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EXISTENCE OF SOLUTIONS FOR NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION WITH ANALYTIC SEMIGROUP

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Abstract. We prove the existence, uniqueness and continuous dependence on initial data of solutions of nonlinear Volterra integrodifferential equations with nonlocal conditions in an arbitrary Banach space. The results are obtained by using the theory of analytic semigroups and the contraction mapping principle.

1. Introduction

The notion of "nonlocal condition" has been introduced to extend the study of the classical initial value problems, see, for example [2, 3, 6, 9, 13]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problem (IVP for short) was initiated by Byszewski [7]. In [7, 8], Byszewski using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order IVP:

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in [t_0, t_0 + a], \tag{1.1}$$

$$u(t_0) + g(t_1, t_2, \cdots, t_p, u(\cdot)) = u_0, \tag{1.2}$$

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where $0 \le t_0 < t_1 < \cdots < t_p \le t_0 + a, (p \in \mathbb{N}), u_0 \in X$, $-A$ is the infinitesimal generator of C_0 semigroup of $T(t), t \ge 0$ in a Banach space X and $f : [t_0, t_0 + a] \times X \to X$, $g(t_1, t_2, \cdots, t_p, \cdot) : X \to X$ are given functions. The symbol $g(t_1, t_2, \cdots, t_p, u(\cdot))$ is used in the sense that in the place of $'\cdot'$ we can substitute only elements of the set $\{t_1, t_2, \cdots, t_p\}$. For example $g(t_1, t_2, \cdots, t_p, u(\cdot))$ can be defined by the formula

$$g(t_1, t_2, \cdots, t_p, u(\cdot)) = C_1 u(t_1) + C_2 u(t_2) + \cdots + C_p u(t_p),$$

where C_i $(i = 1, 2, \cdots, p)$ are given constants.

In this paper, we discuss the existence and uniqueness of local solution for nonlinear Volterra integrodifferential equation with nonlocal condition of the type:

$$x'(t) + Ax(t) = f\Big(t, x(t), \int_0^t k(t, s, x(s))ds\Big), \quad t \in J = [0, b], \tag{1.3}$$

$$x(0) + g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0. \tag{1.4}$$

In (1.3), we assume that $-A$ is an infinitesimal generator of analytic semigroup $T(t), t \ge 0$, in a Banach space X. We note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A + \alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough, where I is the identity operator. Therefore, we reduce the general case in which $-A$ is the infinitesimal generator of a bounded analytic semigroup and the generator is invertible. For convenience, we suppose that $\|T(t)\| \le \overline{M}$, for $t \ge 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$. For $\alpha > 0$ we define the fractional power $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{(\alpha-1)} T(t) dt,$$

where $\Gamma(\cdot)$ is the gamma function. Since $A^{-\alpha}$ is one to one, $A^{\alpha} = (A^{-\alpha})^{-1}$. For $0 < \alpha \le 1$, A^{α} is closed linear operator whose domain with domain $D(A^{\alpha}) \supset D(A)$ dense in X. The closedness of A^{α} implies that $D(A^{\alpha})$, endowed with the graph norm of A^{α},

$$\|x\|_{A^{\alpha}} = \|x\| + \|A^{\alpha} x\|, \quad x \in D(A^{\alpha}),$$

is a Banach space. Since $0 \in \rho(-A)$, A^{α} is invertible, and its graph norm $\|\cdot\|_{A^{\alpha}}$ is equivalent to the norm

$$\|x\|_{\alpha} = +\|A^{\alpha} x\|.$$

Thus, $D(A^{\alpha})$ equipped with the norm $\|\cdot\|_{\alpha}$, is a Banach space, which we denote by X_{α}. From this definition, it is clear that $0 < \alpha < \beta$ implies $X_{\alpha} \supset X_{\beta}$ and that the embedding of X_{β} in X_{α} is continuous. For basic concepts and applications of this theory, we refer to the reader to A. Pazy [11].

Throughout this paper, we use the notation $J = [0, b]$. Let $f : J \times X_\alpha \times X_\alpha \to X$, $k : J \times J \times X \to X_\alpha$ and $g(t_1, t_2, \cdots, t_p, \cdot) : X_\alpha \to X$ be nonlinear functions.

Many authors have studied the problems such as existence, uniqueness, boundedness and other properties of solutions of these equations (1.3)–(1.4) or their special forms by using various techniques, see [1, 2, 3, 5, 10, 12] and the references cited therein. In an interesting paper [4], Balachandran and Chandrasekaran have studied the existence of local and global solutions of (1.3)–(1.4) when $f = g(t, x(t)) + \int_0^t h\Big(t, s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\Big)ds$. We are motivated by the work of Balachandran and Chandrasekaran in [4] and influenced by the work of Byszewski [7]. The results obtained in this paper generalize the some results of [1, 4].

