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COMMON FIXED POINT FOR MAPPINGS UNDER CONTRACTIVE CONDITION BASED ON ALMOST PERFECT FUNCTIONS AND α-ADMISSIBILITY

Wasfi Shatanawi[1,2] and Kamaleldin Abodayeh[2]

[1]Department of General Sciences, Prince Sultan University,
Riyadh, Saudi Arabia
e-mail: `wshatanawi@psu.edu.sa, swasfi@hu.edu.jo`

[2]Department of Mathematics, The Hashemite University,
P.O. Box 330127, Zarqa 13133, Jordan
e-mail: `kamal@psu.edu.sa`

Abstract. In this paper, we introduce the notion of (α, β, ψ)-contraction for two mappings defined on a set X. We utilize our new definition to formulate and prove many fixed and common fixed point results in the context of metric space. Our results are extension and improvement for many exciting results in the context of metric spaces.

1. Introduction and preliminaries

The usability of fixed point theory in applied sciences made this area very attractive area to many researchers. Many researchers applied many fixed point theorems to prove the uniqueness and existence of many problems. The most remarkable result in fixed point theorems is the Banach fixed point theory [6]. For some generalization of Banach fixed point theory, see for example [16]-[17] and all references cited their.

Samet *et al.* [15] introduced the concept of α-admissible for a single mapping and utilized this concept to introduce and prove some fixed point theorems. While, Abdeljawad [1] introduced the concept of α-admissible for a pair of mappings and proved some fixed point theorems of the type Meir-Keeler.

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[0]Corresponding author: K. Abodayeh(kamal@psu.edu.sa).

Recently, many researchers studied many fixed point theorems in theses two concepts (see [4]-[14]). Karapiner *et al.* [11] introduced the concept of triangular α-admissibility for a single mapping and proved a fixed point theorem of the type Meir-Keeler. Salimi *et al.* [14] introduced the concept of α-admissible for single mapping with respect to the function η and constructed some nice fixed point results. Recently, Hussain *et al.* [10] extended the notion of α-admissible for a pair of mappings to the concept of α-admissible with respect to a function β.

The concept of α-admissible mappings is defined as follows:

Definition 1.1. ([15]) Let $S : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then S is called α-admissible if $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha(Sx, Sy) \geq 1$.

The triangular α-admissibility for a single mapping was given by Karapiner [11] as follows:

Definition 1.2. ([11]) Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is called a triangular α-admissible mapping if it satisfies the following conditions:

(1) T is α-admissible;
(2) If $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $\alpha(x, y) \geq 1$.

Abdeljawad [1] introduced the concept of α-admissible for a pair of mappings as follows:

Definition 1.3. ([1]) Let $S, T : X \to X$ be two mappings and $\alpha : X \times X \to [0, \infty)$ be a function. The pair (S, T) is called α-admissible if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.

The α-admissibility for the pair of two mappings with respect to a function β is given as follows:

Definition 1.4. ([8]) Let $S, T : X \to X$ be two mappings and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. The pair (S, T) is said to be α-admissible with respect to η if for any $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$, we have $\alpha(Sx, Ty) \geq \eta(Sx, Ty)$ and $\alpha(Tx, Sy) \geq \eta(Tx, Sy)$.

The concepts of (α,η)-complete metric spaces and $(\alpha,\eta)-$continuous mappings is given in the following definitions.

Definition 1.5. ([9]) Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. Then X is said to be an $(\alpha, \eta)-$complete metric space if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, converges in X.

Definition 1.6. ([9]) Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. A mapping $T : X \to X$ is said to be an $(\alpha, \eta)-$continuous mapping if each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ implies $Tx_n \to Tx$ as $n \to \infty$.

By defining $\eta : X \times X \to \mathbb{R}^+$ by $\eta(s, t) = 1$, we can reformulate the above definitions as follows:

Definition 1.7. Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Then X is said to be an $\alpha-$complete metric space if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.

Definition 1.8. Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to X$ is said to be an $\alpha-$continuous mapping if each sequence $\{x_n\}$ in X that converges to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies that $Tx_n \to Tx$ as $n \to \infty$.

