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EXISTENCE AND UNIQUENESS OF SUZUKI TYPE RESULT IN S_b-METRIC SPACES WITH APPLICATION TO INTEGRAL EQUATIONS

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Abstract. In this paper we prove a Suzuki type unique common coupled fixed point theorem for two pairs of w-compatible mappings along with $(\psi - \phi)$ - and Rational contraction conditions in S_b-metric spaces. We also furnish an example as well as application to integral equation.

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1. Introduction

In 2008, Suzuki [12] generalized the Banach contraction principle [2].

Theorem 1.1. ([12]) *Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to \left(\frac{1}{2}, 1\right]$ by*

$$\theta(r) = \begin{cases} 1 & if \ \ 0 \le r \le \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2} & if \ \ \frac{(\sqrt{5}-1)}{2} \le r \le \quad 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & if \ \ 2^{-\frac{1}{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Bhaskar and Lakshmikantham [4] introduced the notion of coupled fixed point and they provide some coupled fixed point results also.

Recently Sedghi *et al.* [9] defined S_b-metric spaces using the concept of S-metric spaces [10].

The aim of this paper is to prove Suzuki type unique common coupled fixed point theorem for four mappings satisfying generalized contractive condition in a S_b-metric space. Throughout this paper $\mathcal{R}, \mathcal{R}^+$ and $\mathcal{N}$ denote the set of all real numbers, non-negative real numbers and positive integers, respectively.

First we recall some definitions, lemmas and examples.

Definition 1.2. ([10]) Let X be a non-empty set. A $S-$metric on X is a function $S : X^3 \to [0, +\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$,

$(S1)$ $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \ne y \ne z$,
$(S2)$ $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
$(S3)$ $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

Then the pair (X, S) is called a S-metric space.

Definition 1.3. ([9]) Let X be a non-empty set and $b \ge 1$ be given real number. Suppose that $S : X^3 \to [0, \infty)$ is a function satisfying the following properties:

(S_b1) $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \ne y \ne z$,
(S_b2) $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
(S_b3) $S(x, y, z) \le b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$.

Then the function S is called a S_b-metric on X and the pair (X, S) is called a S_b-metric space.

Remark 1.4. ([9]) It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.

Following example shows that a S_b-metric on X need not be a S-metric on X.

Example 1.5. ([9]) Let (X, S) be a S-metric space, and $S_*(x, y, z) = S(x, y, z)^p$, where $p > 1$ is a real number. Note that S_* is a S_b-metric with $b = 2^{2(p-1)}$. Also, (X, S_*) is not necessarily a S-metric space.

Definition 1.6. ([9]) Let (X, S) be a S_b-metric space. Then, for $x \in X$, $r > 0$ we defined the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows, respectively:

$$\begin{aligned} B_S(x,r) &= \{y \in X : S(y,y,x) < r\}, \\ B_S[x,r] &= \{y \in X : S(y,y,x) \leq r\}. \end{aligned}$$

Lemma 1.7. ([9]) *In a S_b-metric space, we have*

$$S(x, x, y) \leq bS(y, y, x)$$

and

$$S(y, y, x) \leq bS(x, x, y).$$

Lemma 1.8. ([9]) *In a S_b-metric space, we have*

$$S(x, x, z) \leq 2bS(x, x, y) + b^2 S(y, y, z).$$

Definition 1.9. ([9]) If (X, S) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$.
(2) S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S(x_n, x_n, x) < \epsilon$ or $S(x, , x, x_n) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim\limits_{n\to\infty} x_n = x$.

Definition 1.10. ([9]) A S_b-metric space (X, S) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma 1.11. ([9]) *Let (X, S) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x. Then we have*

(i) $\frac{1}{2b} S(y, x, x) \leq \lim\limits_{n\to\infty} \inf S(y, y, x_n) \leq \lim\limits_{n\to\infty} \sup S(y, y, x_n) \leq 2bS(y, y, x)$,
(ii) $\frac{1}{b^2} S(x, x, y) \leq \lim\limits_{n\to\infty} \inf S(x_n, x_n, y) \leq \lim\limits_{n\to\infty} \sup S(x_n, x_n, y) \leq b^2 S(x, x, y)$ *for all $y \in X$.*

In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} S(x_n, x_n, y) = 0$.

Definition 1.12. ([4]) An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.13. ([5]) An element $(x, y) \in X \times X$ is called

(i) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.

(ii) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

2. Main Results

Now, we give our main results. Let Ψ be denotes the set of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying:

(ψ_1) ψ is continuous and monotonically increasing,

(ψ_2) $\psi(at) = a\psi(t)$, where a is constant and $t \in \mathbb{R}^+$.

Let Φ be denotes the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying:

(ϕ_1) ϕ is lower semi continuous,

(ϕ_2) $\phi(t) < t$ for $t > 0$.

Theorem 2.1. *Let (X, S) be a S_b-metric space. Suppose that $A, B : X \times X \to X$ and $P, Q : X \to X$ are satisfied:*

(2.1.1) $A(X \times X) \subseteq Q(X), B(X \times X) \subseteq P(X)$,

(2.1.2) $\{A, P\}$ *and* $\{B, Q\}$ *are w-compatible pairs,*

(2.1.3) *One of $P(X)$ or $Q(X)$ is S_b-complete subspace of X,*

(2.1.4)
$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(A(x,y),A(x,y),Px), S(B(u,v),B(u,v),Qu), \\ S(A(y,x),A(y,x),Py), S(B(v,u),B(v,u),Qv) \end{array}\right\} \le \max\left\{\begin{array}{c} S(Px,Px,Qu), \\ S(Py,Py,Qv) \end{array}\right\}$$

implies that

$$\psi\left(S(A(x,y),A(x,y),B(u,v))\right) \le \frac{1}{5b^{12}}\psi\left(M\left(x,y,u,v\right)\right) - \phi\left(M\left(x,y,u,v\right)\right)$$

for all x, y, u, v in X, where $\psi \in \Psi$, $\phi \in \Phi$ and

$$M\left(x,y,u,v\right) = \max\left\{\begin{array}{c} S(Px,Px,Qu), S(Py,Py,Qv), \\ S(A(x,y),A(x,y),Px), S(A(y,x),A(y,x),Py), \\ S(B(u,v),B(u,v),Qu), S(B(v,u),B(v,u),Qv), \\ \frac{S(A(x,y),A(x,y),Qu)\ S(B(u,v),B(u,v),Px)}{1+S(Px,Px,Qu)}, \\ \frac{S(A(y,x),A(y,x),Qv)\ S(B(v,u),B(v,u),Py)}{1+S(Py,Py,Qv)} \end{array}\right\}.$$

Then A, B, P and Q have a unique common coupled fixed point in $X \times X$.

