Nonlinear Functional Analysis and Applications
Vol. 23, No. 2 (2018), pp. 259-274
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

FIXED POINTS OF CHATTERJEA TYPE MULTI-VALUED F-CONTRACTIONS ON CLOSED BALL

Muhammad Nazam[1], Muhammad Arshad[2], Aftab Hussain[3] and H. G. Hyun[4]

[1]Department of Mathematics and Statistics,
International Islamic University, Islamabad, Pakistan
e-mail: `nazim254.butt@gmail.com`

[2]Department of Mathematics and Statistics,
International Islamic University, Islamabad, Pakistan
e-mail: `marshadzia@iiu.edu.pk`

[3]Department of Basic Sciences and Humanities,
Khawaja Farid University, Rahim Yar Khan, Punjab, Pakistan
e-mail: `aftabshh@gmail.com`

[4]Department of Mathematics Education,
Kyungnam University, Changwon, Gyeongnam, 51767, Korea
e-mail: `hyunhg8285@kyungnam.ac.kr`

Abstract. We introduce the notions of Chatterjea type multi-valued F-contraction and Chatterjea type multi-valued (α, η, GF)-contraction on closed ball and obtain two new fixed point theorems for such kind of contractions in complete metric spaces. Some comparative examples are given to illustrate and to show usefulness of these results among famous fixed point theorems on F-contractions.

---

[0]Received January 17, 2017. Revised December 23, 2017.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Complete metric space, fixed point, multi-valued F-contraction, closed ball.
[0]Corresponding author(s): M. Nazam(nazim254.butt@gmail.com),
H. G. Hyun(hyunhg8285@kyungnam.ac.kr).

1. Introduction and preliminaries

Banach contraction principle states that any contraction on a complete metric space has a unique fixed point. This principle guarantees the existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach contraction principle has been extended and generalized in many directions (see [2, 10, 11, 12, 15]). The fixed point theory of multi-valued contraction mappings using the Hausdorff metric was initiated by Nadler [21], who extended the Banach contraction principle to multi-valued mappings. Since then many authors have studied various fixed point results for multi-valued mappings. The theory of multi-valued mappings has many applications in control theory, convex optimization, differential equations and economics. Recently, Sgroi and Vetro have extended the concept of F-contraction for multi-valued mapping and they proved the following theorem in [24].

Theorem 1.1. ([24]) *Let (X,d) be a complete metric space and $T : X \to CB(X)$. If there exists a mapping $F : \mathbb{R}^+ \to \mathbb{R}$, $\tau > 0$ and real numbers $\alpha, \beta, \gamma, \delta, L \geq 0$ such that*

$$2\tau + F(H(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx))$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha + \beta + \gamma + 2L = 1$ and $\gamma \neq 1$, then T has a fixed point.

From the application point of view, the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not only the entire space X but also merely on a subset Y of X. However, if Y is closed and a Picard iterative sequence $\{x_n\}$ in X converges to some x in X then by imposing a subtle restriction on the choice of x_0, one may force Picard iterative sequence to stay eventually in Y. In this case, closedness of Y coupled with some suitable contractive condition establish the existence of a fixed point of T.

We recall some basic definitions and results which will be used in the sequel. Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$.

Definition 1.2. ([28]) Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be an F-contraction if there exists $\tau > 0$ such that

$$d(T(x),T(y)) > 0 \text{ implies } \tau + F(d(T(x),T(y))) \leq F(d(x,y)), \tag{1.1}$$

$\forall\, x, y \in X$, where $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following properties:

(F_1) : F is strictly increasing.

(F_2) : For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{\to\infty} F(a_n) = -\infty$.

(F_3) : There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+}(\alpha)^\theta F(\alpha) = 0$.

(F_4) : $F(\inf A) = \inf F(A)$ for all $A \subseteq (0,\infty)$ with $\inf A > 0$.

We denote by Δ_F and Δ_{F^*}, the set of all functions satisfying the conditions $(F_1) - (F_3)$ and $(F_1) - (F_4)$, respectively. One can note that $\Delta_{F^*} \subseteq \Delta_F$ and for example, $f(x) = \ln x, g(x) = x + \ln x$ are members of Δ_{F^*}.

Remark 1.3. If F satisfies (F_1), then it satisfies (F_4) if and only if F is right continuous.

Wardowski [28] established the following result using F-contraction:

Theorem 1.4. ([28]) *Let (X,d) be a complete metric space and let $T : X \to X$ be a F-contraction. Then T has a unique fixed point $v \in X$ and for any $x_0 \in X$ the sequence $\{T^n(x_0)\}$ is convergent to v.*

Definition 1.5. ([13]) Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be Chatterjea contraction if it satisfies the following condition:

$$d\left(T\left(x\right), T\left(y\right)\right) \leq \frac{k}{2}\left[d\left(x, T\left(y\right)\right) + d\left(y, T\left(x\right)\right)\right]$$

for all $x, y \in X$ and some $k \in [0,1[$.