The paper is organized as follows. In section 2, we present the preliminaries and hypotheses. Section 3 deals with main results. Finally, in section 4, we discuss an example to illustrate the theory.

2. Preliminaries and Hypotheses

Before proceeding to the main results, we recall some basic definitions and setforth preliminaries, and hypotheses that can be used in our further discussion.

Definition 2.1. A continuous solution $x(t)$ of the integral equation

$$\begin{aligned} x(t) &= T(t)x_0 - T(t)g(t_1, t_2, \cdots, t_p, x(\cdot)) \\ &\quad + \int_0^t T(t-s)f\Big(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\Big)ds, \quad t \in J \end{aligned} \tag{2.1}$$

is called a mild solution of (1.3)–(1.4) on J.

Definition 2.2. A classical solution of the equations (1.3)–(1.4) on J is a function $x \in C(J; X) \bigcap C^1((0, b]; X)$ satisfies 1.3–1.4 on J.

Definition 2.3. A function $x : J \to X$ is said to be a local solution of (1.3)–(1.4) if

- (a) $x : J \to X$ is continuous from J to $D(A)$;
- (b) $x : J \to X$ is differential and satisfies (1.3)–(1.4). If the closed interval J replaced by $[0, \infty)$ then the local solution of (1.3)–(1.4) is called global solution.

Let us list the following hypotheses:

(H_1) $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $T(t), t > 0$, in X.

(H_2) $0 \in \rho(-A)$, the resolvent set of $-A$.

(H_3) For $0 \leq \alpha < 1$, the fractional power A^α satisfies

$$\|A^\alpha T(t)\| \leq C_\alpha t^{-\alpha}, \quad \text{for} \quad t > 0,$$

where C_α is a real constant.

(H_4) For an open subset E of $J \times X_\alpha \times X_\alpha$, $f : E \to X$ satisfies the condition, if for every $(t, x, y) \in E$ there is a neighborhood $U \subset E$ and constants $L_1 \geq 0$, $0 < \Theta \leq 1$, such that

$$\|f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2)\| \leq L_1 \Big[|t_1 - t_2|^\Theta + \|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_\alpha \Big], \tag{2.2}$$

for all $(t_i, x_i, y_i) \in U, i = 1, 2$.

(H_5) For an open subset W of $J \times J \times X_\alpha$, $k : W \to X$ satisfies the condition, if for every $(t, x, y) \in W$ there is a neighborhood $V \subset W$ and constants $L_2 \geq 0$, $0 < \Theta_1,\ \Theta_2 \leq 1$, such that

$$\|k(t_1, s_1, x_1) - k(t_2, s_2, x_2)\| \leq L_2 \Big[|t_1 - t_2|^{\Theta_1} + |s_1 - s_2|^{\Theta_2} + \|x_1 - x_2\|_\alpha \Big], \tag{2.3}$$

for all $(t_i, s_i, x_i) \in V, i = 1, 2$.

(H_6) $g : J^p \times X_\alpha \to X$ and there exists constants $L_3 > 0$ and L_4 such that

$$\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\| \leq L_3, \quad \text{for} \quad 0 \leq t < b$$

and

$$\|g(t_1, t_2, \cdots, t_p, x_1(\cdot)) - g(t_1, t_2, \cdots, t_p, x_2(\cdot))\| \leq L_4 \|x_1 - x_2\|_\alpha.$$

3. Existence of solutions

Theorem 3.1. *Suppose that the hypotheses* (H_1)–(H_6) *hold. Then the initial value problem (1.3)–(1.4) has a unique solution* $x \in C([0, b); X) \bigcap C^1((0, b); X)$.

Proof. Choose $t^* > 0$ and $\delta > 0$ such that estimates (2.2) and (2.3) hold on the sets

$$U = \{(t, x, y) : 0 \leq t \leq t^*, \|x - x_0\|_\alpha \leq \delta, \|y - x_0\|_\alpha \leq \delta\},$$

and

$$V = \{(t, s, x) : 0 \leq t, s \leq t^*, \|x - x_0\|_\alpha \leq \delta\},$$

respectively.

Let

$$M = \max_{0 \le t \le t^*} \|f\Big(t, x_0, \int_0^t k(t,s,x_0)ds\Big)\|.$$

Choose b such that for $0 \le t < b$

$$\|T(t)A^\alpha x_0 - A^\alpha x_0\| < \delta/4,$$
$$\|T(t)A^\alpha g(t_1, t_2, \cdots, t_p.x(\cdot)) - A^\alpha g(t_1, t_2, \cdots, t_p.x(\cdot))\| < \delta/4$$

and

$$0 < b < \min\Big\{t^*, \Big[\frac{\delta(1-\alpha)}{2C_\alpha\Big(L_4\delta + L_1\delta + L_1L_3 + L_1L_2(\delta+L_3)b + M\Big)}\Big]^{\frac{1}{(1-\alpha)}}\Big\}. \tag{3.1}$$