Proof. Let $x_0, y_0 \in X$. From (2.1.1), we can construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} A(x_{2n}, y_{2n}) &= Qx_{2n+1} = z_{2n}, \\ A(y_{2n}, x_{2n}) &= Qy_{2n+1} = w_{2n}, \\ B(x_{2n+1}, y_{2n+1}) &= Px_{2n+2} = z_{2n+1}, \\ B(y_{2n+1}, x_{2n+1}) &= Py_{2n+2} = w_{2n+1},\ n = 0, 1, 2, \cdots . \end{aligned}$$

Case (i) Suppose $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$ for some m.
Assume that $z_{2m+1} \neq z_{2m+2}$ or $w_{2m+1} \neq w_{2m+2}$. Since

$$\frac{1}{8b^3} \min \left\{ \begin{array}{c} S(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), Px_{2m+2}), \\ S(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), Qx_{2m+1}), \\ S(A(y_{2m+2}, x_{2m+2}), A(y_{2m+2}, x_{2m+2}), Py_{2m+2}), \\ S(B(y_{2m+1}, x_{2m+1}), B(y_{2m+1}, x_{2m+1}), Qy_{2m+1}) \end{array} \right\}$$

$$\leq \max \left\{ \ S(Px_{2m+2}, Px_{2m+2}, Qx_{2m+1}), S(Py_{2m+2}, Py_{2m+2}, Qy_{2m+1}) \ \right\},$$

from (2.1.4), we have

$$\begin{aligned} &\psi\left(S(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), B(x_{2m+1}, y_{2m+1}))\right) \\ &\leq \frac{1}{5b^{12}} \psi\left(M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right)\right) \\ &\quad -\phi\left(M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right)\right), \end{aligned}$$

where

$$\begin{aligned} &M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right) \\ &= \max \left\{ \begin{array}{l} S(z_{2m+1}, z_{2m+1}, z_{2m}), S(w_{2m+1}, w_{2m+1}, w_{2m}), \\ S(z_{2m+2}, z_{2m+2}, z_{2m+1}), S(w_{2m+2}, w_{2m+2}, w_{2m+1}), \\ S(z_{2m+1}, z_{2m+1}, z_{2m}), S(w_{2m+1}, w_{2m+1}, w_{2m}), \\ \frac{S(z_{2m+2}, z_{2m+2}, z_{2m+1})\ S(z_{2m+1}, z_{2m+1}, z_{2m})}{1+S(z_{2m+1}, z_{2m+1}, z_{2m})}, \\ \frac{S(w_{2m+2}, w_{2m+2}, w_{2m+1})\ S(w_{2m+1}, w_{2m+1}, w_{2m})}{1+S(w_{2m+1}, w_{2m+1}, w_{2m})} \end{array} \right\} \\ &= \max \left\{ \ S(z_{2m+2}, z_{2m+2}, z_{2m+1}), S(w_{2m+2}, w_{2m+2}, w_{2m+1}) \ \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\psi\left(S(z_{2m+2}, z_{2m+2}, z_{2m+1})\right) \\ &\leq \tfrac{1}{5b^{12}}\ \psi\left(\max \left\{ \begin{array}{c} S(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S(w_{2m+2}, w_{2m+2}, w_{2m+1}) \end{array} \right\}\right) \\ &\quad -\phi\left(\max \left\{ \begin{array}{c} S(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S(w_{2m+2}, w_{2m+2}, w_{2m+1}) \end{array} \right\}\right). \end{aligned}$$

Similarly, we can prove

$$\begin{array}{l}\psi\left(S(w_{2m+2},w_{2m+2},w_{2m+1})\right)\\ \leq \frac{1}{5b^{12}}\ \psi\left(\max\left\{\begin{array}{c}S(z_{2m+2},z_{2m+2},z_{2m+1}),\\ S(w_{2m+2},w_{2m+2},w_{2m+1})\end{array}\right\}\right)\\ \quad -\phi\left(\max\left\{\begin{array}{c}S(z_{2m+2},z_{2m+2},z_{2m+1}),\\ S(w_{2m+2},w_{2m+2},w_{2m+1})\end{array}\right\}\right).\end{array}$$

It follows that

$$\begin{array}{l}\psi\left(\max\left\{\begin{array}{c}S(z_{2m+2},z_{2m+2},z_{2m+1}),\\ S(w_{2m+2},w_{2m+2},w_{2m+1})\end{array}\right\}\right)\\ \leq \frac{1}{5b^{12}}\ \psi\left(\max\left\{\begin{array}{c}S(z_{2m+2},z_{2m+2},z_{2m+1}),\\ S(w_{2m+2},w_{2m+2},w_{2m+1})\end{array}\right\}\right)\\ \quad -\phi\left(\max\left\{\begin{array}{c}S(z_{2m+2},z_{2m+2},z_{2m+1}),\\ S(w_{2m+2},w_{2m+2},w_{2m+1})\end{array}\right\}\right).\end{array}$$

It follows that $z_{2m+2}=z_{2m+1}$ and $w_{2m+2}=w_{2m+1}$. Continuing in this process we can conclude that $z_{2m+k}=z_{2m}$ and $w_{2m+k}=w_{2m}$ for all $k\geq 0$. It follows that $\{z_{2m}\}$ and $\{w_{2m}\}$ are Cauchy sequences.

Case (ii) Assume that $z_{2n}\neq z_{2n+1}$ and $w_{2n}\neq w_{2n+1}$ for all n.
Put $S_n=\max\{S(z_{n+1},z_{n+1},z_n),S(w_{n+1},w_{n+1},w_n)\}$. Since

$$\frac{1}{8b^3}\min\left\{\begin{array}{c}S(A(x_{2n+2},y_{2n+2}),A(x_{2n+2},y_{2n+2}),Px_{2n+2}),\\ S(B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1}),Qx_{2n+1}),\\ S(A(y_{2n+2},x_{2n+2}),A(y_{2n+2},x_{2n+2}),Py_{2n+2}),\\ S(B(y_{2n+1},x_{2n+1}),B(y_{2n+1},x_{2n+1}),Qy_{2n+1})\end{array}\right\}$$

$$\leq \max\left\{\ S(Px_{2n+2},Px_{2n+2},Qx_{2n+1}),S(Py_{2n+2},Py_{2n+2},Qy_{2n+1})\ \right\},$$

from (2.1.4), we have

$$\begin{array}{rl}\psi\left(S(z_{2n+2},z_{2n+2},z_{2n+1})\right) & \leq \frac{1}{5b^{12}}\ \psi\left(M\left(x_{2m+2},y_{2m+2},x_{2m+1},y_{2m+1}\right)\right)\\ & \quad -\phi\left(M\left(x_{2m+2},y_{2m+2},x_{2m+1},y_{2m+1}\right)\right),\end{array}$$

where

$$\begin{array}{l}M\left(x_{2m+2},y_{2m+2},x_{2m+1},y_{2m+1}\right)\\ =\max\left\{\begin{array}{l}S(z_{2n+1},z_{2n+1},z_{2n}),S(w_{2n+1},w_{2n+1},w_{2n}),\\ S(z_{2n+2},z_{2n+2},z_{2n+1}),S(w_{2n+2},w_{2n+2},w_{2n+1}),\\ S(z_{2n+1},z_{2n+1},z_{2n}),S(w_{2n+1},w_{2n+1},w_{2n}),\\ \frac{S(z_{2n+2},z_{2n+2},z_{2n})\ S(z_{2n+1},z_{2n+1},z_{2n+1})}{1+S(z_{2n+1},z_{2n+1},z_{2n})},\\ \frac{S(w_{2n+2},w_{2n+2},w_{2n})\ S(w_{2n+1},w_{2n+1},w_{2n+1})}{1+S(w_{2n+1},w_{2n+1},w_{2n})}\end{array}\right\}\\ =\max\left\{\begin{array}{l}S(z_{2n+1},z_{2n+1},z_{2n}),S(z_{2n+2},z_{2n+2},z_{2n+1}),\\ S(w_{2n+1},w_{2n+1},w_{2n}),S(w_{2n+2},w_{2n+2},w_{2n+1})\end{array}\right\}\\ =\max\left\{\ S_{2n+1},S_{2n}\ \right\}.\end{array}$$