Definition 1.6. ([14]) Let $T : X \to 2^X$ be a multi-valued mapping and $\alpha : X \times X \to \mathbb{R}_0^+$ be a nonnegative mapping. Then we say that T is a multi-valued α-admissible mapping if for $x, y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(u,v) \geq 1$. for all $u \in T(x)$ and $v \in T(y)$.

Example 1.7. ([14]) Let $X = \mathbb{R}$ and $\alpha : X \times X \to \mathbb{R}_0^+$ defined by $\alpha(x,y) = x^2 + y^2$ for all $x, y \in X$. Define the mapping $T : X \to 2^X$ by $T(x) = \left\{\sqrt{|x|}, -\sqrt{|x|}\right\}$. Then T is multi-valued α-admissible.

Definition 1.8. Let $T : X \to 2^X$ and $\alpha, \eta : X \times X \to \mathbb{R}_0^+$ be mappings. Then we say that T is multi-valued α-admissible mapping with respect to η if for $x, y \in X$, $\alpha(x,y) \geq \eta(x,y)$ implies that $\alpha(u,v) \geq \eta(u,v)$ for all $u \in T(x)$ and $v \in T(y)$.

Hussain *et al.* [17] introduced the following family of new functions.

Let Π_G denotes the set of all functions $G : (\mathbb{R}_0^+)^4 \to \mathbb{R}^+$ which satisfy the property:

(G): for all $t_1, t_2, t_3, t_4 \in \mathbb{R}_0^+$, if $t_1 t_2 t_3 t_4 = 0$, then there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Let (X,d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $d(x,A) = \inf\{d(x,y) : y \in A\}$. We denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $K(X)$, the class of all compact subsets of X. Let H be the Hausdorff metric induced by the metric d on X, that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\},$$

for every $A, B \in CB(X)$. If $T : X \to CB(X)$ is a multi-valued mapping, then point $q \in X$ is said to be a fixed point of T if $q \in T(q)$.

Definition 1.9. Let (X, d) be a metric space. Let $T : X \to CB(X)$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be functions. Then we say that T is $(\alpha-\eta)$-continuous multi-valued mapping on $(CB(X), H)$, if for a given $x \in X$ and a sequence $\{x_n\}$ with $x_n \overset{d}{\to} x$ as $n \to \infty$, $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, implies $T(x_n) \overset{H}{\to} T(x)$, that is $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, implies $\lim_{n\to\infty} H(T(x_n), T(x)) = 0$.

The following result play a vital role regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball.

Theorem 1.10. ([19, Theorem 5.1.4]) *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d(T(x), T(y)) \leq kd(x, y),\ \textit{for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, T(x_0)) < (1-k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = T(x^*)$.

2. Multi-valued F-contraction on closed ball

In this section, we shall introduce the Chatterjea type multi-valued F-contraction on closed ball and obtain a fixed point theorem for this contraction in complete metric space and we shall show, through an example, the importance of Theorem 2.2 among other famous fixed point theorems present in literature.

Definition 2.1. Let (X, d) be a metric space. The mapping $T : X \to CB(X)$ is called Chatterjea type multi-valued F-contraction on closed ball, if for all $x, y \in \overline{B(x_0, r)} \subseteq X$, $H(T(x), T(y)) > 0$, then

$$2\tau + F(H(T(x), T(y))) \leq F\left(\frac{k}{2}\left[d(x, T(y)) + d(y, T(x))\right]\right), \qquad (2.1)$$

where $0 \leq k < 1$, $F \in \Delta_{F^*}$ and $\tau > 0$.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a Chatterjea type multi-valued F-contraction on closed ball $\overline{B(x_0, r)}$. Moreover,*

$$d(x_0, x_1) \leq (1 - \lambda)r, \text{ for some } x_1 \in T(x_0) \text{ and } \lambda = \frac{k}{2 - k}, \tag{2.2}$$

Then there exists a fixed point x^* in $\overline{B(x_0, r)}$.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_1 \in X$. If $x_1 \in T(x_1)$, then x_1 is a fixed point of T and we are done. Assume that $x_1 \notin T(x_1)$, then $T(x_0) \neq T(x_1)$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F\left(hH\left(T(x_0), T(x_1)\right)\right) \leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau,$$

choose a point x_1 in X such that $x_1 \in T(x_0)$ and $x_1 \notin T(x_1)$, continuing in this manner, we can define a sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$ and $x_n \notin T(x_n)$ for all $n \geq 0$.

