definition 1.9. ([24]) Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

(F1) F is strictly increasing;
(F2) for all sequence $\alpha_n \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if
$$\lim_{n\to\infty} F(\alpha_n) = -\infty;$$
(F3) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Wordowski [24], defined the class of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ by $\mathcal{F}$ and introduced the notion of $F-$contraction as follows.

Definition 1.10. ([24]) Let (X; d) be a metric space. A self-mapping T on X is called an $F-$contraction if there exists $\tau > 0$ such that for $x, y \in X$

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F \in \mathcal{F}$.

Afterward Secelean [19] established the following lemma and utilized an equivalent but a more simple condition $(F2')$ instead of condition $(F2)$.

Lemma 1.11. ([19]) *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing map and α_n be a sequence of positive real numbers. Then the following assertions hold:*

(a) *if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$ *then* $\lim_{n\to\infty} \alpha_n = 0$*;*

(b) *if* $\inf F = -\infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$*; then* $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

He forwarded the following conditions:

(F2′) $\inf F = -\infty$ or

(F2″) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

Very recently Piri *et al.* [15] replaced the condition $(F3)$ by $(F3')$ in Definition 1.9 due to Wardowski as follows:

(F3′) F is continuous on $(0, \infty)$.

Thus Piri and Kumam [15] established the generalization of result of Wordowski [24] using the conditions $(F1), (F2')$ and $(F3')$.

Throughout our subsequent discussion, We drop-out the condition $(F2')$ and named the contraction as relaxed F-contraction. Thus we denote, the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ by Δ_F which satisfy the conditions $(F1)$ and $(F3')$.

Example 1.12. Let $F_1(\alpha) = \ln(\alpha)$, $F_2(\alpha) = -\frac{1}{\alpha}$, $F_3(\alpha) = -\frac{1}{\alpha} + \alpha$, $F_4(\alpha) = \frac{1}{1-e^{\alpha}}$. Then $F_1, F_2, F_3, F_4 \in \Delta_F$.

Our work is devoted to introduce the $G-l$ cyclic F-contraction in G-metric spaces, by integrating the ideas of Piri *et al.* [15] and Kirk *et al.* [10]. Invoking aforesaid development, some fixed point theorems in the structure of G-metric spaces are proved. During the process, we improve the concept of Piri *et al.* [15] by omitting the condition $(F2')$ and also the technique pointed out in [9] is exploited to obtain such type of fixed point results in G-metric spaces that cannot be obtained from the existing results in the setting of associated metric spaces. Moreover, some innovative techniques are employed to demonstrate the validity of our main result. Recently, authors [21] proved certain fixed point results and recognized some very interesting applications in the setting of G-metric spaces, following-up, application of results of this article to existence of solution of Bessel type boundary value problem is presented.

2. Fixed point theorem and graphical verification

In this section, firstly, we introduce the modified F-contractive mapping named as Boyd-Wong type cyclic F-contractions in G-metric space.

Let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

(1) ϕ is upper semi-continuous, i.e., for any sequence $\{t_n\}$ in $[0, \infty)$ such that $t_n \to t$ as $n \to \infty$, we have $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t)$;

(2) $\phi(t) < t$ for each $t > 0$.

Let Ψ denote the set of all continuous functions $\varphi : (0, \infty) \to (0, \infty)$.

Definition 2.1. Let (X, G) be a G-metric space. Let m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X and $Y = \cup_{i=1}^{m} A_i$. An operator $T : Y \to Y$ is called a $G - l$ cyclic F-contraction in X if the following conditions hold:

(1) $\cup_{i=1}^{m} A_i$ is cyclic representation of Y with respect to T.
(2) There exist $\varphi \in \Psi, \phi \in \Phi$ and $F \in \Delta_F$, such that for the condition $G(Tx, T^{l+1}x, Ty) > 0$, we have

$$\varphi(G(x, T^l x, y)) + F(G(Tx, T^{l+1}x, Ty)) \leq F(\phi(G(x, T^l x, y))), \tag{2.1}$$

for all $x \in A_i, y \in A_{i+l},\ i = 1, 2, \cdots, m,\ l \in N,$ *with* $A_{m+1} = A_1,\ A_{m+2} = A_2\ , A_{m+3} = A_3\ \cdots$.

Remark 2.2. If we set $\varphi(G(x, Tx, y)) = \tau > 0,\ \ l = 1$ and utilizing the fact that $\phi(t) < t$ then Condition (2.1) reduces to cyclic version of Wardowski's F-contraction [24] in the context of G-metric spaces, as follows:

$$\text{If }\ G(Tx, T^2x, Ty) > 0,\ \text{ then }\ \tau + F(G(Tx, T^2x, Ty)) \leq F(G(x, Tx, y)). \tag{2.2}$$

From $(F1)$ and (2.2), it is easy to conclude that F-contraction T is a cyclic contractive mapping in the setting of G-metric spaces.

Our main result runs as follows.