Therefore

$$\begin{aligned}\psi\left(S(z_{2n+2},z_{2n+2},z_{2n+1})\right) &\leq \frac{1}{5b^{12}}\,\psi\left(\max\left\{S_{2n+1},S_{2n}\right\}\right)\\ &\quad -\phi\left(\max\left\{S_{2n+1},S_{2n}\right\}\right).\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\psi\left(S(w_{2n+2},w_{2n+2},w_{2n+1})\right) &\leq \frac{1}{5b^{12}}\,\psi\left(\max\left\{S_{2n+1},S_{2n}\right\}\right)\\ &\quad -\phi\left(\max\left\{S_{2n+1},S_{2n}\right\}\right).\end{aligned}$$

Thus

$$\psi\left(S_{2n+1}\right) \leq \tfrac{1}{5b^{12}}\,\psi\left(\max\left\{\ S_{2n+1},S_{2n}\ \right\}\right)-\phi\left(\max\left\{\ S_{2n+1},S_{2n}\ \right\}\right).$$

If S_{2n+1} is maximum, then we get a contradiction so that S_{2n} is maximum. Thus

$$\begin{aligned}\psi\left(S_{2n+1}\right) &\leq \frac{1}{5b^{12}}\,\psi\left(S_{2n}\right)-\phi\left(S_{2n}\right) \\ &< \psi\left(S_{2n}\right).\end{aligned} \tag{2.1}$$

Similarly we can conclude that $\psi\left(S_{2n}\right)<\psi\left(S_{2n-1}\right)$. Since ψ is nondecreasing and continuous, it is clear that $\{S_n\}$ is a non-increasing sequence of non-negative real numbers and must converges to a real number say $k\geq 0$. Suppose $k>0$. Letting $n\to\infty$, in (2.1), we have

$$\psi(k)\leq\frac{1}{5b^{12}}\,\psi(k)-\phi(k)<\psi(k).$$

This is a contradiction. Hence $k=0$. Thus, we have

$$\lim_{n\to\infty}S(z_{n+1},z_{n+1},z_n)=0 \tag{2.2}$$

and

$$\lim_{n\to\infty}S(w_{n+1},w_{n+1},w_n)=0. \tag{2.3}$$

Now we prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in (X,S). On contrary we suppose that $\{z_{2n}\}$ and $\{w_{2n}\}$ are not Cauchy. Then there exist $\epsilon>0$ and monotonically increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that for $n_k>m_k$,

$$\max\{S(z_{2m_k},z_{2m_k},z_{2n_k}),S(w_{2m_k},w_{2m_k},w_{2n_k})\}\geq\epsilon \tag{2.4}$$

and

$$\max\{S(z_{2m_k},z_{2m_k},z_{2n_{k-2}}),S(w_{2m_k},w_{2m_k},w_{2n_{k-2}})\}<\epsilon. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\begin{aligned}
\epsilon \leq\ & M_k = \max\{S(z_{2m_k}, z_{2m_k}, z_{2n_k}), S(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
\leq\ & 2b \max\{S(z_{2m_k}, z_{2m_k}, z_{2m_k+2}), S(w_{2m_k}, w_{2m_k}, w_{2m_k+2})\} \\
& + b^2 \max\{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}), S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}))\} \\
\leq\ & 2b\,(2b \max\{S(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\}) \\
& + 2b\left(b^2 \max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k+2}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k+2})\}\right) \\
& + b^2\,(2b \max\{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}) \\
& + b^2\left(b^2 \max\{S(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k})\}\right) \\
=\ & 4b^3 \max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
& + 2b^4 \max\{S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\} \\
& + 2b^3 \max\{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\} \\
& + b^4 \max\{S(z_{2n_k+1}, z_{2n_k}, z_{2n_k}), S(w_{2n_k+1}, w_{2n_k}, w_{2n_k})\}.
\end{aligned}$$

Letting $k \to \infty$ and apply ψ on both sides, we have that

$$\begin{aligned}
& \psi\left(\frac{\epsilon}{2b^3}\right) \\
& \leq \lim_{k\to\infty} \psi\left(\max\{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}\right).
\end{aligned} \tag{2.6}$$

Now first we claim that

$$\begin{aligned}
& \frac{1}{8b^3} \min \left\{ \begin{array}{c} S(A(x_{2m_k+2}, y_{2m_k+2}), A(x_{2m_k+2}, y_{2m_k+2}), Px_{2m_k+2}), \\ S(B(x_{2n_k+1}, y_{2n_k+1}), B(x_{2n_k+1}, y_{2n_k+1}), Qx_{2n_k+1}), \\ S(A(y_{2m_k+2}, x_{2m_k+2}), A(y_{2m_k+2}, x_{2m_k+2}), Py_{2m_k+2}), \\ S(B(y_{2n_k+1}, x_{2n_k+1}), B(y_{2n_k+1}, x_{2n_k+1}), Qy_{2n_k+1}) \end{array} \right\} \\
& \leq \max \left\{ \begin{array}{c} S(Px_{2m_k+2}, Px_{2m_k+2}, Qx_{2n_k+1}), \\ S(Py_{2m_k+2}, Py_{2m_k+2}, Qy_{2n_k+1}) \end{array} \right\}.
\end{aligned} \tag{2.7}$$

On contrary, suppose that

$$\begin{aligned}
& \tfrac{1}{8b^3} \min \left\{ \begin{array}{c} S(A(x_{2m_k+2}, y_{2m_k+2}), A(x_{2m_k+2}, y_{2m_k+2}), Px_{2m_k+2}), \\ S(B(x_{2n_k+1}, y_{2n_k+1}), B(x_{2n_k+1}, y_{2n_k+1}), Qx_{2n_k+1}), \\ S(A(y_{2m_k+2}, x_{2m_k+2}), A(y_{2m_k+2}, x_{2m_k+2}), Py_{2m_k+2}), \\ S(B(y_{2n_k+1}, x_{2n_k+1}), B(y_{2n_k+1}, x_{2n_k+1}), Qy_{2n_k+1}) \end{array} \right\} \\
& > \max \left\{ \begin{array}{c} S(Px_{2m_k+2}, Px_{2m_k+2}, Qx_{2n_k+1}), \\ S(Py_{2m_k+2}, Py_{2m_k+2}, Qy_{2n_k+1}) \end{array} \right\}.
\end{aligned}$$

Now from (2.4), we have

$$\begin{aligned}
\epsilon &\leq \max\{S(z_{2m_k}, z_{2m_k}, z_{2n_k}), S(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leq 2b^2 \max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + b^2 \max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \\
&< 2b^2 \max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + b^2 \frac{1}{8b^3} \min \left\{ \begin{array}{c} S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\ S(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), \\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}) \end{array} \right\}.
\end{aligned}$$

Letting $k \to \infty$, we have $\epsilon \leq 0$. It is a contradiction. Hence the claim is holds, that is, (2.7) holds.