First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$ by using mathematical induction method. From (2.2), we have

$$d(x_0, x_1) \leq (1 - \lambda)r < r, \tag{2.3}$$

for some $x_1 \in T(x_0)$, which shows that $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_j \in \overline{B(x_0, r)}$ for some $j \in N$. Since $T(x_0) \neq T(x_1)$, from (2.1), we obtain

$$2\tau + F(H(T(x_0), T(x_1))) \leq F\left(\frac{k}{2}\left[d(x_0, T(x_1)) + d(x_1, T(x_0))\right]\right).$$

Since,

$$d(x_1, T(x_1)) \leq H(T(x_0), T(x_1)) < hH(T(x_0), T(x_1)),$$

by condition (F_1), we have

$$\begin{aligned} F(d(x_1, T(x_1))) &\leq F\left(hH(T(x_0), T(x_1))\right) \\ &\leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau. \end{aligned} \tag{2.4}$$

By (F_4) we can write (note that $d(x_1, T(x_1)) > 0$)

$$F(d(x_1, T(x_1))) = \inf_{y \in T(x_1)} F(d(x_1, y))$$

and by (2.4), we have

$$\inf_{y \in T(x_1)} F(d(x_1, y)) \leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau. \tag{2.5}$$

By (2.5), there exists $x_2 \in T(x_1)$ such that

$$F(d(x_1, x_2)) \leq F\left(hH(T(x_0), T(x_1))\right) \leq F\left(H(T(x_0), T(x_1))\right) + \tau.$$

Thus,

$$2\tau + F(d(x_1, x_2)) \leq 2\tau + F\left(H(T(x_0), T(x_1))\right) + \tau,$$

it implies that,

$$\begin{aligned}\tau + F(d(x_1,x_2)) &\leq F\left(\frac{k}{2}\left[d(x_0,x_2)+d(x_1,T(x_0))\right]\right)\\ &\leq F\left(\frac{k}{2}\left[d(x_0,x_2)\right]\right),\end{aligned}$$

where $d(x_1,T(x_0))=0$. Since F is strictly increasing, we have

$$d(x_1,x_2) < \frac{k}{2}\left[d(x_0,x_1)+d(x_1,x_2)\right],$$

it implies that

$$d(x_1,x_2) < \frac{k}{2-k}d(x_0,x_1).$$

Thus, for $0<\lambda=\frac{k}{2-k}<1$ we have,

$$d(x_1,x_2)<\lambda d(x_0,x_1).$$

Repeating these steps for $x_3,x_4,\cdots,x_j$, we obtain

$$d(x_j,x_{j+1})<\lambda^j d(x_0,x_1). \tag{2.6}$$

Now, using triangle inequality and (2.6), we have

$$\begin{aligned}d(x_0,x_{j+1}) &\leq d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_j,x_{j+1})\\ &< d(x_0,x_1)\left[1+\lambda+\lambda^2+\cdots+\lambda^j\right]\\ &\leq (1-\lambda)r\frac{(1-\lambda^{j+1})}{1-\lambda}<r.\end{aligned}$$

This implies that $x_{j+1}\in\overline{B(x_0,r)}$. Hence $x_n\in\overline{B(x_0,r)}$ for all $n\in N$. Since,

$$d(x_n,T(x_n))\leq H(T(x_{n-1}),T(x_n))<hH(T(x_{n-1}),T(x_n)),$$

by (F_1), we have

$$\begin{aligned}F(d(x_n,T(x_n))) &\leq F\left(hH(T(x_{n-1}),T(x_n))\right)\\ &\leq F\left(H\left(T(x_{n-1}),T(x_n)\right)\right)+\tau. \end{aligned}\tag{2.7}$$

By (F_4), we can write (note that $d(x_n,T(x_n))>0$)

$$F(d(x_n,T(x_n)))=\inf_{y\in T(x_n)}F(d(x_n,y))$$

and by (2.7), we have

$$\inf_{y\in T(x_n)}F(d(x_n,y))\leq F\left(H\left(T(x_{n-1}),T(x_n)\right)\right)+\tau. \tag{2.8}$$

By (2.8), there exists $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leq hH(T(x_{n-1}), T(x_n)).$$

Now, since $x_n \notin T(x_n)$, condition (2.1) implies

$$2\tau + F(d(x_n, x_{n+1})) \quad \leq \quad 2\tau + F\left(H(T(x_{n-1}), T(x_n))\right) + \tau$$

and so,

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) \quad &\leq \quad F\left(\frac{k}{2}\left[d(x_{n-1}, T(x_n)) + d(x_n, T(x_{n-1}))\right]\right) \\ &\leq \quad F\left(\frac{k}{2}\left[d(x_{n-1}, x_{n+1})\right]\right) \\ &\leq \quad F\left(\frac{k}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\right) \\ &\leq \quad F\left(\frac{k}{2}\left[d(x_{n-1}, x_n) + \frac{k}{2-k} d(x_{n-1}, x_n)\right]\right) \\ &\leq \quad F\left(\frac{k}{2-k} d(x_{n-1}, x_n)\right) < F\left(d(x_{n-1}, x_n)\right). \end{aligned}$$