Theorem 2.3. *Let (X, G) be a G- complete metric space. Let $T : Y \to Y$ be a continuous $G - l$ cyclic F-contraction on X. Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\cap_{i=1}^{m} A_i$.*

Proof. For $x_0 \in A_1$, we construct the sequence $\{x_n\}$ by $Tx_n = x_{n+1}$ for $n \in N \cup \{0\}$. Since T is cyclic, $x_0 \in A_1, x_1 = Tx_0 \in A_2, x_2 = Tx_1 \in A_3, \cdots,$ and so on. Notice that if $x_{n_0} = x_{n_0+l}$ for some $n_0 \in N \cup \{0\}$ and $l \in N$. Then obviously T has a fixed point and the proof is completed. Thus we assume that $x_n \neq x_{n+l}$ for every $n \in N \cup \{0\}, l \in N$. Therefore we have $G(x_n, x_{n+l}, x_{n+l}) > 0$. Utilizing the fact that $Y = \cup_{i=1}^{m} A_i$ then for each $n \in N \cup \{0\}$, there exist $i_n \in \{1, 2, \cdots, m\}$ such that $x_n \in A_{i_n},\ x_{n+1} = Tx_n \in T(A_{i_n}) \subseteq A_{i_n+1}, \cdots$ and with the similar approach we can find that $x_n \in A_{i_n+l}$. Then employing condition (2.1) with $x = x_{n-1}$ and $y = x_{n+l-1}$, one acquires

$$\begin{aligned} F(G(x_n, x_{n+l}, x_{n+l})) =& F(G(Tx_{n-1}, T^{l+1}x_{n-1}, Tx_{n+l-1})) \\ \leq& F(\phi(G(x_{n-1}, T^l x_{n-1}, x_{n+l-1}))) \\ &- \varphi((x_{n-1}, T^l x_{n-1}, x_{n+l-1})) \\ <& F(\phi(G(x_{n-1}, T^l x_{n-1}, x_{n+l-1}))) \\ =& F(\phi(G(x_{n-1}, x_{n+l-1}, x_{n+l-1}))) \\ <& F(G(x_{n-1}, x_{n+l-1}, x_{n+l-1})). \end{aligned} \tag{2.3}$$

In view of (F_1), it yields that for all $n \in N \cup \{0\}$, $l \in N$,

$$G(x_n, x_{n+l}, x_{n+l}) < G(x_{n-1}, x_{n+l-1}, x_{n+l-1}). \tag{2.4}$$

So $\{G(x_n, x_{n+l}, x_{n+l})\}$ is a decreasing sequence in R^+ and is bounded below at 0, consequently it is convergent to some point, say $p \in R^+$. Now we assert that $p = 0$.

Suppose to contrary that $p > 0$. Letting $n \to \infty$ in (2.3) and utilizing $(F3^{'})$, we have

$$\begin{aligned} F(p) &\leq F(\phi(p)) - \varphi(p) \\ &< F(p) - \varphi(p), \end{aligned}$$

which leads to contradiction. Then we must have $p = 0$, i.e.,

$$\lim_{n\to\infty} G(x_n, x_{n+l}, x_{n+l}) = 0. \tag{2.5}$$

Next, we claim that $\{x_n\}$ is a G-Cauchy sequence.

Suppose that $\{x_n\}$ is not G-Cauchy. Then there exists $\epsilon > 0$ we can find two sub-sequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that for every $n(k) \geq m(k) > k$,

$$G(x_{m(k)}, x_{m(k)+l}, x_{n(k)+l}) = G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l}) \geq \epsilon. \tag{2.6}$$

Now corresponding to $m(k)$, we can select $n(k)$ in such a manner that it is the smallest integer with $n(k) > m(k)$ satisfying (2.6). Consequently, we have

$$G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l-1}) < \epsilon. \tag{2.7}$$

By using Lemma 1.7 and property $(G5)$, we have

$$\begin{aligned}\epsilon \leq& G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l})\\ \leq& G(x_{n(k)+l}, x_{n(k)+l-1}, x_{n(k)+l-1}) + G(x_{n(k)+l-1}, T^l x_{m(k)}, x_{m(k)})\\ \leq& G(x_{n(k)+l}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+l-1}, x_{n(k)+l-1})\\ &+ G(x_{n(k)+l-1}, T^l x_{m(k)}, x_{m(k)})\\ \leq& G(x_{n(k)+l-1}, T^l x_{m(k)}, x_{m(k)}) + 2G(x_{n(k)}, x_{n(k)+l}, x_{n(k)+l})\\ &+ G(x_{n(k)}, x_{n(k)+l-1}, x_{n(k)+l-1})\\ \leq& G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l-1}) + S_{n(k)+l-1} + 2S_{n(k)+l}.\end{aligned}$$

Thus we obtain

$$\epsilon \leq G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l}) \leq \epsilon + S_{n(k)+l-1} + 2S_{n(k)+l}, \tag{2.8}$$

where $S_{n(k)+l} = G(x_{n(k)}, x_{n(k)+l}, x_{n(k)+l})$. Letting $k \to \infty$ in (2.8) and utilizing (2.5), it implies that

$$\lim_{n\to\infty} G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l}) = \epsilon. \tag{2.9}$$

Consider the following and on utilizing Lemma 1.7 and $(G5)$

$$\begin{aligned}&G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l})\\ &\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, T^l x_{m(k)}, x_{n(k)+l})\\ &= G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{n(k)+l}, x_{m(k)-1}, T^l x_{m(k)})\\ &\leq 2G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{n(k)+l}, x_{n(k)}, x_{n(k)})\\ &\quad + G(x_{n(k)}, x_{n(k)+l-1}, x_{n(k)+l-1}) + G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)})\\ &\leq 2G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + 2G(x_{n(k)}, x_{n(k)+l}, x_{n(k)+l})\\ &\quad + G(x_{n(k)}, x_{n(k)+l-1}, x_{n(k)+l-1}) + G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)})\\ &\leq 2S_{m(k)} + 2S_{n(k)+l} + S_{n(k)+l-1} + G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)}).\end{aligned}$$

Which on making $k \to \infty$, reduces to

$$\epsilon \leq \lim_{n\to\infty} G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)}). \tag{2.10}$$

By the similar approach as above, we can get

$$\begin{aligned}G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)}) \leq& 2S_{n(k)+l-1} + S_{n(k)+l} + S_{m(k)-1}\\ &+ G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l}).\end{aligned} \tag{2.11}$$