Now from (2.1.4), we have

$$\begin{aligned}
&\psi\left(S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1})\right) \\
&\leq \frac{1}{5b^{12}} \psi \left(\max \left\{ \begin{array}{l} S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\ S(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), \\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\ \frac{S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \end{array} \right\} \right) \\
&\quad - \phi \left(\max \left\{ \begin{array}{l} S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\ S(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), \\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\ \frac{S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \end{array} \right\} \right).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\psi\left(S(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1})\right)\\
&\le \tfrac{1}{5b^{12}}\psi\left(\max\left\{\begin{array}{l}
S(z_{2m_k+1},z_{2m_k+1},z_{2n_k}),\\
S(w_{2m_k+1},w_{2m_k+1},w_{2n_k}),\\
S(z_{2m_k+2},z_{2m_k+2},z_{2m_k+1}),\\
S(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1}),\\
S(z_{2n_k+1},z_{2n_k+1},z_{2n_k}),\\
S(w_{2n_k+1},w_{2n_k+1},w_{2n_k}),\\
\frac{S(z_{2m_k+2},z_{2m_k+2},z_{2n_k})\ S(z_{2n_k+1},z_{2n_k+1},z_{2m_k+1})}{1+S(z_{2m_k+1},z_{2m_k+1},z_{2n_k})},\\
\frac{S(w_{2m_k+2},w_{2m_k+2},w_{2n_k})\ S(w_{2n_k+1},w_{2n_k+1},w_{2m_k+1})}{1+S(w_{2m_k+1},w_{2m_k+1},w_{2n_k})}
\end{array}\right\}\right)\\
&\quad-\phi\left(\max\left\{\begin{array}{l}
S(z_{2m_k+1},z_{2m_k+1},z_{2n_k}),S(w_{2m_k+1},w_{2m_k+1},w_{2n_k}),\\
S(z_{2m_k+2},z_{2m_k+2},z_{2m_k+1}),\\
S(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1}),\\
S(z_{2n_k+1},z_{2n_k+1},z_{2n_k}),S(w_{2n_k+1},w_{2n_k+1},w_{2n_k}),\\
\frac{S(z_{2m_k+2},z_{2m_k+2},z_{2n_k})\ S(z_{2n_k+1},z_{2n_k+1},z_{2m_k+1})}{1+S(z_{2m_k+1},z_{2m_k+1},z_{2n_k})},\\
\frac{S(w_{2m_k+2},w_{2m_k+2},w_{2n_k})\ S(w_{2n_k+1},w_{2n_k+1},w_{2m_k+1})}{1+S(w_{2m_k+1},w_{2m_k+1},w_{2n_k})}
\end{array}\right\}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\psi\left(\max\left\{S(z_{2m_k+2},z_{2m_k+2},z_{2m_k+1}),S(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1})\right\}\right)\\
&\le \frac{1}{5b^{12}}\ \psi\left(\max\left\{\begin{array}{l}
S(z_{2m_k+1},z_{2m_k+1},z_{2n_k}),\\
S(w_{2m_k+1},w_{2m_k+1},w_{2n_k}),\\
S(z_{2m_k+2},z_{2m_k+2},z_{2m_k+1}),\\
S(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1}),\\
S(z_{2n_k+1},z_{2n_k+1},z_{2n_k}),S(w_{2n_k+1},w_{2n_k+1},w_{2n_k}),\\
\frac{S(z_{2m_k+2},z_{2m_k+2},z_{2n_k})\ S(z_{2n_k+1},z_{2n_k+1},z_{2m_k+1})}{1+S(z_{2m_k+1},z_{2m_k+1},z_{2n_k})},\\
\frac{S(w_{2m_k+2},w_{2m_k+2},w_{2n_k})\ S(w_{2n_k+1},w_{2n_k+1},w_{2m_k+1})}{1+S(w_{2m_k+1},w_{2m_k+1},w_{2n_k})}
\end{array}\right\}\right)\\
&\quad-\phi\left(\max\left\{\begin{array}{l}
S(z_{2m_k+1},z_{2m_k+1},z_{2n_k}),\\
S(w_{2m_k+1},w_{2m_k+1},w_{2n_k}),\\
S(z_{2m_k+2},z_{2m_k+2},z_{2m_k+1}),\\
S(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1}),\\
S(z_{2n_k+1},z_{2n_k+1},z_{2n_k}),\\
S(w_{2n_k+1},w_{2n_k+1},w_{2n_k}),\\
\frac{S(z_{2m_k+2},z_{2m_k+2},z_{2n_k})\ S(z_{2n_k+1},z_{2n_k+1},z_{2m_k+1})}{1+S(z_{2m_k+1},z_{2m_k+1},z_{2n_k})},\\
\frac{S(w_{2m_k+2},w_{2m_k+2},w_{2n_k})\ S(w_{2n_k+1},w_{2n_k+1},w_{2m_k+1})}{1+S(w_{2m_k+1},w_{2m_k+1},w_{2n_k})}
\end{array}\right\}\right).
\end{aligned}
\tag{2.8}
$$

But

$$\begin{aligned}
&\max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\}\\
&\le 2b\max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad + b^2\max\{S(z_{2m_k}, z_{2m_k}, z_{2n_k}), S(w_{2m_k}, w_{2m_k}, w_{2n_k})\}\\
&\le 2b\max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad + b^2\left(2b\max\{S(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\}\right)\\
&\quad + b^2\left(b^2\max\{S(z_{2n_k-2}, z_{2n_k-2}, z_{2n_k}), S(w_{2n_k-2}, w_{2n_k-2}, w_{2n_k})\}\right)\\
&< 2b\max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad + 2b^3\epsilon + b^4(2b\max\{S(z_{2n_k-2}, z_{2n_k-2}, z_{2n_k-1}), S(w_{2n_k-2}, w_{2n_k-2}, w_{2n_k-1})\})\\
&\quad + b^4\left(b^2\max\{S(z_{2n_{k-1}}, z_{2n_k-1}, z_{2n_k}), S(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k})\}\right)\\
&\le 2b\max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad + 2b^3\epsilon + b^7\max\{S(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\}\\
&\quad + 2b^6\max\{S(z_{2n_k-1}, z_{2n_k-1}, z_{2n_k-2}), S(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k-2})\}.
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty}\max\{S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \le 2b^3\epsilon.$$

Also, we have

$$\begin{aligned}
&\lim_{k\to\infty}\frac{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\\
&\le \lim_{k\to\infty}\frac{\begin{bmatrix}\left[2bS(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}) + b^2S(z_{2m_{k+1}}, z_{2m_{k+1}}, z_{2n_k})\right]\\ \left[2bS(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}) + b^2S(z_{2n_k}, z_{2n_k}, z_{2m_k+1})\right]\end{bmatrix}}{1+S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\\
&\le \lim_{k\to\infty}\frac{b^5S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})\ S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}{1+S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\\
&\le \lim_{k\to\infty} b^5S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})\\
&\le 2b^8\epsilon.
\end{aligned}$$

Similarly, we obtain that

$$\lim_{k\to\infty}\frac{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \le 2b^8\epsilon.$$

Letting $k \to \infty$ in (2.8). Then we have

$$\begin{aligned}
&\lim_{k\to\infty} \psi\left(\max\left\{S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\right\}\right)\\
&\leq \frac{1}{5b^{12}}\psi\left(\max\{2b^3\epsilon, 0, 0, 0, 0, 2b^8\epsilon, 2b^8\epsilon\}\right)\\
&\quad -\lim_{k\to\infty}\phi\left(\max\left\{\begin{array}{l}
S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}),\\
S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}),\\
S(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}),\\
S(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}),\\
S(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}),\\
S(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}),\\
\frac{S(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})},\\
\frac{S(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\ S(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})}
\end{array}\right\}\right)\\
&\leq \frac{1}{5b^{12}}\psi\left(2b^8\epsilon\right). \qquad (2.9)
\end{aligned}$$