Thus, we get

$$F\left(d(x_n, x_{n+1})\right) < F\left(d(x_{n-1}, x_n)\right) - \tau. \tag{2.9}$$

By (F_1), we have

$$F\left(d(x_{n-1}, x_n)\right) < F\left(d(x_{n-1}, x_n)\right) \leq F\left(d(x_{n-2}, x_{n-1})\right) - \tau.$$

By (2.9), we obtain

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_0, x_1)\right) - n\tau. \tag{2.10}$$

By (2.10), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

By the property (F_3), there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \left((d(x_n, x_{n+1}))^{\kappa} F\left(d(x_n, x_{n+1})\right)\right) = 0. \tag{2.12}$$

Following (2.10), for all $n \in \mathbb{N}$, we obtain

$$(d(x_n, x_{n+1}))^{\kappa} \left(F\left(d(x_n, x_{n+1})\right) - F\left(d(x_0, x_1)\right)\right) \leq -\left(d(x_n, x_{n+1})\right)^{\kappa} n\tau \leq 0. \tag{2.13}$$

By (2.11), (2.12) and letting $n \to \infty$, in (2.13), we have

$$\lim_{n\to\infty} \left(n\left(d(x_n, x_{n+1})\right)^{\kappa}\right) = 0. \tag{2.14}$$

Since (2.14) holds, there exists $n_1 \in \mathbb{N}$, such that $n\left(d(x_n, x_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$, that is, for all $n \geq n_1$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}}. \tag{2.15}$$

Using (2.15), we get for $m > n \geq n_1$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \dfrac{1}{i^{\frac{1}{\kappa}}}$ leads to $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B(x_0, r)}, d\right)$. Since $\left(\overline{B(x_0, r)}, d\right)$ is a complete metric space, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$.

In order to prove that $x^* \in T(x^*)$, there are two cases:

Case I: Assume that T is continuous. Then, the sequence $\{T(x_i)\}_{i=1}^{\infty}$ converges to $T(x^*)$. Since $x_i \in T(x_{i-1})$ for all i, it follows that $x^* \in T(x^*)$. Hence x^* is a fixed point of T.

Case II: Let T is not continuous. Then, we assume that $H(T(x_n), T(x^*) > 0$, otherwise the result is obvious. Using contractive condition (2.1), we obtain

$$2\tau + F\left(H(T(x_n), T(x^*))\right) \leq F\left(\frac{k}{2}\left[d\left(x_n, T(x^*)\right) + d\left(x^*, T(x_n)\right)\right]\right).$$

Since F is right continuous, so

$$d(x_{n+1}, T(x^*)) \leq H(T(x_n), T(x^*)) < hH(T(x_n), T(x^*)),$$

which implies that

$$F(d(x_{n+1}, T(x^*)) < F\left(hH(T(x_n), T(x^*))\right) \leq F\left(H(T(x_n), T(x^*))\right) + \tau.$$

Thus, we have

$$2\tau + F(d(x_{n+1}, T(x^*)) \quad < \quad 2\tau + F\left(H(T(x_n), T(x^*))\right) + \tau$$

and so,

$$\tau + F(d(x_{n+1}, T(x^*)) \quad < \quad F\left(\frac{k}{2}\left[d(x_n, T(x^*)) + d(x^*, T(x_n))\right]\right),$$

which implies

$$d\left(x_{n+1},T(x^*)\right)<\frac{k}{2}\left[d\left(x_n,T(x^*)\right)+d\left(x^*,x_{n+1}\right)\right].$$

Letting $n\to\infty$, we get

$$d(x^*,T(x^*))<\frac{k}{2}d(x^*,T(x^*)).$$

Hence, we have

$$\left(1-\frac{k}{2}\right)d(x^*,T(x^*))<0,$$

this implies that $d(x^*,T(x^*))=0$. Since T is closed, thus, $x^*\in T(x^*)$ which completes the proof. $\square$

Following example shows that the contractive condition (2.1) holds on closed ball $\overline{B(x_0,r)}$, whereas it does not hold true on the whole space.