Letting $k \to \infty$ on the both sides, we have

$$\lim_{n\to\infty} G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)}) \leq \epsilon. \tag{2.12}$$

From (2.10) and (2.12), we obtain that

$$\lim_{n\to\infty} G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)}) = \epsilon. \tag{2.13}$$

Arguing as above, we have the following

$$\begin{aligned} G(x_{n(k)+l-1}, x_{m(k)-1}, T^l x_{m(k)}) &\leq 2S_{m(k)+l} + S_{m(k)+l-1} \\ &\quad + G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}). \end{aligned}$$

Letting $n \to \infty$ and utilizing (2.13), we have

$$\epsilon \leq \lim_{n\to\infty} G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}) \tag{2.14}$$

also considering the same approach, we have

$$\begin{aligned} G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}) &\leq S_{m(k)+l+1} + 2S_{m(k)+l-1} \\ &\quad + G(x_{m(k)-1}, T^l x_{m(k)}, x_{n(k)+l-1}). \end{aligned}$$

Taking $n \to \infty$, we have

$$\lim_{n\to\infty} G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}) \leq \epsilon. \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$\lim_{n\to\infty} G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}) = \epsilon. \tag{2.16}$$

Employing (2.1) with $x = x_{m(k)-1}$ and $y = x_{n(k)+l-1}$, we have

$$\begin{aligned} &\psi(G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1})) \\ &+ F(G(Tx_{m(k)-1}, T^{l+1} x_{m(k)-1}, Tx_{n(k)+l-1})) \\ &\leq F(\phi(G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}))). \end{aligned}$$

This implies that

$$\begin{aligned} F(G(x_{m(k)}, T^l x_{m(k)}, x_{n(k)+l})) &\leq F(\phi(G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1}))) \\ &\quad - \varphi(G(x_{m(k)-1}, T^l x_{m(k)-1}, x_{n(k)+l-1})). \end{aligned}$$

Taking $k \to \infty$ and using $(F3')$, (2.9), (2.16) and semi-continuity of ϕ, we have

$$\begin{aligned} F(\epsilon) &\leq F(\phi(\epsilon)) - \varphi(\epsilon) \\ &< F(\epsilon) - \varphi(\epsilon), \end{aligned}$$

which is a contradiction, and this show that $\{x_n\}$ is a G-Cauchy sequence. Since (X, G) is G-complete, it is G-convergent to a limit $w \in X$. Since $x_0 \in A_1$, the subsequence $\{x_{m(n-1)}\}_{n=1}^{\infty} \in A_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^{\infty} \in A_2$ and continuing in this process we find that $\{x_{mn-1}\}_{n=1}^{\infty} \in A_m$. All the m subsequences are G-convergent in the G-closed sets A_i and consequently, they are convergent to the same limit $w \in \cap_{i=1}^{m} A_i$. Application of continuity of T gives

$$\begin{aligned}G(w, Tw, w) &= \lim_{n\to\infty} G(x_n, Tx_n, x_{n+1})\\ &= \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\\ &= G(w, w, w)\\ &= 0.\end{aligned}$$

Which amounts to say that

$$Tw = w.$$

Therefore w is a fixed point of T.

In order to prove uniqueness of fixed point, suppose that $u, w \in \cap_{i=1}^{m} A_i$ are two fixed points of T such that $u \neq w$. Then, from $G(Tu, T^{l+1}u, Tw) > 0$, we have

$$\begin{aligned}F(G(u, u, w)) &= F(G(Tu, T^{l+1}u, Tw))\\ &\leq F(\phi G(u, T^{l}u, w)) - \varphi(G(u, T^{l}u, w))\\ &= F(\phi G(u, u, w)) - \varphi(G(u, u, w))\\ &< F(G(u, u, w)) - \varphi(G(u, u, w)),\end{aligned}$$

which is a contradiction. Then we must have $u = w$. Hence T has a unique fixed point in $\cap_{i=1}^{m} A_i$. This concludes the proof. □

Next, we furnish an illustrative example which demonstrates the validity of the hypotheses and degree of generality of Theorem 2.3.

Example 2.4. For $X = [0, 1]$, let

$$G(x, y, z) = \begin{cases} 0, \text{ if and only if } x = y = z, \\ \max\{x, y\} + \max\{y, z\} + \max\{x, z\}, \text{ otherwise.} \end{cases}$$

Then (X, G) is a complete G-metric space. Let A_1 and A_2 be two closed subsets of X, defined as $A_1 = [0, \frac{1}{2}]$ and $A_2 = [0, \frac{2}{3}]$. Then $Y := A_1 \cup A_2 = [0, \frac{2}{3}]$. Define the mapping $T : Y \to Y$ by $Ty = \frac{y^3}{1+y}$. Then we know that T is continuous, $T(A_1) = T([0, \frac{1}{2}]) = [0, \frac{1}{12}] \subset [0, \frac{2}{3}] = A_2$ and $T(A_2) = T([0, \frac{2}{3}]) = [0, \frac{8}{45}] \subset [0, \frac{1}{2}] = A_1$. Consequently $Y = A_1 \cup A_2$ has a cyclic representation of T. Let $F(\alpha) = \ln(\alpha)$ for all $\alpha \in R^{+}$. Taking $\varphi : (0, \infty) \to (0, \infty)$ by $\varphi(t) = \frac{t}{200}$ and $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(t) = \frac{50t}{51}$. In order to verify the Condition (2.1), we notice that for $x \in A_1$ and $y \in A_2$,

$$G(Tx, T^{l+1}x, Ty) > 0$$

if and only if

$$(x = 0 \wedge y > 0) \vee (x > 0 \wedge y = 0) \vee (x > 0 \wedge y > 0).$$

Case-I. For $x = 0, y > 0$, we have

$$G(x, T^l x, y) = G(0, 0, y) = 0 + y + y = 2y,$$
$$G(Tx, T^{l+1}x, Ty) = G(0, 0, \frac{y^3}{1+y}) = \frac{2y^3}{1+y}.$$