Abodayeh *et al.* [2] introduced the notion of almost perfect function as follows:

Definition 1.9. ([2]) A nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an almost perfect function if ψ satisfies the following conditions:

(1) $\psi(t) = 0$ if an only if $t = 0$.
(2) If (t_n) is a sequence in $[0, +\infty)$ with $\psi(t_n) \to 0$, then $t_n \to 0$.

In this paper, we utilized the concept of almost perfect function and the concept α-admissibility to construct and prove many fixed and common fixed point results in the setting of metric spaces.

2. Main result

We start our work by introducing the notion of triangular α-admissible for pair of self mappings T and S on a set X.

Definition 2.1. Let $S, T : X \to X$ be two mappings and $\alpha : X \times X \to \mathbb{R}$ be a function. The pair (S, T) is said to be triangular α-admissible if the following conditions hold:

(1) If $x, y \in X$ with $\alpha(x, y) \geq 1$, then $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.
(2) If $x, y, z \in X$ with $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $\alpha(x, y) \geq 1$.

Now, we introduce the notion of triangular α-admissible with respect to another function β for the pair of self mappings S and T on a set X.

Definition 2.2. Let $S,T : X \to X$ be two mappings and $\alpha, \beta : X \times X \to \mathbb{R}$ be two functions. The pair (S,T) is said to be triangular α-admissible with respect to β if the following conditions hold:

(1) If $x, y \in X$ with $\alpha(x,y) \geq \beta(x,y)$, then $\alpha(Sx,Ty) \geq \beta(Sx,Ty)$ and $\alpha(Tx,Sy) \geq \beta(Tx,Sy)$;
(2) If $\alpha(x,z) \geq \beta(x,z)$ and $\alpha(z,y) \geq \beta(z,y)$, then $\alpha(x,y) \geq \beta(x,y)$.

In order to make our work visible, we introduce the following definition:

Definition 2.3. Let (X,d) be a metric space. Let $S,T : X \to X$ be two mappings and $\alpha, \beta : X \times X \to [0,+\infty)$. The pair (S,T) is called (α,β,ψ)-contractive if there exist $k \in [0,1)$ and an almost perfect function ψ such that for $x, y \in X$ with $\alpha(x,y) \geq \beta(x,y)$, we have

$$\psi(d(Sx,Ty)) \leq \max\{k\psi(d(x,y)), k\psi(d(x,Sx)), k\psi(d(y,Ty))\}.$$

Now, we introduce and prove our main result:

Theorem 2.4. *Let (X,d) be an (α,β)-complete bounded metric space, where $\alpha, \beta : X \times X \to \mathbb{R}$ be two functions. Let $S,T : X \to X$ be (α,β)-continuous mappings. Assume the following conditions hold:*

(1) *(S,T) is (α,β,ψ)-contractive.*
(2) *(S,T) is triangular α-admissible with respect to β.*
(3) *There exists $x_0 \in X$ such that $\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0)$ and $\alpha(TSx_0, Sx_0) \geq \beta(TSx_0, Sx_0)$.*

Then the two mappings S and T have a common fixed point.

Proof. By Condition (3), we choose $x_0 \in X$ with $\alpha(Sx_0, TSx_1) \geq \beta(Sx_0, TSx_1)$ and $\alpha(TSx_0, Sx_1) \geq \beta(Tx_0, Sx_1)$. Define a sequence $\{x_n\}$ in X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n \in \mathbb{N}$. Since the pair (S,T) is α-admissible with respect to β, we have

$$\alpha(x_1,x_2) = \alpha(Sx_0,Tx_1) \geq \beta(Sx_0,Tx_1) = \beta(x_1,x_2)$$

and

$$\alpha(x_2,x_1) = \alpha(Tx_1,Sx_0) \geq \beta(Tx_1,Sx_0) = \beta(x_2,x_1).$$

Again, by using α-admissible property with respect to β, we have

$$\alpha(x_2,x_3) = \alpha(Tx_1,Sx_2) \geq \beta(Tx_1,Sx_2) = \beta(x_2,x_3)$$

and

$$\alpha(x_3,x_2) = \alpha(Sx_2,Tx_1) \geq \beta(Sx_2,Tx_1) = \beta(x_3,x_2).$$

Repeating the above process for n-times, we have $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ and $\alpha(x_{n+1}, x_n) \geq \beta(x_{n+1}, x_n)$.