Example 2.3. Let $X=\mathbb{R}_0^+$ and $d\left(x,y\right)=\left|x-y\right|$. Then (X,d) is a complete metric space. Define a mapping $T:X\to CB(X)$ by

$$T\left(x\right)=\begin{cases}\left[0,\frac{x}{4}\right], & \text{if } x\in[0,1];\\ \left[x-\frac{1}{2},x-\frac{1}{4}\right], & \text{if } x\in(1,\infty).\end{cases}$$

Set $\tau=\ln(\sqrt{2})$, $k=\dfrac{3}{10}$, $x_0=\dfrac{1}{2}$, $r=\dfrac{1}{2}$, then $\overline{B(x_0,r)}=[0,1]$. If $F(\alpha)=\ln(\alpha)$, $\alpha>0$ and $\tau>0$, then for $x_1=\frac{1}{8}\in T(x_0)$,

$$d(x_0,T(x_0))=\inf_{y\in T(x_0)}d(x_0,y)<d(x_0,x_1)=\left|\frac{1}{2}-\frac{1}{8}\right|=\frac{3}{8}<(1-\lambda)r.$$

For $x,y\in\overline{B(x_0,r)}$, the inequality

$$\left|\frac{x}{4}-\frac{y}{4}\right|<\frac{k}{2}\left[\left|x-\frac{y}{4}\right|+\left|y-\frac{x}{4}\right|\right],$$

holds. Thus,

$$H(T(x),T(y))<\frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right],$$

which implies

$$2\tau+\ln\left(H(T(x),T(y))\right)\leq\ln\left(\frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right]\right).$$

That is,

$$2\tau+F\left(H(T(x),T(y))\right)\leq F\left(\frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right]\right).$$

Now if $x = 100, y = 10 \in (1, \infty)$, then

$$\begin{aligned} H(T(x), T(y)) &= \left| x - \frac{1}{4} - y + \frac{1}{4} \right| = |x - y| \\ &\geq \frac{k}{2} \left[d(x, T(y)) + d(y, T(x) \right] \end{aligned}$$

and consequently, contractive condition (2.1) does not hold on X. Hence, hypotheses of Theorem 2 hold on closed ball and $x = 0$ is a fixed point of T in $\overline{B(x_0, r)}$.

Corollary 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a Chatterjea type F-contraction on closed ball $\overline{B(x_0, r)}$ in complete metric space. If*

$$d(x_0, T(x_0)) \leq (1 - \lambda)r, \ \textit{where } \lambda = \frac{k}{2 - k}, \tag{2.16}$$

then there exists a point x^* in $\overline{B(x_0, r)}$ such that $T(x^*) = x^*$.

3. Multi-valued (α, η, GF)-contraction on closed ball

This section contains introduction of Chatterjea type multi-valued (α, η, GF)-contraction on closed ball, a new fixed point theorem for this contraction in complete metric space and an illustrative example which explains usefulness of Theorem 3 among other prominent fixed point theorems present in literature.

Definition 3.1. Let (X, d) be a metric space. Suppose that $\alpha, \eta : X \times X \to \mathbb{R}_0^+$ are two functions. The mapping $T : X \to CB(X)$ is called a Chatterjea type multi-valued (α, η, GF)-contraction on closed ball, if for all $x, y \in \overline{B(x_0, r)} \subseteq X$ with $\eta(x, y) \leq \alpha(x, y)$ and $H(T(x), T(y)) > 0$, we have

$$2\tau(G) + F\left(H(T(x), T(y))\right) \leq F\left(\frac{k}{2}\left[d(x, T(y)) + d(y, T(x)\right]\right), \tag{3.1}$$

where $\tau(G) = G(d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x)))$, $0 \leq k < 1, G \in \Pi_G$ and $F \in \Delta_{F^*}$.

Theorem 3.2. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a Chatterjea type multi-valued (α, η, GF)-contraction on closed ball $\overline{B(x_0, r)}$ satisfying the following conditions:*

1. *T is a multi-valued α-admissible mapping with respect to η,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, u_0) \geq \eta(x_0, u_0)$ for all $u_0 \in T(x_0)$,*
3. *$d(x_0, x_1) \leq (1 - \lambda)r$, for some $x_1 \in T(x_0)$ and $\lambda = \frac{k}{2-k}$.*

Then there exists a fixed point x^* of T in $\overline{B(x_0, r)}$.

Proof. Let $x_0 \in X$ be an arbitrary point such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ for all $x_1 \in T(x_0)$. Since T is a multi-valued α-admissible mapping with respect to η, for $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$

$$\alpha(x_1, x_2) \geq \eta(x_1, x_2) \text{ for all } x_1 \in T(x_0) \text{ and } x_2 \in T(x_1).$$

Continuing in this process, we can define a sequence $\{x_n\} \subset X$ such that

$$x_n \notin T(x_n), x_{n+1} \in T(x_n)$$

and

$$\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \tag{3.2}$$

for all $x_{n-1} \in T(x_{n-2})$ and $x_n \in T(x_{n-1})$.