Consider the left hand side(LHS) and right hand side(RHS) of (2.1):

$$\begin{aligned}\varphi(G(x, T^l x, y)) + F(G(Tx, T^{l+1}x, Ty)) &= \varphi(2y) + F(\frac{2y^3}{1+y}) \\ &= \frac{y}{100} + \ln(\frac{2y^3}{1+y})\end{aligned}$$

and

$$\begin{aligned}F(\phi(G(x, T^l x, y))) &= F(\phi(2y)) \\ &= F(\frac{50(2y)}{51}) \\ &= \ln(\frac{100y}{51}).\end{aligned}$$

Following figure shows that the RHS expression (with blue curve/surface) dominates the LHS expression (with purple curve/surface) for $x, y \in [0, 1]$, which validates our inequality.

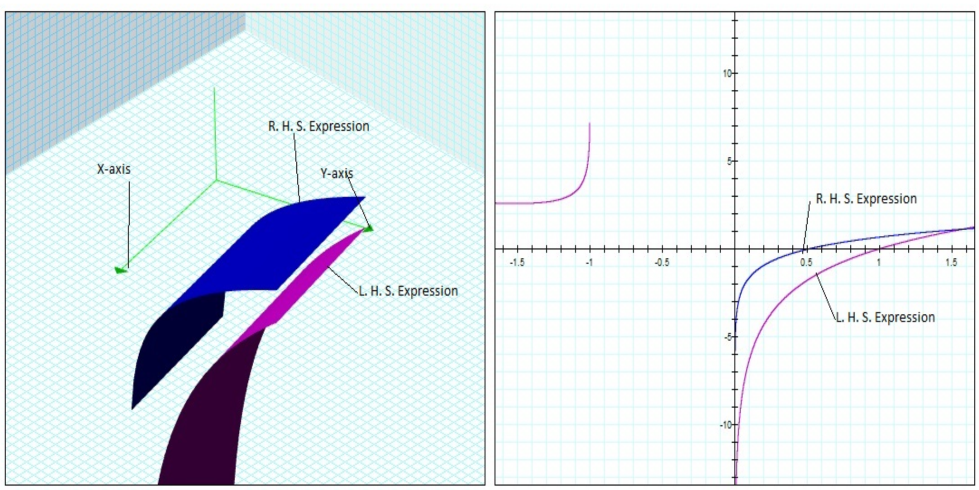

FIGURE 1. Plot of inequality for Case 1

Case-II. For $x > 0$ and $y = 0$, we have

$$\begin{aligned} G(x, T^l x, y) =& 2x + T^l x, \\ G(Tx, T^{l+1}x, Ty) =& 2Tx + T^{l+1}x. \end{aligned}$$

Now, we consider the sides of (2.1) for different values of $l \in N$.
If $l = 1$, then LHS is

$$\begin{aligned} &\varphi(G(x, Tx, y)) + F(G(Tx, T^2x, Ty)) \\ &= \varphi(2x + Tx) + F(2Tx + T^2x) \\ &= \frac{2x + \frac{x^3}{1+x}}{200} + \ln\left(2\frac{x^3}{1+x} + \frac{(\frac{x^3}{1+x})^3}{1 + \frac{x^3}{1+x}}\right) \end{aligned}$$

and RHS is

$$\begin{aligned} F(\phi(G(x, T^l x, y))) &= F(\phi(2x + Tx)) \\ &= \ln(\frac{50(2x + \frac{x^3}{1+x})}{51}). \end{aligned}$$

Subsequent figure demonstrates that the RHS with red surface dominates the purple surface, i.e. LHS and they interchange their domination after $x = 1.58$, this amounts to say that for $x, y \in [0, 1]$, inequality (2.1) is satisfied.