Let $B = C(J;X)$ be the Banach space with usual supremum norm which we denote by $\|\cdot\|_B$. Define a mapping $F: B \to B$ by

$$\begin{aligned}(Fy)(t) &= T(t)A^\alpha x_0 - T(t)A^\alpha g(t_1, t_2, \cdots, t_p, A^{-\alpha}y(\cdot)) \\ &\quad + \int_0^t T(t-s)A^\alpha f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)ds.\end{aligned} \tag{3.2}$$

Obviously, $(Fy)(0) = A^\alpha x_0 - A^\alpha g$. Let S be the nonempty closed and bounded subset of B defined by

$$S = \{y \in B : y(0) = A^\alpha x_0 - A^\alpha g, \quad \|y(t) - (A^\alpha x_0 - A^\alpha g)\| \le \delta\}.$$

For $y \in S$, we have

$$\begin{aligned}
&\|(Fy)(t) - (A^\alpha x_0 - A^\alpha g)\| \\
&\le \|T(t)A^\alpha x_0 - A^\alpha x_0\| \\
&\quad + \|T(t)A^\alpha g(t_1, t_2, \cdots, t_p, A^{-\alpha}y(\cdot)) - A^\alpha g(t_1, t_2, \cdots, t_p, A^{-\alpha}y(\cdot))\| \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| \|f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)\|ds \\
&\le \delta/4 + \delta/4 + \int_0^t \|A^\alpha T(t-s)\|\Big[\|f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big) \\
&\quad - f\Big(s, x_0, \int_0^s k(s,\tau,x_0)d\tau\Big)\|\Big]ds \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| \|f\Big(s, x_0, \int_0^s k(s,\tau,x_0)d\tau\Big)\|ds \\
&\le \delta/2 + \int_0^t \|A^\alpha T(t-s)\| L_1\Big[\|A^{-\alpha}y(s) - (x_0 - g) - g\|_\alpha \\
&\quad + \int_0^s L_2\|A^{-\alpha}y(\tau) - (x_0-g) - g\|_\alpha d\tau\Big]ds + \int_0^t MC_\alpha(t-s)^{-\alpha}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta/2 + \int_0^t \|A^\alpha T(t-s)\| L_1 \Big[\delta + L_3 + L_2(\delta + L_3)b\Big] ds \\
&\quad + MC_\alpha b^{(1-\alpha)}(1-\alpha)^{-1} \\
&\leq \delta/2 + C_\alpha (t-s)^{(1-\alpha)}(1-\alpha)^{-1} L_1 \Big[\delta + L_3 + L_2(\delta + L_3)b\Big] \\
&\quad + MC_\alpha b^{(1-\alpha)}(1-\alpha)^{-1} \\
&\leq \delta/2 + C_\alpha \Big[L_1\delta + L_1 L_3 + L_1 L_2(\delta + L_3)b + M\Big] b^{(1-\alpha)}(1-\alpha)^{-1} \\
&< \delta/2 + C_\alpha \Big[L_4\delta + L_1\delta + L_1 L_3 + L_1 L_2(\delta + L_3)b + M\Big] b^{(1-\alpha)}(1-\alpha)^{-1} \\
&< \delta/2 + \delta/2 \\
&= \delta. \qquad (3.3)
\end{aligned}
$$

Therefore, F maps S into itself. Moreover, if $y_1, y_2 \in S$, then

$$
\begin{aligned}
&\|(Fy_1)(t) - (Fy_2)(t)\| \\
&\leq \|T(t)\Big[A^\alpha g(t_1, t_2, \cdots, t_p, A^{-\alpha} y_1(\cdot)) - A^\alpha g(t_1, t_2, \cdots, t_p, A^{-\alpha} y_2(\cdot))\Big]\| \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| \Big[\|f\Big(s, A^{-\alpha} y_1(s), \int_0^s k(s,\tau, A^{-\alpha} y_1(\tau)) d\tau\Big) \\
&\quad - f\Big(s, A^{-\alpha} y_2(s), \int_0^s k(s,\tau, A^{-\alpha} y_2(s)) d\tau\Big)\|\Big] ds \\
&\leq \|T(t)A^\alpha\| \Big[\|g(t_1, t_2, \cdots, t_p, A^{-\alpha} y_1(\cdot)) - g(t_1, t_2, \cdots, t_p, A^{-\alpha} y_2(\cdot))\|\Big] \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| L_1 \Big[\|A^{-\alpha} y_1(s) - A^{-\alpha} y_2(s)\|_\alpha \\
&\quad + \int_0^s \|k(s,\tau, A^{-\alpha} y_1(\tau)) - k(s,\tau, A^{-\alpha} y_2(\tau))\|_\alpha d\tau\Big] ds \\
&\leq C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1} L_4 \|A^{-\alpha} y_1(\cdots) - A^{-\alpha} y_2(\cdots)\|_\alpha \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| L_1 \Big[\|y_1(s) - y_2(s)\| + \int_0^s L_2 \|y_1(\tau) - y_2(\tau)\| d\tau\Big] ds \\
&\leq C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1} L_4 \|y_1 - y_2\|_B \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| L_1 \Big[\|y_1 - y_2\|_B + \int_0^s L_2 \|y_1 - y_2\|_B d\tau\Big] ds \\
&\leq C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1} L_4 \|y_1 - y_2\|_B \\
&\quad + C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1} L_1 \Big[\|y_1 - y_2\|_B + L_2 b \|y_1 - y_2\|_B\Big] \\
&\leq C_\alpha \Big[L_4 + L_1 + L_1 L_2 b\Big] b^{(1-\alpha)}(1-\alpha)^{-1} \|y_1 - y_2\|_B, \qquad (3.4)
\end{aligned}
$$