Now if $x_1 \in T(x_1)$, then x_1 is a fixed point of T. So, we assume that $x_0 \neq x_1$, then $T(x_0) \neq T(x_1)$. Since F is right continuous, there exists a real number $h > 1$ such that

$$F\left(hH\left(T(x_0), T(x_1)\right)\right) \leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau(G).$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, T(x_n)) = 0$, then x_n is a fixed point of T, so we are done. We assume that $d(x_n, T(x_n)) > 0$, for all $n \in \mathbb{N}$. First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. From hypothesis (3) we obtain,

$$d(x_0, x_1) \leq (1 - \lambda)r < r \text{ for some } x_1 \in T(x_0), \tag{3.3}$$

which shows that $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_j \in \overline{B(x_0, r)}$ for some $j \in N$. Then from (3.1), we obtain

$$2\tau(G) + F(H(T(x_0), T(x_1))) \leq F\left(\frac{k}{2}\left[d(x_0, T(x_1)) + d(x_1, T(x_0))\right]\right).$$

Since,

$$d(x_1, T(x_1)) \leq H(T(x_0), T(x_1)) < hH(T(x_0), T(x_1)),$$

by (F_1), we have

$$\begin{aligned} F(d(x_1, T(x_1))) &\leq F\left(hH(T(x_0), T(x_1))\right) \\ &\leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau(G). \end{aligned} \tag{3.4}$$

By (F_4), we can write (note that $d(x_1, T(x_1)) > 0$)

$$F(d(x_1, T(x_1))) = \inf_{y \in T(x_1)} F(d(x_1, y))$$

and by (3.4), we have

$$\inf_{y \in T(x_1)} F(d(x_1, y)) \leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau(G). \tag{3.5}$$

By (3.5), there exists $x_2 \in T(x_1)$ such that

$$F(d(x_1, x_2)) \leq F\left(hH(T(x_0), T(x_1))\right) \leq F\left(H(T(x_0), T(x_1))\right) + \tau(G).$$

Thus, we have

$$2\tau(G) + F(d(x_1, x_2)) \le 2\tau(G) + F\left(H(T(x_0), T(x_1))\right) + \tau(G),$$

which implies that

$$\tau(G) + F(d(x_1, x_2)) \le F\left(\frac{k}{2}\left[d(x_0, x_2) + d(x_1, T(x_0))\right]\right),$$

where $\tau(G) = G(d(x_1, T(x_1)), d(x_2, T(x_2)), d(x_1, T(x_2)), 0)$. Thus by property (G), there exists $\tau > 0$ such that $\tau(G) = \tau$. Therefore, we get

$$\tau + F(d(x_1, x_2)) \le F\left(\frac{k}{2}\left[d(x_0, x_2)\right]\right).$$

Since F is strictly increasing, we have

$$d(x_1, x_2) < \frac{k}{2}\left[d(x_0, x_1) + d(x_1, x_2)\right],$$

it implies that

$$d(x_1, x_2) < \frac{k}{2-k} d(x_0, x_1).$$

Thus, for $0 < \lambda = \frac{k}{2-k} < 1$ we have

$$d(x_1, x_2) < \lambda d(x_0, x_1).$$

Continuing in this process, for $x_3, x_4, \cdots, x_j$, we obtain

$$d(x_j, x_{j+1}) < \lambda^j d(x_0, x_1). \tag{3.6}$$

Now, using triangle inequality and (3.6), we have

$$\begin{aligned} d(x_0, x_{j+1}) &\le d(x_0, x_1) + d\left(x_1, x_2\right) + \cdots + d(x_j, x_{j+1}) \\ &< d(x_0, x_1)\left[1 + \lambda + \lambda^2 + \cdots + \lambda^j\right] \\ &\le (1-\lambda) r \frac{(1-\lambda^{j+1})}{1-\lambda} < r. \end{aligned}$$

This implies that $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$.
Now, following the proof of the Theorem 2.2, we obtain for $m > n \ge n_1$,

$$d(x_n, x_m) \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

The convergence of the series $\sum_{i=n}^{\infty} \dfrac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B(x_0, r)}, d\right)$. Since $\left(\overline{B(x_0, r)}, d\right)$ is a complete metric space, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$.

Next, in order to prove that x^* is a fixed point of T, there are two cases:

Case I: Let T is (α, η)-continuous. Then, since $x_n \to x^*$ as $n \to \infty$ and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, $T(x_n) \overset{H}{\to} T(x^*)$, that is,

$$\lim_{n\to\infty} d(x_n, x^*) = 0$$

and

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} H(T(x_n), T(x^*)) = 0$. Hence x^* is a fixed point of T.