Now letting $n \to \infty$ in (2.6), from (2.2), (2.3) and (2.9), we have

$$\psi\left(\frac{\epsilon}{2b^3}\right) \leq \frac{1}{5b^{12}}\psi\left(2b^8\epsilon\right).$$

This is a contradiction. Hence $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b-Cauchy sequences in (X, S). In addition,

$$\begin{aligned}
&\max\{S(z_{2n+1}, z_{2n+1}, z_{2m+1}), S(w_{2n+1}, w_{2n+1}, w_{2m+1})\}\\
&\leq 2b\max\{S(z_{2n+1}, z_{2n+1}, z_{2n}), S(w_{2n+1}, w_{2n+1}, w_{2n})\}\\
&\quad +b\max\{S(z_{2m+1}, z_{2m+1}, z_{2n}), S(w_{2m+1}, w_{2m+1}, w_{2n})\}\\
&\leq 2b\max\{S(z_{2n+1}, z_{2n+1}, z_{2n}), S(w_{2n+1}, w_{2n+1}, w_{2n})\}\\
&\quad +2b^2\max\{S(z_{2m+1}, z_{2m+1}, z_{2m}), S(w_{2m+1}, w_{2m+1}, w_{2m})\}\\
&\quad +b^2\max\{S(z_{2n}, z_{2n}, z_{2m}), S(w_{2n}, w_{2n}, w_{2m})\}.
\end{aligned}$$

It is clear that

$$S(z_{2n+1}, z_{2n+1}, z_{2m+1}) < \epsilon$$

and

$$S(w_{2n+1}, w_{2n+1}, w_{2m+1}) < \epsilon.$$

Therefore $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also S_b-Cauchy sequences in (X, S). Thus $\{z_n\}$ and $\{w_n\}$ are S_b-Cauchy sequences in (X, S).

Suppose $P(X)$ is an S_b- complete subspace of (X, S). Then the sequences $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are convergent to α and β in $P(X)$. Thus there exists a and b in $P(X)$ such that

$$\lim_{n\to\infty} z_n = \alpha = Pa \quad \text{and} \quad \lim_{n\to\infty} w_n = \beta = Pb. \qquad (2.10)$$

Before going to prove the common coupled fixed point for the mappings A, B, P and Q, first we claim that for each $n \geq 1$ at least one of the following assertion is hold.

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} \leq \max\left\{\ S(\alpha,\alpha,z_{2n}), S(\beta,\beta,w_{2n})\ \right\}$$

or

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \leq \max\left\{\ S(\alpha,\alpha,z_{2n-2}), S(\beta,\beta,w_{2n-2})\ \right\}.$$

On contrary suppose that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} > \max\left\{\ S(\alpha,\alpha,z_{2n}), S(\beta,\beta,w_{2n})\ \right\}$$

and

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} > \max\left\{\ S(\alpha,\alpha,z_{2n-1}), S(\beta,\beta,w_{2n-1})\ \right\}.$$

Now, we know that

$$\begin{aligned}
&\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \\
&\leq \min\left\{\begin{array}{c} 2bS(z_{2n}, z_{2n}, \alpha) + b^2 S(\alpha,\alpha,z_{2n-1}), \\ 2bS(w_{2n}, w_{2n}, \beta) + b^2 S(\beta,\beta,z_{2n-1}) \end{array}\right\} \\
&\leq 2b^2 \max\left\{\begin{array}{c} S(\alpha,\alpha,z_{2n}), \\ S(\beta,\beta,w_{2n}) \end{array}\right\} + b^2 \max\left\{\begin{array}{c} S(\alpha,\alpha,z_{2n-1}), \\ S(\beta,\beta,z_{2n-1}) \end{array}\right\} \\
&< \frac{1}{4b}\min\left\{\begin{array}{c} S(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} + \frac{1}{8b}\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \\
&\leq \frac{1}{4b}\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} + \frac{1}{8b}\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \\
&= \frac{3}{8b}\min\left\{\begin{array}{c} S(z_{2n}, z_{2n}, z_{2n-1}), \\ S(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\}.
\end{aligned}$$

This is a contradiction. Hence our assertion is true.

First, we suppose that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} \leq \max\left\{\ S(\alpha,\alpha,z_{2n}), S(\beta,\beta,w_{2n})\ \right\}.$$

Now we have to prove that $A(a,b) = \alpha$ and $A(b,a) = \beta$. On contrary, suppose that $A(a,b) \neq \alpha$ or $A(b,a) \neq \beta$. Since

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(A(a,b), A(a,b), \alpha), \\ S(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S(A(b,a), A(b,a), \beta), \\ S(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} \leq \max\left\{\ S(\alpha,\alpha,z_{2n}), S(\beta,\beta,w_{2n})\ \right\},$$

from (2.1.4), definition of ψ and Lemma 1.11, we have

$$
\begin{aligned}
&\psi\left(\tfrac{1}{2b}S(A(a,b),A(a,b),\alpha\right) \\
&\leq \lim_{n\to\infty}\inf\psi\left(S(A(a,b),A(a,b),B(x_{2n+1},y_{2n+1})\right) \\
&\leq \tfrac{1}{5b^{12}}\lim_{n\to\infty}\inf\psi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,z_{2n}),S(\beta,\beta,w_{2n}), \\ S(A(a,b),A(a,b),\alpha),S(A(b,a),A(b,a),\beta), \\ S(z_{2n+1},z_{2n+1},z_{2n}),S(w_{2n+1},w_{2n+1},w_{2n}), \\ \frac{\left[\begin{array}{l} S(A(a,b),A(a,b),Qx_{2n+1}) \\ \times S(z_{2n+1},z_{2n+1},\alpha)\end{array}\right]}{1+S(\alpha,\alpha,Qx_{2n+1})}, \\ \frac{\left[\begin{array}{l} S(A(b,a),A(b,a),w_{2n}) \\ \times S(w_{2n+1},w_{2n+1},\beta)\end{array}\right]}{1+S(\beta,\beta,Qy_{2n+1})}\end{array}\right\}\right) \\
&\quad -\lim_{n\to\infty}\inf\ \phi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,z_{2n}),S(\beta,\beta,w_{2n}), \\ S(A(a,b),A(a,b),\alpha),S(A(b,a),A(b,a),\beta), \\ S(z_{2n+1},z_{2n+1},z_{2n}),S(w_{2n+1},w_{2n+1},w_{2n}), \\ \frac{\left[\begin{array}{l} S(A(a,b),A(a,b),Qx_{2n+1}) \\ \times S(z_{2n+1},z_{2n+1},\alpha)\end{array}\right]}{1+S(\alpha,\alpha,Qx_{2n+1})}, \\ \frac{\left[\begin{array}{l} S(A(b,a),A(b,a),w_{2n}) \\ \times S(w_{2n+1},w_{2n+1},\beta)\end{array}\right]}{1+S(\beta,\beta,Qy_{2n+1})}\end{array}\right\}\right) \\
&\leq \tfrac{1}{5b^{12}}\ \psi\left(\max\left\{\ 0,0,S(A(a,b),A(a,b),\alpha),S(A(b,a),A(b,a),\beta),0,0,0,0\ \right\}\right) \\
&= \tfrac{1}{5b^{12}}\ \psi\left(\max\left\{\ S(A(a,b),A(a,b),\alpha),S(A(b,a),A(b,a),\beta)\ \right\}\right).
\end{aligned}
$$