and using (3.1), the equation (3.4) implies that

$$\|(Fy_1)-(Fy_2)\|_B \le \frac{1}{2}\|y_1-y_2\|_B.$$

Hence, by the contraction mapping theorem, the mapping F has a unique fixed point $y \in S$. This fixed point satisfies the integral equation

$$y(t) = T(t)A^\alpha x_0 - T(t)A^\alpha g(t_1,t_2,\cdots,t_p,A^{-\alpha}y(\cdot)) + \int_0^t A^\alpha T(t-s) f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)ds. \tag{3.5}$$

From (2.2), (2.3) and the continuity of y it follows that

$$t \to f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,s,A^{-\alpha}y(s))ds\Big)$$

and $t \to k(t,s,A^{-\alpha}y(s))$ are continuous on J and therefore, there exist constants N and K such that

$$\|f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,s,A^{-\alpha}y(s))ds\Big)\| \le N, \tag{3.6}$$

and

$$\|k(t,s,A^{-\alpha}y(s))\| \le K. \tag{3.7}$$

Note that for every β satisfying $0<\beta<1-\alpha$ and every $0<h<1$, we have by Theorem 2.6.13 in Pazy [11] that

$$\|[T(h)-I]A^\alpha T(t-s)\| \le C_\beta h^\beta \|A^{\alpha+\beta}T(t-s)\| \le rh^\beta (t-s)^{-(\alpha+\beta)}, \tag{3.8}$$

for some $r>0$. If $0<t<t+h\le b$, then we have

$$\begin{aligned}
&\|y(t+h)-y(t)\| \\
&\le \|[T(h)-I]A^\alpha T(t)x_0\| + \|[T(h)-I]A^\alpha T(t) g(t_1,t_2,\cdots,t_p,A^{-\alpha}y(\cdot))\| \\
&\quad + \int_0^t \|[T(h)-I]A^\alpha T(t-s)\| \|f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)\|ds \\
&\quad + \int_t^{t+h} \|A^\alpha T(t+h-s)\| \|f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)\|ds \\
&= I_1+I_2+I_3+I_4.
\end{aligned} \tag{3.9}$$

Using (H_6) and (3.6), we find that

$$\begin{aligned}
I_1 &\le rt^{-(\alpha+\beta)}h^\beta \le M_1h^\beta, \\
I_2 &\le rL_3t^{-(\alpha+\beta)}h^\beta \le M_2h^\beta, \\
I_3 &\le rNh^\beta \int_0^t (t-s)^{-(\alpha+\beta)}ds \le M_3h^\beta,
\end{aligned}$$

$$I_4 \leq C_\alpha N \int_t^{t+h} (t+h-s)^{-\alpha} ds \leq M_4 h^\beta.$$

Here, M_1 and M_2 depends on t and vanish at $t \to 0$, but M_3 and M_4 can be selected to be independent of $t \in J$. Combining (3.9) with these estimates it follows that for every $t^{'}$ there is a constant C_1 such that

$$\|y(t) - y(s)\| \leq C_1 |t-s|^\beta \quad \text{for} \quad 0 \leq t^{'} \leq t, s \leq b$$

and therefore, this implies that y is locally Hölder continuous on $(0, b]$. The local Hölder continuity of $t \to f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,s,A^{-\alpha}y(s))ds\Big)$ follows from