Case II: We assume that $d(x_n, T(x^*) > 0$, otherwise x^* is a fixed point of T. From contractive condition (3.1), we obtain

$$F\left(d(x_n, T(x^*))\right) \leq F\left(\frac{k}{2}\left[d\left(x_{n-1}, T(x^*)\right) + d\left(x^*, T(x_{n-1})\right)\right]\right) - \tau(G),$$

where $\tau(G) = G(d(x_{n-1}, x_n), d(x^*, T(x^*)), d(x_{n-1}, T(x^*)), d(x^*, x_n))$. Since F is continuous, we have

$$\begin{aligned} F\left(\lim_{n\to\infty} d(x_n, T(x^*))\right) \quad &\leq \quad F\left(\frac{k}{2}\left[\lim_{n\to\infty} d\left(x_{n-1}, T(x^*)\right) + \lim_{n\to\infty} d\left(x^*, x_n\right)\right]\right) \\ &\quad - \lim_{n\to\infty} \tau(G), \end{aligned}$$

which gives,

$$d(x^*, T(x^*)) \quad < \quad \frac{k}{2} d(x^*, T(x^*)).$$

That is,

$$\left(1 - \frac{k}{2}\right) d(x^*, T(x^*)) \quad < \quad 0.$$

This implies that $d(x^*, T(x^*)) = 0$. Consequently, x^* is a fixed point of T in $\overline{B(x_0, r)}$. This completes the proof. □

Example 3.3. Let $X = \mathbb{R}_0^+$ and d be the usual metric on X. Define $T : X \to X$, $\alpha : X \times X \to [0, +\infty)$, $\eta : X \times X \to \mathbb{R}^+$, $G : (\mathbb{R}_0^+)^4 \to \mathbb{R}^+$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} \left[0, \frac{5x}{19}\right] & \text{if } x \in [0, 1], \\ \\ \left[x - \frac{2}{3}, x - \frac{1}{3}\right] & \text{if } x \in (1, \infty), \end{cases}$$

$$\alpha(x, y) = \begin{cases} e^{x+y} & \text{if } x \in [0, 1], \\ \frac{1}{3} & \text{otherwise}, \end{cases}$$

$\eta(x,y)=\frac{1}{2}$ for all $x,y\in X$, $G(t_1,t_2,t_3,t_4)=\tau>0$ and $F(t)=\ln(t)$ with $t>0$. Set $k=\frac{4}{5}$, $x_0=\frac{1}{2}$, $r=\frac{1}{2}$, then $\overline{B(x_0,r)}=[0,1]$. Now

$$d\left(\frac{1}{2},T\left(\frac{1}{2}\right)\right)<\left|\frac{1}{2}-\frac{5}{38}\right|<r.$$

For $x,y\in\overline{B(x_0,r)}$, we know that $\alpha(x,y)=e^{x+y}\geq\frac{1}{2}=\eta(x,y)$.

On the other hand, for all $x\in[0,1]$ $T(x)\in[0,1]$, we have $\alpha(T(x),T(y))\geq\eta(T(x),T(y))$. Moreover, for $x\neq y$, $H(T(x),T(y))=\left|\frac{5x}{19}-\frac{5y}{19}\right|>0$. Clearly, $\alpha(0,T(0))\geq\eta(0,T(0))$. Hence, we have

$$H(T(x),T(y))=\left|\frac{5x}{19}-\frac{5y}{19}\right|=\frac{5}{19}\left|x-y\right|.$$

For $x,y\in\overline{B(x_0,r)}$, the inequality

$$\frac{5}{19}\left|x-y\right|<\frac{k}{2}\left[\left|x-\frac{5y}{19}\right|+\left|y-\frac{5x}{19}\right|\right]$$

holds. Thus, we have

$$H(T(x),T(y))<\frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right].$$

Consequently, we obtain that

$$2\tau+\ln\left(H(T(x),T(y))\right)\leq\ln\left(\frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right]\right),$$

which implies

$$2\tau+F\left(H(T(x),T(y))\right)\leq F\left(\frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right]\right).$$

If $x\notin\overline{B(x_0,r)}$ or $y\notin\overline{B(x_0,r)}$, then $\alpha(x,y)=\frac{1}{3}\ngeq\frac{1}{2}=\eta(x,y)$. Moreover, if $x=100,y=10\in(1,\infty)$, then

$$\begin{aligned}H(T(x),T(y)) &= \left|x-\frac{1}{3}-y+\frac{1}{3}\right|=\left|x-y\right|\\ &\geq \frac{k}{2}\left[d(x,T(y))+d(y,T(x)\right].\end{aligned}$$

Therefore, the contractive condition (3.1) does not hold on X. Hence, hypotheses of Theorem 3.2 hold on closed ball and $x=0$ is a fixed point of T in $\overline{B(x_0,r)}$.