Using the property of triangular α-admissibility with respect to β, we can deduce that for any $n, m \in \mathbb{N}$ with $m > n$, we have $\alpha(x_n, x_m) \geq \beta(x_n, x_m)$ and $\alpha(x_m, x_n) \geq \beta(x_m, x_n)$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{2n_0} = x_{2n_0+1}$, then $x_{2n_0} = Sx_{2n_0}$. Hence x_{2n_0} is a fixed point of S. Since $\alpha(x_{2n_0}, x_{2n_0+1}) \geq \beta(x_{2n_0}, x_{2n_0+1})$, we have

$$\begin{aligned}
\psi(d(x_{2n_0+1}, x_{2n_0+2})) &= \psi(d(Sx_{2n_0}, Tx_{2n_0+1}) \\
&\leq \max\{k\psi(d(x_{2n_0}, x_{2n_0+1})), k\psi(d(x_{2n_0}, Sx_{2n_0})), \\
&\quad k\psi(d(x_{2n_0+1}, Tx_{2n_0+1}))\} \\
&= k\psi(d(x_{2n_0+1}, x_{2n_0+2})).
\end{aligned}$$

Since $k \in [0, 1)$, we conclude that $\psi(d(x_{2n_0+1}, x_{2n_0+2})) = 0$. Using the properties of ψ, we conclude that $x_{2n_0} = x_{2n_0+1} = x_{2n_0+2}$. Thus x_{2n_0} is a common fixed point of S and T. Similarly, we can show that if $x_{2n_0+1} = x_{2n_0+2}$, then x_{2n_0+1} is a common fixed point of S and T. Thus, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Now given $n \in \mathbb{N}$. Then

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(Sx_{2n}, Tx_{2n+1})) \\
&\leq \max\{k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n}, Sx_{2n})), \\
&\quad k\psi(d(x_{2n+1}, Tx_{2n+1}))\} \\
&= \max\{k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2}))\}.
\end{aligned}$$

If $\max\{k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2}))\} = k\psi(d(x_{2n+1}, x_{2n+2}))$, we conclude that $x_{2n+1} = x_{2n+2}$, which is a contradiction. So, we have

$$\max\{k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2}))\} = k\psi(d(x_{2n}, x_{2n+1})).$$

Therefore,

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq k\psi(d(x_{2n}, x_{2n+1})). \tag{2.1}$$

Note that,

$$\begin{aligned}
\psi(d(x_{2n}, x_{2n+1})) &= \psi(d(Tx_{2n-1}, Sx_{2n})) \\
&= \psi(d(Sx_{2n}, Tx_{2n-1})) \\
&\leq \max\{k\psi(d(x_{2n}, x_{2n-1})), k\psi(d(x_{2n}, Sx_{2n})), \\
&\quad k\psi(d(x_{2n-1}, Tx_{2n-1}))\} \\
&= \max\{k\psi(d(x_{2n}, x_{2n-1})), k\psi(d(x_{2n}, x_{2n+1}))\}.
\end{aligned}$$

If $\max\{k\psi(d(x_{2n}, x_{2n-1})), k\psi(d(x_{2n}, x_{2n+1}))\} = k\psi(d(x_{2n}, x_{2n+1}))$, we conclude that $x_{2n} = x_{2n+1}$, which is a contradiction. So

$$\max\{k\psi(d(x_{2n}, x_{2n-1})), k\psi(d(x_{2n}, x_{2n+1}))\} = k\psi(d(x_{2n}, x_{2n-1})).$$

Therefore

$$\psi(d(x_{2n},x_{2n+1})) \leq k\psi(d(x_{2n},x_{2n-1})). \tag{2.2}$$

By combining (2.1) and (2.2), we conclude that

$$\psi(d(x_n,x_{n+1})) \leq k\psi(d(x_{n-1},x_n)). \tag{2.3}$$

Repeating (2.3) n-times, we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\leq k\psi(d(x_{n-1},x_n))\\ &\leq k^2\psi(d(x_{n-2},x_{n-1}))\\ &\vdots\\ &\leq k^n\psi(d(x_0,x_1)).\end{aligned} \tag{2.4}$$

Letting $n \to +\infty$ in (2.4), we get

$$\lim_{n\to+\infty}\psi(d(x_n,x_{n+1})) = 0.$$

Using the properties of almost perfect mappings, we conclude that

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0. \tag{2.5}$$

The next step is to show that $\{x_n\}$ is a Cauchy sequence in X. To do this, let $n,m \in \mathbb{N}$ with $m > n$. We have four cases to prove the result.