$$\begin{aligned}
&\|f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,\tau,A^{-\alpha}y(\tau))d\tau\Big) - f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)\| \\
&\leq L_1\Big[|t-s|^\Theta + \|A^{-\alpha}y(t) - A^{-\alpha}y(s)\|_\alpha \\
&\quad + \|\int_0^t k(t,\tau,A^{-\alpha}y(\tau))d\tau - \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\|\Big] \\
&\leq L_1\Big[|t-s|^\Theta + \|y(t)-y(s)\| + \int_0^s \|k(t,\tau,A^{-\alpha}y(\tau)) - k(s,\tau,A^{-\alpha}y(\tau))\|_\alpha d\tau \\
&\quad + \int_s^t \|k(t,\tau,A^{-\alpha}y(\tau))\|_\alpha d\tau\Big] \\
&\leq L_1\Big[|t-s|^\Theta + C_1|t-s|^\beta \\
&\quad + \int_0^s L_2\Big(|t-s|^{\Theta_1} + |\tau-\tau|^{\Theta_2} + \|A^{-\alpha}y(\tau) - A^{-\alpha}y(\tau)\|_\alpha\Big)d\tau + \int_s^t K d\tau\Big] \\
&\leq L_1\Big[|t-s|^\Theta + C_1|t-s|^\beta + L_2|t-s|^{\Theta_1} b + K(t-s)\Big] \\
&\leq L_1\Big[|t-s|^\Theta + C_1|t-s|^\beta + L_2|t-s|^{\Theta_1} b + K(t-s)^{(1-\beta)}(t-s)^\beta\Big] \\
&\leq L_1\Big[1 + C_1 + L_2 b + K b^{(1-\beta)}\Big]|t-s|^\gamma \\
&\leq C_2|t-s|^\gamma,
\end{aligned}$$

where $C_2 = L_1\Big[1 + C_1 + L_2 b + K b^{(1-\beta)}\Big]$ and $0 < \gamma < 1$. Let y be a solution of (3.5). Consider the inhomogeneous initial value problem

$$x'(t) + Ax(t) = f\Big(t, A^{-\alpha}y(t), \int_0^t k(t,s,A^{-\alpha}y(s))ds\Big), \quad t \in J = [0,b], \tag{3.10}$$

$$x(0) + g(t_1, t_2, \cdots, t_p, A^{-\alpha}y(\cdot)) = x_0. \tag{3.11}$$

This problem has a unique solution $x \in C^1((0,b];X)$, which is given by

$$x(t) = T(t)x_0 - T(t)g(t_1, t_2, \cdots, t_p, A^{-\alpha}y(\cdot))$$

$$+\int_0^t T(t-s)f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)ds. \tag{3.12}$$

For $t>0$, each term of (3.12) is in $D(A)$ and a fortiori in $D(A^\alpha)$. Operating on both sides of (3.12) with A^α we find that

$$A^\alpha x(t) = T(t)A^\alpha x_0 - T(t)A^\alpha g(t_1,t_2,\cdots,t_p,A^{-\alpha}y(\cdot))$$
$$+\int_0^t T(t-s)A^\alpha f\Big(s, A^{-\alpha}y(s), \int_0^s k(s,\tau,A^{-\alpha}y(\tau))d\tau\Big)ds. \tag{3.13}$$

From (3.5), the right side of (3.13) equals $y(t)$ and therefore, $x(t) = A^{-\alpha}y(t)$ and by (3.12), $x \in C^1((0,b];X)$ is a solution of (1.3)–(1.4). The uniqueness of x follows from the uniqueness of the solutions of (3.5) and (3.10)–(3.11). Thus, the Theorem 3.1 is proved. □

In order to establish the global existence of classical solutions to (1.3), we need the following lemma.

Lemma 3.2 ([1],p.185)**.** *Let* $\phi(t,s)\ge 0$ *be continuous on* $0\le s\le t\le T<\infty$. *If there are positive constants* A, B *and* α *such that*

$$\phi(t,s)\le A + B\int_s^t (t-\sigma)^{\alpha-1}\phi(\sigma,s)d\sigma,$$

for $0\le s\le t\le T$, *then there is a constant* C *such that*

$$\phi(t,s)\le C.$$

The following theorem establishes the existence of global solutions of (1.3)-(1.4).

Theorem 3.3. *Suppose that* $-A$ *is the infinitesimal generator of an analytic semigroup* $T(t)$ *satisfying* $\|T(t)\|\le \overline{M}$, *for* $t\ge 0$ *and* $0\in\rho(-A)$. *Let the hypotheses* (H_4) *and* (H_5) *be satisfied with* $J=[0,\infty)$. *Moreover, if there are continuous nondecreasing functions* $p_1,p_2:[0,\infty)\to\mathbb{R}^+$ *such that*

$$\|f(t,x,y)\|\le p_1(t)[\|x\|_\alpha + \|y\|_\alpha]$$

and

$$\|k(t,s,x)\|\le p_2(t)\|x\|_\alpha$$

for $t\ge 0$, $x,y\in X_\alpha$, *then the initial value problem (1.3)–(1.4) has a unique solution* x *which exists for all* $t\ge 0$.