Corollary 3.4. *Let (X,d) be a complete metric space. Let $T:X\to X$ be a Chatterjea type (α,η,GF)-contraction mapping on a closed ball $\overline{B(x_0,r)}$ satisfying the following assertions:*

(1) *T is an α-admissible mapping with respect to η;*

(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, T(x_0)) \geq \eta(x_0, T(x_0))$*;*
(3) $d(x_0, T(x_0)) \leq (1-\lambda)r$*, where* $\lambda = \frac{k}{2-k}$*.*

Then there exists a unique point x^* *in* $\overline{B(x_0, r)}$ *such that* $T(x^*) = x^*$*.*

References

[1] M. Abbas, B. Ali and S. Romaguera, *Fixed and periodic points of generalized contractions in metric spaces*, Fixed Point Theoryand Appl., **2013**: 243 (2013).

[2] R.P. Agarwal, D. O'Regan and N. Shahzad, *Fixed point theorems for generalized contractive maps of Mei-Keeler type*, Mathematische Nachrichten, **276** (2004), 3-12.

[3] J. Ahmad, A. Al-Rawashdeh and A. Azam, *Some New Fixed Point Theorems for Generalized Contractions in Complete Metric Spaces*, Fixed Point Theory and Appl., **2015**:80 (2015).

[4] Ö. Acar, G. Durmaz and G. Minak, *Generalized multivalued F-contractions on complete metric spaces*, Bull. of the Iranian Math. Soc., **40** (2014), 1469-1478.

[5] M. Arshad, A. Shoaib and I. Beg, *Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space*, Fixed Point Theory and Appl., **2013**:115 (2013).

[6] M. Arshad, A. Shoaib and P. Vetro, *Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered dislocated metric spaces*, J. of Funct. Spaces and Appl., article ID 638181, **2013** (2013).

[7] A. Augustynowicz, *Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function*, Serdica Math. Jour., **23** (1997), 203-210.

[8] R. Batra and S. Vashistha, *Fixed points of an F-contraction on metric spaces with a graph*, Int. J. Comput. Math., **91** (2014), 1-8.

[9] R. Batra, S, Vashistha and R. Kumar, *A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance*, J. Math. Comput. Sci., **4(5)** (2014), 826-833.

[10] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. of Math., **19(1)** (2003), 722.

[11] V. Berinde, *Iterative approximation of fixed points*, Springer-Verlag, Berlin Heidelberg, 2007.

[12] D.W. Boyd and J.S.W. Wong, *On nonlinear contractions*, Proc. of the Amer. Math. Soc., **20** (1969), 458464.

[13] S.K. Chatterjea, *Fixed point theorems*, Compts. Rend. Acad. Bulgare Sc. **25** (1972), 727-730.

[14] B.S. Choudhury and C. Bandyopadhyay, *A new multivalued contraction and stability of its fixed point sets*, J. of the Egyptian Math. Soc., **23** (2015), 321-325.

[15] Lj.B. Ciric, *A generalization of Banachs contraction principle*, Proc. of the Amer. Math. Soc., **45** (1974), 267273.

[16] M. Cosentino and P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-Type*, Filomat, **28(4)** (2014), 715-722.

[17] N. Hussain and P. Salimi, *Suzuki-wardowski type fixed point theorems for* α*-GF-contractions*, Taiwanese J. Math., **18** (2014), 1879-1895.

[18] N. Hussain, E. Karapınar, P. Salimi and F. Akbar, α*-admissible mappings and related fixed point theorems*, J. Inequal. Appl., **114** (2013), 1-11.

[19] E. Kryeyszig, Introductory functional analysis with applications, John Wiley & Sons, New York, (Wiley Classics Library Edition) 1989.

[20] G. Minak, A. Halvaci and I. Altun, *Ćirić type generalized $F-$contractions on complete metric spaces and fixed point results*, Filomat, **28(6)** (2014), 1143-1151.

[21] SB. Nadler, *Multivalued contraction mappings*, Pac. J. Math., **30** (1969), 475-488.

[22] D. ÓRegan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. of Math. Anal. and Appl., **341** (2008), 1241-1252.

[23] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl., **2014**:210 (2014).

[24] M. Sgroi and C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat, **27(7)** (2013), 1259-1268.

[25] P. Salimi, A. Latif and N. Hussain, *Modified $\alpha - \psi$-contractive mappings with applications*, Fixed Point Theory Appl., **2013**:151 (2013).

[26] NA. Secelean, *Iterated function systems consisting of F-contractions*, Fixed Point Theory Appl., **2013**:227 (2013).

[27] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for $\alpha - \psi$-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154-2165.

[28] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012**:94 (2012).