Similarly, we have

$$\psi\left(\frac{1}{2b}S(A(b,a),A(b,a),\beta\right) \leq \frac{1}{5b^{12}}\psi\left(\max\left\{\begin{array}{c} S(A(a,b),A(a,b),\alpha), \\ S(A(b,a),A(b,a),\beta)\end{array}\right\}\right).$$

Thus

$$
\begin{aligned}
&\psi\left(\tfrac{1}{2b}\max\left\{\begin{array}{c} S(A(a,b),A(a,b),\alpha), \\ S(A(b,a),A(b,a),\beta)\end{array}\right\}\right) \\
&\leq \tfrac{1}{5b^{12}}\psi\left(\max\left\{\begin{array}{c} S(A(a,b),A(a,b),\alpha), \\ S(A(b,a),A(b,a),\beta)\end{array}\right\}\right).
\end{aligned}
$$

By the definition of ψ, it follows that $A(a,b)=\alpha=Pa$ and $A(b,a)=\beta=Pb$. Since (A,P) is w-compatible pair, we have $A(\alpha,\beta)=P\alpha$ and $A(\beta,\alpha)=P\beta$.

From the definition of S_b-metric it is clear that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),P\alpha),S(A(\beta,\alpha),A(\beta,\alpha),P\beta) \\ S(B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1}),Qx_{2n+1}), \\ S(B(y_{2n+1},x_{2n+1}),B(y_{2n+1},x_{2n+1}),Qy_{2n+1}) \end{array}\right\}$$

$$\leq \max\left\{\; S(P\alpha,P\alpha,Qx_{2n+1}),S(P\beta,P\beta,Qy_{2n+1})\;\right\}.$$

From (2.1.4), by the definition of ψ and Lemma 1.11, we have

$$\begin{aligned}
&\psi\left(\tfrac{1}{2b}S(A(\alpha,\beta),A(\alpha,\beta),\alpha\right) \\
&\leq \tfrac{1}{5b^{12}}\ \limsup_{n\to\infty}\ \psi\left(\max\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),z_{2n}), \\ S(A(\beta,\alpha),A(\beta,\alpha),w_{2n}), \\ S(z_{2n+1},z_{2n+1},z_{2n}), \\ S(w_{2n+1},w_{2n+1},w_{2n}), \\ S(z_{2n+1},z_{2n+1},A(\alpha,\beta)), \\ S(w_{2n+1},w_{2n+1},A(\beta,\alpha)), \end{array}\right\}\right) \\
&\qquad -\limsup_{n\to\infty}\ \phi\left(\max\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),z_{2n}), \\ S(A(\beta,\alpha),A(\beta,\alpha),w_{2n}), \\ S(z_{2n+1},z_{2n+1},z_{2n}), \\ S(w_{2n+1},w_{2n+1},w_{2n}), \\ S(z_{2n+1},z_{2n+1},A(\alpha,\beta)), \\ S(w_{2n+1},w_{2n+1},A(\beta,\alpha)), \end{array}\right\}\right) \\
&\leq \tfrac{1}{5b^{12}}\ \psi\left(\max\left\{\begin{array}{c} 2bS(A(\alpha,\beta),A(\alpha,\beta),\alpha),2bS(A(\beta,\alpha),A(\beta,\alpha),\beta), \\ 0,0,b^2S(\alpha,\alpha,A(\alpha,\beta)),b^2S(\beta,\beta,A(\beta,\alpha)), \end{array}\right\}\right) \\
&\leq \tfrac{1}{5b^{12}}\ \psi\left(2b^2\max\left\{\; S(A(\alpha,\beta),A(\alpha,\beta),\alpha),S(A(\beta,\alpha),A(\beta,\alpha),\beta)\;\right\}\right).
\end{aligned}$$

Similarly, we have that

$$\psi\left(\frac{1}{2b}S(A(\alpha,\beta),A(\alpha,\beta),\alpha\right)\leq\frac{1}{5b^{12}}\,\psi\left(2b^2\max\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array}\right\}\right).$$

Thus

$$\begin{aligned}
&\psi\left(\tfrac{1}{2b}\max\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array}\right\}\right) \\
&\leq \tfrac{1}{5b^{12}}\ \psi\left(2b^2\max\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array}\right\}\right).
\end{aligned}$$

By the definition of ψ, it follows that $A(\alpha,\beta)=\alpha=P\alpha$ and $A(\beta,\alpha)=\beta=P\beta$. Therefore (α,β) is common coupled fixed point of A and P. Since $A(X\times X)\subseteq Q(X)$, there exist x and y in X such that $A(\alpha,\beta)=\alpha=Qx$ and

$A(\beta,\alpha)=\beta=Qy$. Since

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),P\alpha),\\ S(A(\beta,\alpha),A(\beta,\alpha),P\beta)\\ S(B(x,y),B(x,y),Qx),\\ S(B(y,x),B(y,x),Qy)\end{array}\right\}\le\max\left\{\begin{array}{c} S(P\alpha,P\alpha,Qx),\\ S(P\beta,P\beta,Qy)\end{array}\right\},$$

from (2.1.4), we have

$$\begin{aligned}\psi\left(S(A(\alpha,\beta),A(\alpha,\beta),B(x,y))\right) &\le \tfrac{1}{5b^{12}}\ \psi\left(b\max\left\{\begin{array}{c} S(\alpha,\alpha,B(x,y)),\\ S(\beta,\beta,B(y,x))\end{array}\right\}\right)\\ &\quad -\phi\left(\max\left\{\begin{array}{c} S(B(x,y),B(x,y),\alpha),\\ S(B(y,x),B(y,x),\beta)\end{array}\right\}\right).\end{aligned}$$

Similarly

$$\begin{aligned}\psi\left(S(\beta,\beta,B(y,x))\right) &\le \tfrac{1}{5b^{12}}\ \phi\left(b\max\left\{\begin{array}{c} S(\alpha,\alpha,B(x,y)),\\ S(\beta,\beta,B(y,x))\end{array}\right\}\right)\\ &\quad -\phi\left(\max\left\{\begin{array}{c} S(B(x,y),B(x,y),\alpha),\\ S(B(y,x),B(y,x),\beta)\end{array}\right\}\right).\end{aligned}$$

Thus

$$\begin{aligned}\psi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,B(x,y)),\\ S(\beta,\beta,B(y,x))\end{array}\right\}\right) &\le \tfrac{1}{5b^{12}}\phi\left(b\max\left\{\begin{array}{c} S(\alpha,\alpha,B(x,y)),\\ S(\beta,\beta,B(y,x))\end{array}\right\}\right)\\ &\quad -\phi\left(\max\left\{\begin{array}{c} S(B(x,y),B(x,y),\alpha),\\ S(B(y,x),B(y,x),\beta)\end{array}\right\}\right).\end{aligned}$$

It follows that $B(x,y)=\alpha=Qx$ and $B(y,x)=\beta=Qy$. Since (B,Q) is w-compatible pair, we have $B(\alpha,\beta)=Q\alpha$, and $B(\beta,\alpha,)=Q\beta$.
Since