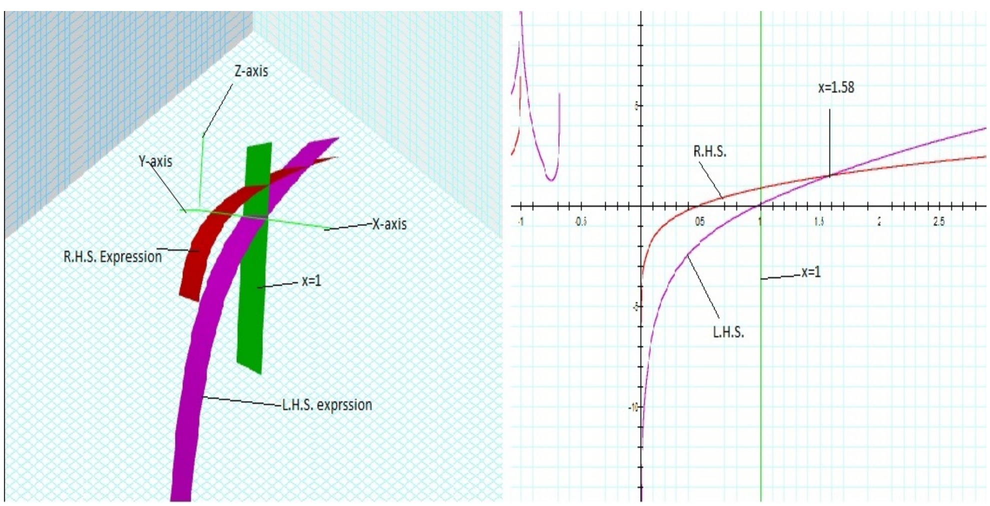

FIGURE 2. Plot of inequality for Case 2, when $l = 1$

If $l = 2$, then LHS is

$$\begin{aligned}
&\varphi(G(x, T^2x, y)) + F(G(Tx, T^3x, Ty)) \\
&= \varphi(2x + T^2x) + F(2Tx + T^3x) \\
&= \frac{2x + \frac{(\frac{x^3}{1+x})^3}{1+\frac{x^3}{1+x}}}{200} + \ln\left(2\frac{x^3}{1+x} + \frac{(\frac{(\frac{x^3}{1+x})^3}{1+\frac{x^3}{1+x}})^3}{1 + \frac{(\frac{x^3}{1+x})^3}{1+\frac{x^3}{1+x}}}\right).
\end{aligned}$$

and RHS is

$$\begin{aligned}
F(\phi(G(x, T^2x, y))) &= F(\phi(2x + T^2x)) \\
&= \ln\left(\frac{50\left(2x + \frac{(\frac{x^3}{1+x})^3}{1+\frac{x^3}{1+x}}\right)}{51}\right).
\end{aligned}$$

Following figure demonstrates that RHS with red surface dominates the purple surface i.e. LHS and they interchange their domination after $x = 1.5974$, this shows that for $x, y \in [0, 1]$, inequality (2.1) is satisfied.

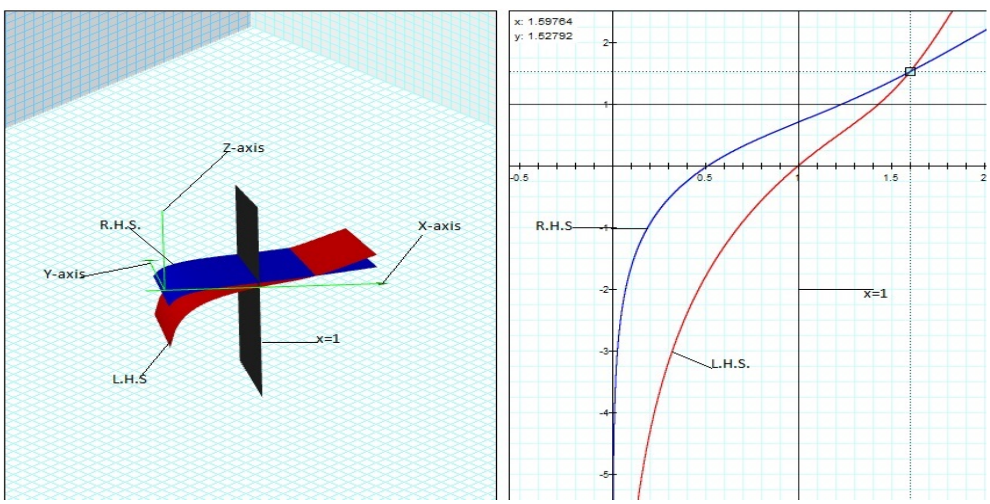

Figure 3. Plot of inequality for Case 2, when $l = 2$

With the similar approach one can verify that for all $l \in N$, inequality (2.1) is satisfied in this case.

Case-III. Let $x > 0$ and $y > 0$. Now, following sub-cases arise.

(i) If $x < y$, then inequality (2.1) is represented by following inequality:

$$\frac{x+2y}{200} + \ln(\frac{x^3}{1+x} + 2\frac{y^3}{1+y}) \leq \ln(\frac{50(x+2y)}{51})$$

which is verified by following figure.

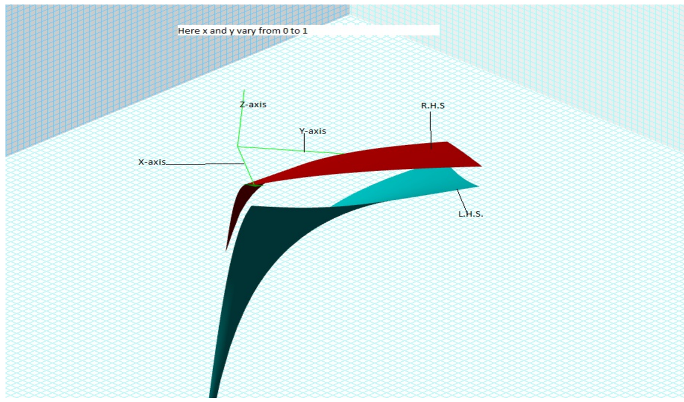

FIGURE 4. Plot of inequality for Case 3(i)

(ii) If $T^l x < x < y$, then inequality (2.1) becomes

$$\frac{2x+y}{200} + \ln(2\frac{x^3}{1+x} + \frac{y^3}{1+y}) \leq \ln(\frac{50(2x+y)}{51})$$

and following figure verifies above inequality and also shows that inequality holds good up to $x = 1.59$.

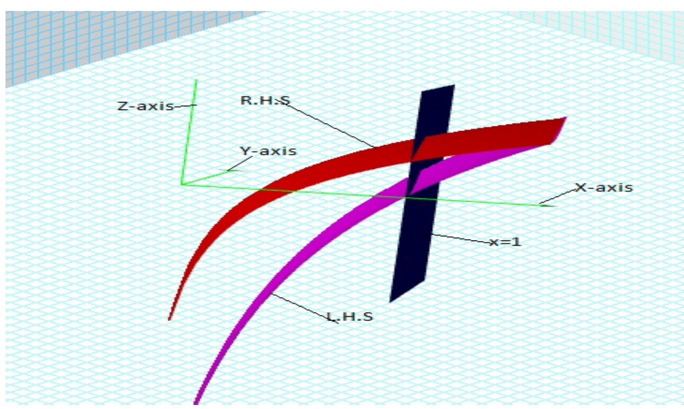

FIGURE 5. Plot of inequality for Case 3(ii)

(iii) If $y < T^l x$, then this case is similar to Case 2, and hence verified.
(iv) If $x = y$, then inequality (2.1) takes the form

$$\frac{3x}{200} + \ln(3\frac{x^3}{1+x}) \leq \ln(\frac{50(3x)}{51}).$$

Subsequent figure shows the validity of above inequality, which is valid up to $x = 1.56$.