Proof. Applying Theorem 3.1 we can continue the solution of (1.3)–(1.4) as long as $\|x\|_\alpha$ is bounded. It is therefore sufficient to show that if x exist on $[0,b)$ then $\|x(t)\|_\alpha$ is bounded $t\uparrow b$. Since

$$A^\alpha x(t) = T(t)A^\alpha x_0 - T(t)A^\alpha g(t_1,t_2,\cdots,t_p,x(\cdot))$$

$$+ \int_0^t A^\alpha T(t-s) f\Big(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\Big) ds. \tag{3.14}$$

Taking norm on both sides of equation (3.14) and using the properties of $T(t)$ and A that they commute, $T(t) \le \overline{M}$, $A^\alpha T(t) \le C_\alpha t^{-\alpha}$, for $t \ge 0$ and hypotheses (H_4) and (H_5), we get

$$\begin{aligned}
\|A^\alpha x(t)\| &\le \|T(t)A^\alpha x_0\| + \|T(t)\|\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\| \\
&\quad + \int_0^t \|A^\alpha T(t-s)\| \|f\Big(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\Big)\| ds \\
&\le \overline{M}\|A^\alpha x_0\| + \overline{M}\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\| \\
&\quad + \int_0^t C_\alpha (t-s)^{-\alpha} p_1(s)\Big[\|x(s)\|_\alpha + \int_0^s \|k(s,\tau,x(\tau))\| d\tau\Big] ds \\
&\le \overline{M}\|A^\alpha x_0\| + \overline{M}\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\| \\
&\quad + \int_0^t C_\alpha (t-s)^{-\alpha} p_1(s)\Big[\|x(s)\|_\alpha + \int_0^s p_2(s)\|x(\tau)\|_\alpha d\tau\Big] ds \\
&\le \overline{M}\|A^\alpha x_0\| + \overline{M}\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\| \\
&\quad + C_\alpha \int_0^t (t-s)^{-\alpha} P(s)\Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau\Big] ds \\
&\le \overline{M}\|A^\alpha x_0\| + \overline{M}\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\| \\
&\quad + C_\alpha P \int_0^t (t-s)^{-\alpha} \Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau\Big] ds
\end{aligned}$$

that is,

$$\|x(t)\|_\alpha \le C_3 + C_4 \int_0^t (t-s)^{-\alpha}\Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau\Big] ds, \tag{3.15}$$

where $C_3 = \overline{M}\|A^\alpha x_0\| + \overline{M}\|A^\alpha g(t_1, t_2, \cdots, t_p, x(\cdot))\|$, $C_4 = C_\alpha P$ and $P = \sup\{p_1(t), p_2(t)\}$. Integrating (3.15) over $(0, t)$ and changing the order of integration, we get

$$\begin{aligned}
&\int_0^t \|x(\xi)\|_\alpha d\xi \\
&\le C_3 b + C_4 \int_0^t \int_s^t (\xi - s)^{-\alpha}\Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau\Big] d\xi ds \\
&\le C_3 b + C_4 \int_0^t (t-s)^{(1-\alpha)}(1-\alpha)^{-1}\Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau\Big] ds \\
&\le C_3 b + \frac{C_4 b}{(1-\alpha)} \int_0^t (t-s)^{-\alpha}\Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau\Big] ds
\end{aligned}$$

$$\leq C_5 + C_6 \int_0^t (t-s)^{-\alpha} \Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau \Big] ds, \tag{3.16}$$

for some positive constants C_5 and C_6, depending on α and b. Adding (3.15) and (3.16), we have

$$\begin{aligned} &\|x(t)\|_\alpha + \int_0^t \|x(\xi)\|_\alpha d\xi \\ &\leq C_7 + C_8 \int_0^t (t-s)^{-\alpha} \Big[\|x(s)\|_\alpha + \int_0^s \|x(\tau)\|_\alpha d\tau \Big] ds, \end{aligned} \tag{3.17}$$

for some positive constants C_7 and C_8, depending on α and b. Define

$$z(t) = \|x(t)\|_\alpha + \int_0^t \|x(\xi)\|_\alpha d\xi. \tag{3.18}$$

Using (3.18), the equation (3.17) becomes

$$z(t) \leq C_7 + C_8 \int_0^t (t-s)^{-\alpha} z(s) ds. \tag{3.19}$$

Applying the Lemma 3.2 to (3.19), we obtain

$$z(t) \leq C \quad \text{on} \quad [0, b).$$

Therefore,

$$\|x(t)\|_\alpha + \int_0^t \|x(\xi)\|_\alpha d\xi = z(t) \leq C,$$

which yields

$$\|x(t)\|_\alpha \leq C.$$

This completes the proof of the Theorem 3.3. □

Theorem 3.4. *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq \overline{M}$, for $t \geq 0$ and $0 \in \rho(-A)$. Let the hypotheses $(H_3) - (H_6)$ be satisfied and $x_0 \in X_\alpha$. Suppose that the functions $x_1(t)$ and $x_2(t)$ satisfy the equation (1.3) for $0 \leq t \leq b < \infty$ with*