Case I: n is odd and m is even. Here, $n = 2t+1$ and $m = 2t+s+1$ for some $t,s \in \mathbb{N}$, where s is odd.
Since $\alpha(x_{2t+1},x_{2t+1+s}) \geq \beta(x_{2t+1},x_{2t+1+s})$, we have

$$\begin{aligned}\psi(d(x_n,x_m)) &= \psi(d(x_{2t+1},x_{2t+s+1}))\\ &= \psi(d(Sx_{2t},Tx_{2t+s}))\\ &\leq \max\{k\psi(d(x_{2t},x_{2t+s})),k\psi(d(x_{2t},Sx_{2t})),\\ &\quad k\psi(d(x_{2t+s},Tx_{2t+s}))\}\\ &= \max\{k\psi(d(x_{2t},x_{2t+s})),k\psi(d(x_{2t},x_{2t+1})),\\ &\quad k\psi(d(x_{2t+s},x_{2t+s+1}))\}\\ &= \max\{k\psi(d(x_{2t},x_{2t+s})),k\psi(d(x_{2t},x_{2t+1}))\}\\ &\leq \max\{k\psi(d(x_{2t},x_{2t+s})),k^{2t+1}\psi(d(x_0,x_1))\}\\ &= \max\{k\psi(d(x_{n-1},x_{n-1+s})),k^n\psi(d(x_0,x_1))\}.\end{aligned} \tag{2.6}$$

Since $\alpha(x_{2t}, x_{2t+s}) \geq \beta(x_{2t}, x_{2t+s})$, we have

$$\begin{aligned}
\psi(d(x_{n-1}, x_{n-1+s})) &= \psi(d(x_{2t}, x_{2t+s})) \\
&= \psi(d(Tx_{2t-1}, Sx_{2t+s-1})) \\
&= \psi(d(Sx_{2t+s-1}, Tx_{2t-1})) \\
&\leq \max\{k\psi(d(x_{2t-1}, x_{2t+s-1})), k\psi(d(x_{2t-1}, Tx_{2t-1})), \\
&\quad k\psi(d(x_{2t+s-1}, Sx_{2t+s-1}))\} \\
&\leq \max\{k\psi(d(x_{2t-1}, x_{2t+s-1})), k\psi(d(x_{2t-1}, x_{2t})), \\
&\quad k\psi(d(x_{2t+s-1}, x_{2t+s}))\} \\
&= \max\{k\psi(d(x_{2t-1}, x_{2t+s-1})), k\psi(d(x_{2t-1}, x_{2t}))\} \\
&\leq \max\{k\psi(d(x_{2t-1}, x_{2t+s-1})), k^{2t}\psi(d(x_0, x_1))\} \\
&= \max\{k\psi(d(x_{n-2}, x_{n-2+s})), k^{n-1}\psi(d(x_0, x_1))\}. \quad (2.7)
\end{aligned}$$

By (2.6) and (2.7), we conclude that

$$\begin{aligned}
\psi(d(x_n, x_m)) &\leq \max\{k\psi(d(x_{n-1}, x_{n-1+s})), k^n\psi(d(x_0, x_1))\} \\
&\leq \max\{k^2\psi(d(x_{n-2}, x_{n-2+s})), k^n\psi(d(x_0, x_1))\} \\
&\vdots \\
&\leq \max\{k^n\psi(d(x_0, x_s)), k^n\psi(d(x_0, x_1))\}. \quad (2.8)
\end{aligned}$$

On letting $n \to +\infty$ in (2.8), we conclude that

$$\lim_{n,m\to+\infty} \psi(d(x_n, x_m)) = 0.$$

Since ψ is an almost perfect function, we obtain

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0. \quad (2.9)$$

Case II: n and m are both odd. By triangular inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{m-1}) + d(x_{m-1}, x_m).$$