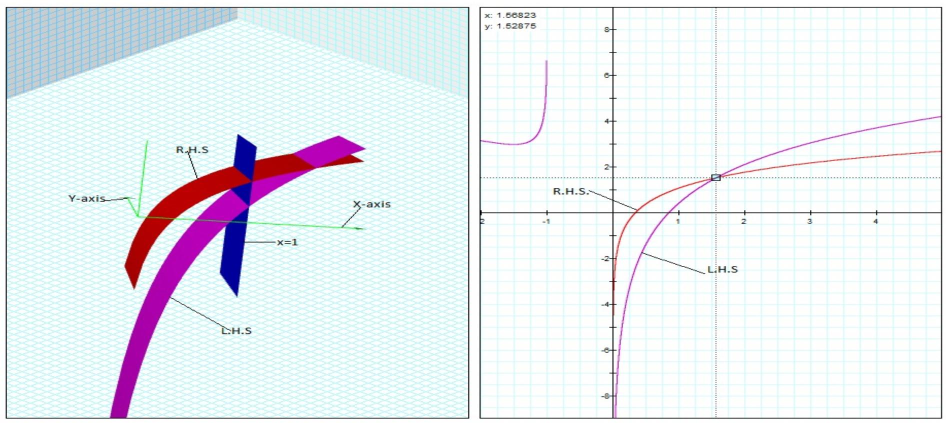

FIGURE 6. Plot of inequality for Case 3(iv)

Thus, for all possible cases, we immediately conclude that

$$\psi(G(x, Tx, y)) + F(G(Tx, T^2x, Ty)) \leq F(\phi(G(x, Tx, y))),$$

for all $x \in A_1$ and $y \in A_2$. Thus all the conditions of Theorem 2.3 are satisfied and T has a fixed point $x = 0 \in A_1 \cap A_2$, which is indeed unique in $[0, 1]$ and shown by the following figure.

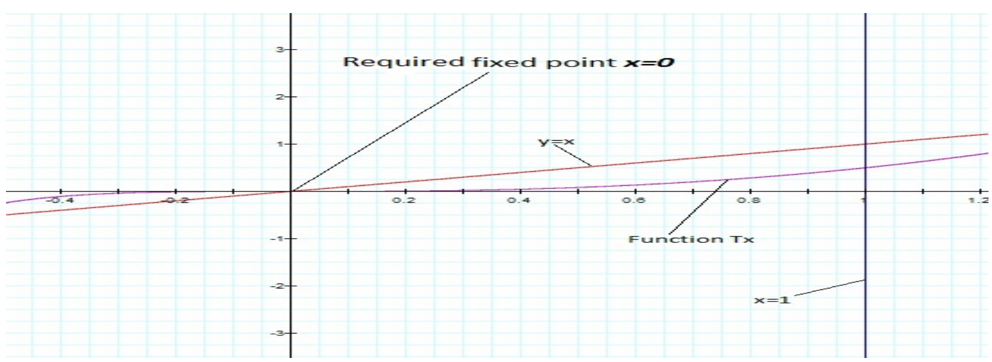

FIGURE 7. Unique Fixed Point of Tx in [0,1]

If we put $\varphi(t) = \tau > 0$, in Theorem 2.3, resulting following corollary which can be demonstrated as $G - l$ cyclic version of Wardowski's Theorem [24], in the context of G-metric spaces.

Corollary 2.5. *Let (X, G) be a complete G-metric space and $\{A_i\}_{i=1}^m$ be a finite family of nonempty G-closed subsets of X with $Y := \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a map satisfying $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1, A_{m+2} = A_2 \cdots$. Suppose that there exists $\phi \in \Phi$ and $F \in \Delta_F$ such that for $G(Tx, T^{l+1}x, Ty) > 0$,*

$$\tau + F(G(Tx, T^{l+1}x, Ty)) \leq F(\phi(G(x, T^l x, y))),$$

for all $x \in A_i$ and $y \in A_{i+l}$. Then T has a unique fixed point in $\cap_{i=1}^m A_i$.

If we set $l = 1$ in Corollary 2.5, then cyclic Boyd-Wong type fixed point result through F-contractive mapping in the setting of G-metric spaces is obtained.

Corollary 2.6. *Let (X, G) be a complete G-metric space and $\{A_i\}_{i=1}^m$ be a finite family of nonempty G-closed subsets of X with $Y := \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a map satisfying $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$. Suppose that there exists $\phi \in \Phi$ and $F \in \Delta_F$ such that for $G(Tx, T^2x, Ty) > 0$,*

$$\tau + F(G(Tx, T^2x, Ty)) \leq F(\phi(G(x, Tx, y))),$$

for all $x \in A_i$ and $y \in A_{i+1}$. Then T has a unique fixed point in $\cap_{i=1}^m A_i$.

Utilizing the fact that $\phi(t) < t$ in Corollary 2.6, Wardowski type cyclic fixed point theorem in the framework of G- metric spaces is attained.