$$x_1(0) = g(t_1, t_2, \cdots, t_p, x_1(\cdot)) = {x_0}^*$$

and

$$x_2(0) = g(t_1, t_2, \cdots, t_p, x_2(\cdot)) = {x_0}^{**},$$

respectively and $x_1(t), x_2(t) \in X_\alpha$. Then, we have

$$\|x_1(t) - x_2(t)\|_\alpha \leq C.$$

Proof. Let the functions $x_1(t)$ and $x_2(t)$ satisfy the equation (1.3) for $0 \le t \le b < \infty$ with

$$x_1(0) = g(t_1, t_2, \cdots, t_p, x_1(\cdot)) = x_0{}^*$$

and

$$x_2(0) = g(t_1, t_2, \cdots, t_p, x_2(\cdot)) = x_0{}^{**},$$

respectively and $x_1(t), x_2(t) \in X_\alpha$. Then by Theorem 3.3, we obtain

$$\begin{aligned} A^\alpha x_1(t) &= T(t)A^\alpha x_0{}^* - T(t)A^\alpha g(t_1, t_2, \cdots, t_p, x_1(\cdot)) \\ &\quad + \int_0^t A^\alpha T(t-s) f\Big(s, x_1(s), \int_0^s k(s,\tau,x_1(\tau))d\tau\Big)ds \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} A^\alpha x_2(t) &= T(t)A^\alpha x_0{}^{**} - T(t)A^\alpha g(t_1, t_2, \cdots, t_p, x_2(\cdot)) \\ &\quad + \int_0^t A^\alpha T(t-s) f\Big(s, x_2(s), \int_0^s k(s,\tau,x_2(\tau))d\tau\Big)ds. \end{aligned} \tag{3.21}$$

Using hypotheses and properties of $T(t)$ and A, we have

$$\begin{aligned} &\|A^\alpha x_1(t) - A^\alpha x_2(t)\| \\ &\le \|T(t)A^\alpha x_0{}^* - T(t)A^\alpha x_0{}^{**}\| \\ &\quad + \|T(t)A^\alpha g(t_1, t_2, \cdots, t_p, x_1(\cdot)) - T(t)A^\alpha g(t_1, t_2, \cdots, t_p, x_2(\cdot)) \\ &\quad + \int_0^t \|A^\alpha T(t-s)\| \Big[\|f\Big(s, x_1(s), \int_0^s k(s,\tau,x_1(\tau))d\tau\Big) \\ &\quad - f\Big(s, x_2(s), \int_0^s k(s,\tau,x_2(\tau))d\tau\Big)\|\Big]ds \\ &\le \overline{M}\|x_0 * - x_0{}^{**}\|_\alpha + \overline{M}C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1}L_4\|x_1(t) - x_2(t)\|_\alpha \\ &\quad + C_\alpha \int_0^t (t-s)^{-\alpha} L_1 \Big[\|x_1(s) - x_2(s)\|_\alpha + L_2 \int_0^s \|x_1(\tau) - x_2(\tau)\|_\alpha d\tau\Big]ds. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|x_1(t) - x_2(t)\|_\alpha \\ &\le \frac{\overline{M}}{\overline{C}}\|x_0 * - x_0{}^{**}\|_\alpha + \frac{C_\alpha L_1}{\overline{C}} \int_0^t (t-s)^{-\alpha} \\ &\quad \times \Big[\|x_1(s) - x_2(s)\|_\alpha + L_2 \int_0^s \|x_1(\tau) - x_2(\tau)\|_\alpha d\tau\Big]ds, \end{aligned} \tag{3.22}$$

where $\overline{C} = \Big[1 - \overline{M}C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1}L_4\Big]$, $\overline{M}C_\alpha b^{(1-\alpha)}(1-\alpha)^{-1}L_4 < 1$. Define $m(t) = \|x_1(t) - x_2(t)\|_\alpha$. Then from equation (3.22), we get

$$m(t) \leq \frac{\overline{M}}{\overline{C}}\|x_0* - {x_0}^{**}\|_\alpha + \frac{C_\alpha L_1}{\overline{C}}\int_0^t (t-s)^{-\alpha}\Big[m(s) + L_2\int_0^s m(\tau)d\tau\Big]ds. \tag{3.23}$$

Integrating (3.23) over $(0,t)$ and changing the order of integration, we obtain

$$\begin{aligned}\int_0^t m(\xi)d\xi \leq{}& \frac{\overline{M}b}{\overline{C}}\|x_0* - {x_0}^{**}\|_\alpha \\ &+ \frac{C_\alpha L_1}{\overline{C}}\int_0^t\int_s^t (\xi-s)^{-\alpha}\Big[m(s) + L_2\int_0^s m(\tau)d\tau\Big]d\xi ds \\ \leq{}& \frac{\overline{M}b}{\overline{C}}\|x_0* - {x_0}^{**}\|_\alpha \\ &+ \frac{C_\alpha L_1 b}{\overline{C}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\Big[m(s) + L_2\int_0^s m(\tau)d\tau\Big]ds. \end{aligned} \tag{3.24}$$