Letting $n, m \to +\infty$ in above inequality and using Case I and (2.5), we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0.$$

Case III: n and m are both even. By triangular inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m).$$

Letting $n, m \to +\infty$ in above inequality and using Case I and (2.5), we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0.$$

Case IV: n is even and m is odd. By triangular inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m-1}) + d(x_{m-1}, x_m) = 0.$$

Letting $n, m \to +\infty$ in above inequality and using Case I and (2.5), we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0.$$

Combining all cases together, we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is an $(\alpha,\beta)-$complete metric space, then there exists $x \in X$ such that $x_n \to x$. Using $(\alpha,\beta)-$continuity of the mappings S and T, we deduce that $Sx_{2n} \to Sx$ and $Tx_{2n+1} \to Tx$. Using the uniqueness of limit, we have $Sx = x$ and $Tx = x$. Thus x is a common fixed point of S and T. □

Now, we recall the definition of altering distance:

Definition 2.5. ([12]) A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if ψ satisfies the following conditions:

(1) $\psi(t) = 0$ if and only if $t = 0$.
(2) ψ is continuous and nondecreasing.

Corollary 2.6. *Let (X, d) be an (α,β)-complete bounded metric space, where $\alpha, \beta : X \times X \to \mathbb{R}$ are two functions. Let $S, T : X \to X$ be (α,β)-continuous mappings. Assume that the pair (S, T) is triangular α-admissible with respect to β. Moreover, assume there exist an altering distance function ψ and $k \in [0, 1)$ such that for $x, y \in X$ with $\alpha(x, y) \geq \beta(x, y)$,*

$$\psi(d(Sx, Ty)) \leq \max\{k\psi(d(x, y)), k\psi(d(x, Sx)), k\psi(d(y, Ty))\}.$$

If there exists $x_0 \in X$ such that

$$\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0) \quad \text{and} \quad \alpha(TSx_0, Sx_0) \geq \beta(Sx_0, TSx_0),$$

then S and T have a common fixed point.

Proof. The proof follows from Theorem 2.4 by noting that every altering distance function is an almost perfect function. □

Corollary 2.7. *Let (X, d) be an (α,β)-complete bounded metric space, where $\alpha, \beta : X \times X \to \mathbb{R}$ are two functions. Let $S, T : X \to X$ be (α,β)-continuous mappings. Assume that the pair (S, T) is triangular α-admissible with respect to β. Moreover, assume there exist an almost perfect function ψ and $a, b, c \in [0, 1)$ with $a + b + c < 1$ such that for $x, y \in X$ with $\alpha(x, y) \geq \beta(x, y)$,*

$$\psi(d(Sx, Ty)) \leq a\psi(d(x, y)) + b\psi(d(x, Sx)) + c\psi(d(y, Ty)).$$

If there exists $x_0 \in X$ such that $\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0)$ and $\alpha(TSx_0, Sx_0) \geq \beta(TSx_0, Sx_0)$, then S and T have a common fixed point.

Proof. Note that $a\psi(d(x,y)) + b\psi(d(x,Sx)) + c\psi(d(y,Ty)) \leq (a+b+c)\max\{\psi(d(x,y)), \psi(d(x,Sx)), \psi(d(y,Ty))\}$. The proof follows from Theorem 2.4 by taking $k = a+b+c$ and noting that S and T satisfy all the hypotheses of Theorem 2.4. □

Corollary 2.8. *Let (X,d) be an (α,β)-complete bounded metric space, where $\alpha, \beta : X \times X \to \mathbb{R}$ be two functions. Let $S,T : X \to X$ be (α,β)-continuous mappings. Assume that the pair (S,T) is triangular α-admissible with respect to β. Moreover, assume there exist an altering distance function ψ and $a,b,c \in [0,1)$ with $a+b+c < 1$ such that for $x,y \in X$ with $\alpha(x,y) \geq \beta(x,y)$,*

$$\psi(d(Sx,Ty)) \leq a\psi(d(x,y)) + b\psi(d(x,Sx)) + c\psi(d(y,Ty)).$$

If there exists $x_0 \in X$ such that $\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0)$ and $\alpha(TSx_0, Sx_0) \geq \beta(TSx_0, Sx_0)$, then S and T have a common fixed point.