Corollary 2.7. *Let (X, G) be a complete G-metric space and $\{A_i\}_{i=1}^m$ be a finite family of nonempty G-closed subsets of X with $Y := \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a map satisfying $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$. Suppose that there exists $\phi \in \Phi$ and $F \in \Delta_F$ such that for $G(Tx, T^2x, Ty) > 0$,*

$$\tau + F(G(Tx, T^2x, Ty)) \leq F((G(x, Tx, y))),$$

for all $x \in A_i$ and $y \in A_{i+1}$. Then T has a unique fixed point in $\cap_{i=1}^m A_i$.

3. Existence theorem to Bessel-type boundary value problem

In this section, we use the fixed point results discussed earlier, to establish the existence of solution of the following Bessel-type boundary value problem.

$$\begin{cases} t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} = K(t, u(t)); \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $K : [0, 1] \times R^+ \to R$ is a continuous function. This problem is equivalent to the integral equation:

$$u(t) = \int_0^1 G(t,s)K(s,u(s))ds, \quad t \in [0,1], \tag{3.2}$$

where $G(t,s)$ is the Green's function

$$G(t,s) = \begin{cases} \frac{s}{2t}(1-t^2), & 0 \le s < t \le 1, \\ \frac{t}{2s}(1-s^2), & 0 \le t < s \le 1. \end{cases} \tag{3.3}$$

Let $X = C([0,1], R^+)$ be the set of all non-negative continuous real functions defined on $[0,1]$. For an arbitrary $u \in X$, we define

$$\|u\| = \sup_{t\in[0,1]} \{|u(t)|\}. \tag{3.4}$$

Define $G : X \times X \times X \to R^+$ by

$$G(u,v,w) = \max\{\|u-v\|, \|v-w\|, \|w-u\|\}, \tag{3.5}$$

where $\|u\|$ is defined by (3.4). Then clearly (X, G) is a complete G-metric space.

Consider the self-map $T : X \to X$ defined by

$$Tu(t) = \int_0^1 G(t,s)K(s,u(s))ds, \ t \in [0,1]. \tag{3.6}$$

Then clearly u^* is a solution of (3.2) if and only if u^* is a fixed point of T.

Now we prove the existence theorem of the Bessel-type boundary value problem by using the fixed point theorem.

Theorem 3.1. *Suppose the following hypotheses hold:*

(i) *K is a nonincreasing function.*

(ii) *There exists $\tau > 0$ such that*

$$|K(s,u) - K(s,v)| \le e^{-\tau}|u-v|,$$

for all $s \in [0,1]$ and $u, v \in R^+$.

(iii) *There exist $\xi, \eta \in X$ such that $\xi(t) \le \eta(t)$ for $t \in [0,1]$ and that $T\xi(t) \le \eta(t)$ and $T\eta(t) \ge \xi(t)$ for $t \in [0,1]$.*

Then the integral equation (3.2) has a solution $u^* \in X$ and it belongs to

$$A = \{u \in X : \xi(t) \le u(t) \le \eta(t), \ \ t \in [0,1]\}.$$

Proof. In order to prove the existence of a fixed point of T, we construct closed subsets A_1 and A_2 of X as follows.

$$A_1 = \{u \in X : u(t) \le \eta(t), \ \ t \in [0,1]\}$$

and

$$A_2 = \{u \in X : \xi(t) \le u(t), ; \ \ t \in [0,1]\}.$$

Let $u \in A_1$, that is

$$u(s) \leq \eta(s), \quad for \;\; all \;\; s \in [0,1].$$

Since $G(t,s) \geq 0$ for all $t, s \in [0,1]$, from hypotheses (i) and (iii), it is deduced that

$$\int_0^1 G(t,s)K(s,u(s))ds \geq \int_0^1 G(t,s)K(s,\eta(s))ds \geq \xi(t), \quad t \in [0,1].$$

Hence, we have $Tu \in A_2$. This implies that $T(A_1) \subseteq A_2$. Using the similar approach as above, the other inclusion is proved. Hence $Y := A_1 \cup A_2$ is a cyclic representation of Y with respect to T.

Let $u \in A_1$ and $v \in A_2$. Then clearly $Tu(t) \neq Tv(t)$ and we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &\leq \int_0^1 G(t,s)|K(s,u(s)) - K(s,v(s))|ds \\
&\leq \int_0^1 G(t,s)e^{-\tau}|u(s) - v(s)|ds \\
&\leq e^{-\tau}\|u - v\| \int_0^1 G(t,s)ds.
\end{aligned}$$

This implies that

$$|Tu(t) - Tv(t)| \leq e^{-\tau}\|u - v\| \int_0^1 G(t,s)ds$$

or equivalently,

$$\|Tu(t) - Tv(t)\| \leq e^{-\tau}\|u - v\| \int_0^1 G(t,s)ds. \tag{3.7}$$

Similarly, we can derive that

$$\|T^2u(t) - Tv(t)\| \leq e^{-\tau}\|Tu - v\| \int_0^1 G(t,s)ds \tag{3.8}$$

and

$$\|Tu(t) - T^2u(t)\| \leq e^{-\tau}\|u - Tu\| \int_0^1 G(t,s)ds. \tag{3.9}$$

From (3.7), (3.8) and (3.9), we can get

$$\begin{aligned}
&\max\{\|Tu(t) - Tv(t)\|, \|T^2u(t) - Tv(t)\|, \|Tu(t) - T^2u(t)\|\} \\
&\leq e^{-\tau}\max\{\|u - v\|, \|Tu - v\|, \|u - Tu\|\} \int_0^1 G(t,s)ds.
\end{aligned}$$