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),P\alpha),\\ S(A(\beta,\alpha),A(\beta,\alpha),P\beta)\\ S(B(\alpha,\beta),B(\alpha,\beta),Q\alpha),\\ S(B(\beta,\alpha),B(\beta,\alpha),Q\beta)\end{array}\right\}\le\max\left\{\begin{array}{c} S(P\alpha,P\alpha,Q\alpha),\\ S(P\beta,P\beta,Q\beta)\end{array}\right\},$$

from (2.1.4) we have

$$\begin{aligned}&\psi\left(S(A(\alpha,\beta),A(\alpha,\beta),B(\alpha,\beta))\right)\\ &\le \tfrac{1}{5b^{12}}\ \psi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,B(\alpha,\beta)),S(\beta,\beta,B(\beta,\alpha)),\\ S(B(\alpha,\beta),B(\alpha,\beta),\alpha),S(B(\beta,\alpha),B(\beta,\alpha),\beta)\end{array}\right\}\right)\\ &\quad -\phi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,B(\alpha,\beta)),S(\beta,\beta,B(\beta,\alpha)),\\ S(B(\alpha,\beta),B(\alpha,\beta),\alpha),S(B(\beta,\alpha),B(\beta,\alpha),\beta)\end{array}\right\}\right)\\ &\le \tfrac{1}{5b^{12}}\ \psi\left(b\max\left\{\ S(\alpha,\alpha,B(\alpha,\beta)),S(\beta,\beta,B(\beta,\alpha))\ \right\}\right).\end{aligned}$$

Similarly

$$\psi\left(S(\beta,\beta,B(\beta,\alpha))\right) \le \frac{1}{5b^{12}}\ \psi\left(b\max\left\{\ S(\alpha,\alpha,B(\alpha,\beta)), S(\beta,\beta,B(\beta,\alpha))\ \right\}\right).$$

Thus

$$\psi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,B(\alpha,\beta)),\\ S(\beta,\beta,B(\beta,\alpha))\end{array}\right\}\right) \le \tfrac{1}{5b^{12}}\ \psi\left(b\max\left\{\begin{array}{c} S(\alpha,\alpha,B(\alpha,\beta)),\\ S(\beta,\beta,B(\beta,\alpha))\end{array}\right\}\right).$$

It implies that $B(\alpha,\beta)=\alpha=Q\alpha$ and $B(\beta,\alpha)=\beta=Q\beta$. Therefore (α,β) is common coupled fixed point of A,B,P and Q.

To prove the uniqueness, let us take (α^1,β^1) is another common coupled fixed point of A,B,P and Q. Since

$$\begin{array}{l}\tfrac{1}{8b^3}\min\left\{\begin{array}{c} S(A(\alpha,\beta),A(\alpha,\beta),P\alpha), S(A(\beta,\alpha),A(\beta,\alpha),P\beta),\\ S(B(\alpha^1,\beta^1),B(\alpha^1,\beta^1),Q\alpha^1), S(B(\beta^1,\alpha^1),B(\beta^1,\alpha^1),Q\beta^1)\end{array}\right\}\\ \le \max\left\{\ S(P\alpha,P\alpha,Q\alpha^1), S(P\beta,P\beta,Q\beta^1)\ \right\},\end{array}$$

from (2.1.4) we have

$$\begin{aligned}\psi\left(S(\alpha,\alpha,\alpha^1)\right) &= \psi\left(S(A(\alpha,\beta),A(\alpha,\beta),B(\alpha^1,\beta^1))\right)\\ &\le \tfrac{1}{5b^{12}}\ \psi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1), S(\alpha,\alpha,\alpha),\\ S(\beta,\beta,\beta), S(\alpha^1,\alpha^1,\alpha^1), S(\beta^1,\beta^1,\beta^1),\\ \frac{S(\alpha,\alpha,\alpha^1)S(\alpha^1,\alpha^1,\alpha)}{1+S(\alpha,\alpha,\alpha^1)}, \frac{S(\beta,\beta,\beta^1)S(\beta^1,\beta^1,\beta)}{1+S(\beta,\beta,\beta^1)}\end{array}\right\}\right)\\ &\quad -\phi\left(\max\left\{\begin{array}{c} S(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1), S(\alpha,\alpha,\alpha),\\ S(\beta,\beta,\beta), S(\alpha^1,\alpha^1,\alpha^1), S(\beta^1,\beta^1,\beta^1),\\ \frac{S(\alpha,\alpha,\alpha^1)S(\alpha^1,\alpha^1,\alpha)}{1+S(\alpha,\alpha,\alpha^1)}, \frac{S(\beta,\beta,\beta^1)S(\beta^1,\beta^1,\beta)}{1+S(\beta,\beta,\beta^1)}\end{array}\right\}\right)\\ &\le \tfrac{1}{5b^{12}}\ \psi(b\max\{S(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1)\}).\end{aligned}$$

Similarly, we have

$$\psi\left(S(\beta,\beta,\beta^1)\right) \le \frac{1}{5b^{12}}\ \psi(b\max\{S(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1)\}).$$

Thus

$$\psi\left(\max\left\{\ S(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1)\ \right\}\right) \le \tfrac{1}{5b^{12}}\ \psi(b\max\{S(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1)\}).$$

It implies that $\alpha=\alpha^1$ and $\beta=\beta^1$. Hence (α,β) is the unique common coupled fixed point of A,B,P and Q.

Similarly the remaining proof also follows when the Sub case(b) holds. That is,

$$\tfrac{1}{8b^3}\min\left\{\begin{array}{c} S(z_{2n},z_{2n},z_{2n-1}),\\ S(w_{2n},w_{2n},w_{2n-1})\end{array}\right\} \le \max\left\{\ S(\alpha,\alpha,z_{2n-1}), S(\beta,\beta,w_{2n-1})\ \right\}.$$

□

Theorem 2.2. *Let (X, S) be a complete S_b-metric space. Suppose that $A : X \times X \to X$ is a mapping satisfying*

$$\frac{1}{8b^3}\min\left\{\begin{array}{l} S(A(x,y),A(x,y),x), \\ S(A(u,v),A(u,v),u), \end{array}\right\} \leq \max\left\{\ S(x,x,u), S(y,y,v)\ \right\}$$

which implies that

$$\psi\left(S(A(x,y),A(x,y),A(u,v))\right) \leq \frac{1}{5b^{12}}\psi\left(M\left(x,y,u,v\right)\right) - \phi\left(M\left(x,y,u,v\right)\right),$$

for all x, y, u, v in X, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is linear and monotonically increasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semicontinuous, $\psi(0) = \phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$ and

$$M\left(x,y,u,v\right) = \max\left\{\begin{array}{c} S(x,x,u), S(y,y,v), S(A(x,y),A(x,y),x), \\ S(A(y,x),A(y,x),y), S(A(u,v), \\ A(u,v),u), S(A(v,u),A(v,u),v), \\ \frac{S(A(x,y),A(x,y),u)\ S(A(u,v),A(u,v),x)}{1+S(x,x,u)}, \\ \frac{S(A(y,x),A(y,x),v)\ S(A(v,u),A(v,u),y)}{1+S(y,y,v)} \end{array}\right\}.$$

Then A has a unique coupled fixed point in $X \times X$.