Proof. The proof follows from Corollary 2.7 by noting that every altering distance function is an almost perfect function. □

On a set X, define the function $\beta : X \times X \to \mathbb{R}$ by $\beta(s,t) = 1$. Then we have the following corollaries:

Corollary 2.9. *Let (X,d) be an α-complete metric space, where $\alpha : X \times X \to \mathbb{R}$ is a function. Let $S,T : X \to X$ be α-continuous mappings. Assume that the following conditions hold:*

(1) *There exist an almost function ψ and $k \in [0,1)$ such that if $x,y \in X$ with $\alpha(x,y) \geq 1$, then*

$$\psi(d(Sx,Ty)) \leq \max\{k\psi(d(x,y)), k\psi(d(x,Sx)), k\psi(d(y,Ty))\}.$$

(2) *(S,T) is triangular α-admissible.*

(3) *There exists $x_0 \in X$ such that $\alpha(Sx_0, TSx_0) \geq 1$ and $\alpha(TSx_0, Sx_0) \geq 1$.*

Then the two mappings S and T have a common fixed point.

Corollary 2.10. *Let (X,d) be an α-complete bounded metric space, where $\alpha : X \times X \to \mathbb{R}$ is a function. Let $S,T : X \to X$ be α-continuous mappings. Assume that the pair (S,T) is triangular α-admissible. Moreover, assume that there exist an altering distance function ψ and $k \in [0,1)$ such that for $x,y \in X$ with $\alpha(x,y) \geq 1$,*

$$\psi(d(Sx,Ty)) \leq \max\{k\psi(d(x,y)), k\psi(d(x,Sx)), k\psi(d(y,Ty))\}.$$

If there exists $x_0 \in X$ such that $\alpha(Sx_0, TSx_0) \geq 1$ and $\alpha(TSx_0, Sx_0) \geq 1$, then S and T have a common fixed point.

Corollary 2.11. *Let (X,d) be an α-complete bounded metric space, where $\alpha : X \times X \to \mathbb{R}$ is a function. Let $S,T : X \to X$ be (α,β)-continuous mappings. Assume that the pair (S,T) is triangular α-admissible. Moreover, assume that there exist an almost perfect function ψ and $a,b,c \in [0,1)$ with $a+b+c<1$ such that for $x,y \in X$ with $\alpha(x,y) \geq 1$,*

$$\psi(d(Sx,Ty)) \leq a\psi(d(x,y)) + b\psi(d(x,Sx)) + c\psi(d(y,Ty)).$$

If there exists $x_0 \in X$ such that $\alpha(Sx_0,TSx_0) \geq \beta(Sx_0,TSx_0)$ and $\alpha(TSx_0,Sx_0) \geq \beta(TSx_0,Sx_0)$, then S and T have a common fixed point.

Proof. Note that $a\psi(d(x,y)) + b\psi(d(x,Sx)) + c\psi(d(y,Ty)) \leq (a+b+c) \max\{\psi(d(x,y)), \psi(d(x,Sx)), \psi(d(y,Ty))\}$. The proof follows from Theorem 2.4 by taking $k=a+b+c$ and noting that S and T satisfy all the hypotheses of Theorem 2.4. □

Corollary 2.12. *Let (X,d) be an α-complete bounded metric space, where $\alpha : X \times X \to \mathbb{R}$ is a function. Let $S,T : X \to X$ be (α,β)-continuous mappings. Assume that the pair (S,T) is triangular α-admissible. Moreover, assume there exist an altering distance function ψ and $a,b,c \in [0,1)$ with $a+b+c<1$ such that for $x,y \in X$ with $\alpha(x,y) \geq 1$,*

$$\psi(d(Sx,Ty)) \leq a\psi(d(x,y)) + b\psi(d(x,Sx)) + c\psi(d(y,Ty)).$$

If there exists $x_0 \in X$ such that $\alpha(Sx_0,TSx_0) \geq \beta(Sx_0,TSx_0)$ and $\alpha(TSx_0,Sx_0) \geq \beta(TSx_0,Sx_0)$, then S and T have a common fixed point.

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