This leads to say that

$$G(Tu, T^2u, Tv) \leq e^{-\tau}G(u, Tu, v) \int_0^1 G(t,s)ds.$$

It is clear that

$$\int_0^1 G(t,s)ds = -t\ \log\sqrt{t} < \frac{1}{10},\ \ for\ \ all\ \ t \in [0,1].$$

Thus, we have

$$G(Tu, T^2u, Tv) \leq e^{-\tau}\frac{G(u, Tu, v)}{10}$$

Taking the logarithm on the both sides, we can obtain

$$\ln(G(Tu, T^2u, Tv)) \leq \ln(e^{-\tau}\frac{G(u, Tu, v)}{10}),$$

that is,

$$\tau + \ln(G(Tu, T^2u, Tv)) \leq \ln(\frac{G(u, Tu, v)}{10}).$$

Here, we notice that the function $F : R^+ \to R$ defined by $F(\alpha) = \ln(\alpha)$ for each $\alpha \in C([0,1], R)$ and for $\tau > 0$, is in Δ_F. Consequently all the conditions of Corollary 2.6 are satisfied by operator T with $\phi(t) = \frac{t}{10}$. Therefore, T has a unique fixed point $u^* \in A_1 \cup A_2$, that is, $u^* \in A$ is the unique solution to integral equation (3.2) and hence Bessel type boundary value problem (3.1) has a solution. □

Acknowledgments: This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of Korea(2015R1D1A1A09058177) and was completed while the first author was visiting Department of Mathematics Education, Kyungnam University, Korea.

References

[1] M. Abbas, J.K. Kim and T. Nazir, *Common fixed point of mappings satisfying almost generalized contractive condition in partially ordered G-metric spaces,* J. of Comput. Anal. and Appl., **19** (2015), 928-937.

[2] M. Asadi, E. Karapinar and P. Salimi, *A new approach to G- metric and related fixed point theorems,* Jour. of Inequ. and Appl., **2013:254**, (2013).

[3] M. Asadi and P. Salimi, *Some fixed point and common fixed point theorems on G-metric spaces,* Nonlin. Funct. Anal. and Appl., **21(3)** (2016), 523-530.

[4] H. Aydi, M. Postolache and W. Shatanawi, *Coupled fixed point results for (ψ, φ)-weakly contractive mappings in ordered G-metric spaces,* Comput. Math. Appl., **63(1)** (2012), 298-309.

[5] H. Aydi, B. Damjanovic, B. Samet and W. Shatanawi, *Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces,* Math. Comput. Model., **54** (2011), 2443-2450.

[6] S. Banach, *Sur les operations dons les ensembles abstraits et leur application aux equations integrales,* Fundam. Math., **3** (1922), 133-181.

[7] M. Bousselsal and B. Almohaimeed, *Coincidence and fixed point theorems for a new contraction principle in partially ordered G-metric spacea andappications to ordinary differential equations,* Nonlin. Funct. Anal. and Appl., **20(2)** (2015), 223-245.

[8] M. Jleli and B. Samet, *Remarks on G-metric spaces and fixed point theorems,* Fixed Point Theory Appl., **2012:210,** (2012).

[9] E. Karapinar and R.P. Agarwal, *Further fixed point results on G-metric spaces,* Fixed Point Theory Appl., **2013:154,** (2013).

[10] W.A. Kirk, P.S. Srinavasan and P.Veeramani, *Fixed points for mapping satisfying cyclical contractive conditions,* Fixed Point Theory, **4** (2003), 79-89.

[11] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces.* J. Nonlinear Convex Anal., **7** (2006), 289-297.

[12] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory.* Ph.D. thesis, The University of Newcastle, Australia, (2005).

[13] Z. Mustafa and B. Sims, F̂ixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory Appl., **2009,** Article ID 917175 (2009).

[14] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces.* Fixed Point Theory Appl., **2008,** Article ID 189870 (2008).

[15] H. Piri and P. Kumam, *Some Fixed Point Theorems Concerning F-Contraction in Complete Metric Spaces*, Fixed Point Theory Appl., **2014:210,** (2014).

[16] M. Pacurar and I.A. Rus, *Fixed point theory for cyclic φ- contractions,* Nonlinear Anal., **72** (2010), 1181-1187.

[17] G. Petrusel, *Cyclic representations and periodic points*, Studia Univ. Babes-Bolyai Math., **50** (2005), 107-112.

[18] B. Samet, C. Vetro and F. Vetro, *Remarks on G-metric spaces.* Int. J. Anal., **2013**, Article ID 917158 (2013).

[19] N.A. Secelean, *Iterated function systems consisting of F-contractions.* Fixed Point Theory Appl., **2013,** Article ID 277 (2013). doi:10.1186/1687-1812-2013-277.

[20] D. Singh, V. Joshi, N. Singh and P. Kumam, *Fixed point results for generalized F-contractive and Roger Hardy type F-contractive mappings in G-metric spaces,* RACSAM, **(2016)**. doi:10.1007/s13398-016-0305-3.

[21] D. Singh, V. Joshi, M. Imdad and P. Kumam, *Fixed point theorems via generalized F-contraction with applications to functional equations occurring in dynamic programming,* Jour. Fixed Point Theory Appl., **19(2)** (2017), 1453-1479.

[22] D. Singh, V. Joshi, J.K. Kim, *Some Best Proximity Point and Fixed Point Theorems via Generalized Cyclic Contraction in G-Metric Spaces,* Global J Pure Appl. Math., **11(6)** (2015), 5203-5222.

[23] D. Singh, V. Joshi and J.K. Kim, *Unifying some implicit common fixed point theorems for hybrid pairs of mappings in G-metric spaces through altering distance function,* Appl. Math. Inform. Sci., **10(5)** (2016), 1787-1798.

[24] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory Appl., **2012,** Article ID 94 (2012).