Example 2.3. Let $X = [0,1]$ and $S : X \times X \times X \to \mathcal{R}^+$ by $S(x,y,z) = (|y+z-2x| + |y-z|)^2$. Then S is S_b metric space with $b = 4$. Define $A, B : X \times X \to X$ and $P, Q : X \to X$ by $A(x,y) = \frac{x+y}{4^8\sqrt{6}}$, $B = \frac{x+y}{4^9\sqrt{6}}$, $P(x) = \frac{x}{4}$ and $Q(x) = \frac{x}{16}$. Also define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t$ and $\phi(t) = \frac{t}{30b^{12}}$.

$$\begin{aligned}
&\psi\left(S(A(x,y),A(x,y),B(u,v))\right) \\
&= (|A(x,y) + B(u,v) - 2A(x,y)| + |A(x,y) - B(u,v)|)^2 \\
&= (2\ |A(x,y) - B(u,v)|)^2 \\
&= 4\left|\frac{x+y}{4^7\sqrt{3}} - \frac{u+v}{4^8\sqrt{3}}\right|^2 \\
&= \frac{2}{3}\ \left|\frac{4x-u}{4^9} + \frac{4y-v}{4^9}\right|^2 \\
&\leq \frac{1}{6(4^6)^2}\ \left(\max\left\{\left|\frac{4x-u}{16}\right|, \left|\frac{4y-v}{16}\right|\right\}\right)^2 \\
&\leq \frac{1}{6(4^{12})}\ \max\left\{\left|\frac{x}{4} - \frac{u}{16}\right|^2, \left|\frac{y}{4} - \frac{v}{16}\right|^2\right\}
\end{aligned}$$

$$= \frac{1}{6(4^{12})} \max\left\{ S(Px,Px,Qu), S(Py,Py,Qv), S(A(x,y),A(x,y),Px) \right\}$$

$$\leq \frac{1}{3(b^{10})}\ \psi\left(\max\left\{\begin{array}{c} S(Px,Px,Qu), S(Py,Py,Qv), \\ S(A(x,y),A(x,y),Px), S(A(y,x),A(y,x),Py), \\ S(B(u,v),B(u,v),Qu), S(B(v,u),B(v,u),Qv), \\ \frac{S(A(x,y),A(x,y),Qu)\ S(B(u,v),B(u,v),Px)}{1+S(Px,Px,Qu)}, \\ \frac{S(A(y,x),A(y,x),Qv)\ S(B(v,u),B(v,u),Py)}{1+S(Py,Py,Qv)} \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{c} S(Px,Px,Qu), S(Py,Py,Qv), \\ S(A(x,y),A(x,y),Px), S(A(y,x),A(y,x),Py), \\ S(B(u,v),B(u,v),Qu), S(B(v,u),B(v,u),Qv), \\ \frac{S(A(x,y),A(x,y),Qu)\ S(B(u,v),B(u,v),Px)}{1+S(Px,Px,Qu)}, \\ \frac{S(A(y,x),A(y,x),Qv)\ S(B(v,u),B(v,u),Py)}{1+S(Py,Py,Qv)} \end{array}\right\}\right).$$

It is clear that all conditions of Theorem 2.1 satisfied and $(0,0)$ is a unique common coupled fixed point of A, B, P and Q.

3. Application

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 2.2.

Consider the initial value problem:

$$x^1(t) = f(t, x(t), x(t)),\ t \in I = [0,1],\ x(0) = x_0, \tag{3.1}$$

where $f : I \times \left[\frac{x_0}{4}, \infty\right) \times \left[\frac{x_0}{4}, \infty\right) \to \left[\frac{x_0}{4}, \infty\right)$ and $x_0 \in \mathbb{R}$.

Theorem 3.1. *Consider the initial value problem* (3.1) *with* $f \in C\left(I \times \left[\frac{x_0}{4}, \infty\right) \times \left[\frac{x_0}{4}, \infty\right)\right)$ *and*

$$\int_0^t f(s, x(s), y(s))ds = \frac{1}{\sqrt{6b^4}} \min\left\{ \int_0^t f(s, , x(s), x(s))ds, \int_0^t f(s, y(s), y(s))ds \right\}.$$

Then there exists a unique solution in $C\left(I, \left[\frac{x_0}{4}, \infty\right)\right)$ *for initial value problem* (3.1)*.*

Proof. The integral equation corresponding to initial value problem (3.1) is

$$x(t) = x_0 + \int_0^t f(s, x(s), x(s))ds.$$

Let $X = C\left(I, \left[\frac{x_0}{4}, \infty\right)\right)$ and $S(x,y,z) = (|y+z-2x| + |y-z|)^2 \; for \; x, y \in X$. Define $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$, $\phi(t) = \frac{t}{5b^{12}}$. Define $A : X \times X \to X$ by

$$A(x,y)(t) \quad = \quad x_0 + \int_0^t f(s, x(s), y(s))ds. \tag{3.2}$$

Now

$$\begin{aligned}
&S(A(x,y)(t), A(x,y)(t), A(u,v)(t)) \\
&= \{| \; A(x,y)(t) + A(u,v)(t) - 2A(x,y)(t) \; | + | \; A(x,y)(t) - A(u,v)(t) \; |\}^2 \\
&= 4 \; | \; A(x,y)(t) - A(u,v)(t) \; |^2 \\
&= 4 \left| \int_0^t f(s,x(s),y(s))ds - \int_0^t f(s,u(s),v(s))ds \right|^2 \\
&= \frac{4}{\sqrt{6}b^5} \left| \min\left\{ \begin{array}{l} \int_0^t f(s,x(s),x(s))ds, \\ \int_0^t f(s,y(s),y(s))ds \end{array} \right\} - \min\left\{ \begin{array}{l} \int_0^t f(s,u(s),u(s))ds, \\ \int_0^t f(s,v(s),v(s))ds \end{array} \right\} \right|^2 \\
&\leq \frac{2}{5b^{12}} \left| \max\left\{ \begin{array}{l} \int_0^t f(s,x(s),x(s))ds - \int_0^t f(s,u(s),u(s))ds, \\ \int_0^t f(s,y(s),y(s))ds - \int_0^t f(s,v(s),v(s))ds \end{array} \right\} \right|^2 \\
&= \frac{2}{3\;b^{10}} \max\left\{ \begin{array}{l} \left| \int_0^t f(s,x(s),x(s))ds - \int_0^t f(s,u(s),u(s))ds, \right|^2 \\ \\ \left| \int_0^t f(s,y(s),y(s))ds - \int_0^t f(s,v(s),v(s))ds \right|^2 \end{array} \right\} \\
&= \frac{1}{6\;b^{10}} \max\left\{ \; 2 \; | \; x(t) \; - \; u(t) \; |^2, 2 \; | \; y(t) \; - \; v(t) \; |^2 \; \right\} \\
&= \frac{1}{6\;b^{10}} \max\{S(x,x,u), S(y,y,v)\} \\
&\leq \psi(M(x,u,y,v)) - \phi(M(x,u,y,v)).
\end{aligned}$$

It follows from Theorem 2.2 that A has a unique coupled fixed point in X. □

4. Conclusion

In this attempt, we prove a Suzuki type unique common coupled fixed point theorem for two pairs of w-compatible mappings along with $(\psi - \phi)$ - and Rational contraction conditions in S_b-metric spaces. We also furnish an example as well as application to integral equation.

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