Adding the corresponding sides of of equations (3.23) and (3.24), we have

$$\begin{aligned} &m(t) + \int_0^t m(\xi)d\xi \\ &\leq \frac{\overline{M}}{\overline{C}}(1+b)\|x_0* - {x_0}^{**}\|_\alpha \\ &\quad + \frac{C_\alpha L_1}{\overline{C}}[1 + \frac{b}{(1-\alpha)}]\int_0^t (t-s)^{-\alpha}\Big[m(s) + L_2\int_0^s m(\tau)d\tau\Big]ds \\ &\leq d_1 + d_2\int_0^t (t-s)^{-\alpha}\Big[m(s) + \int_0^s m(\tau)d\tau\Big]ds, \end{aligned} \tag{3.25}$$

where $d_1 = \frac{\overline{M}}{\overline{C}}(1+b)\|x_0* - {x_0}^{**}\|_\alpha$ and

$$d_2 = \max\{\frac{C_\alpha L_1}{\overline{C}}[1 + \frac{b}{(1-\alpha)}], \frac{C_\alpha L_1 L_2}{\overline{C}}[1 + \frac{b}{(1-\alpha)}]\},$$

depending on α and difference estimation of initial data. Let

$$w(t) = m(t) + \int_0^t m(\xi)d\xi.$$

Then equation (3.25) takes the form

$$w(t) \leq d_1 + d_2\int_0^t (t-s)^{-\alpha}w(s)ds. \tag{3.26}$$

Applying the Lemma 3.2 to (3.26), which yields $w(t) \le C$ and consequently, we have

$$\|x_1(t) - x_2(t)\|_\alpha = m(t) \le C,$$

where C depends upon the initial data of solutions of the equation (1.3). This proves the Theorem 3.4. □

4. Application

Now, we give an example to illustrate the application of our results established in previous section. We consider the following boundary value problem

$$\frac{\partial w(t,\xi)}{\partial t} - \frac{\partial^2 w(t,\xi)}{\partial \xi^2} = \mu\Big(t, w(t,\xi), \int_0^t a(t,s,w(s,\xi))ds\Big), \quad t > 0, \quad \xi \in I = [0,\pi], \tag{4.1}$$

$$w(t,0) = w(t,\pi) = 0, \quad t > 0, \tag{4.2}$$

$$w(0,\xi) = x_0(\xi) + \sum_{i=1}^{n} \alpha_i w(t_i,\xi), \quad \xi \in I, \tag{4.3}$$

where $\mu : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $a : J \times J \times \mathbb{R} \to \mathbb{R}$ are continuous and $t_i > 0$, $\alpha_i \in \mathbb{R}$ are prefixed numbers. Let us take $X = L^2([0,\pi])$. We define the operator $A : D(A) \subset X \to X$ by $Aw = -w_{\xi\xi}$, where $D(A) = \{w(\cdot) \in X : w(0) = w(\pi) = 0\}$. Furthermore, A has discrete spectrum, the eigenvalues are $n^2, n \in \mathbb{N}$, with corresponding normalized characteristics vectors $w_n(\xi) := \sqrt{\frac{2}{\pi}}\sin(n\xi)$, $n = 1, 2, 3, \cdots$, and the following conditions hold :

(i) $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(ii) If $w \in D(A)$ then $Aw = \sum_{n=1}^{\infty} n^2 < w, w_n > w_n$.

Hence, A is infinitesimal generator of an analytic semigroup $T(t), t \ge 0$ on X and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, w_n > w_n, \quad w \in X.$$

Define the functions $f : [0,\infty) \times X \times X \to X$, $a : J \times J \times X \to X$, and $g : C(J,X) \to X$ as follows

$$f(t,x,y)\xi = \mu(t, x(t,\xi), y(t,\xi)),$$
$$a(t,s,x(t,\xi)) = k(t,s,x)\xi,$$

$$g(t_1, t_2, \cdots, t_k, x(\cdot))\xi = -\sum_{i=1}^{k} \beta_i x(t_i, \xi),$$

for $t_i > 0$ and $0 \leq \xi \leq \pi$. From the above choices of the functions and generator A, the equations (4.1)–(4.3) can be formulated as an abstract nonlinear Volterra integrodifferential equations (1.3)–(1.4) in a Banach space X. Further, for every $x \in X$,

$$A^{-\frac{1}{2}} w = \sum_{n=1}^{\infty} 1/n < w, w_n > w_n,$$

with $\|A^{-\frac{1}{2}}\| = 1$ and the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n < w, w_n > w_n,$$

on the space $D(A^{\frac{1}{2}}) = \{w \in X : \sum_{n=1}^{\infty} n < w, w_n > w_n \in X\}$. Let X_α denote the space $D(A^\alpha)$ with $\alpha = 1/2$. Under the assumptions that hypotheses (H_4) and (H_5) are satisfied then by Theorems 3.1 and 3.3 there exits a unique global classical solution of the equations (1.3)-(1.4) which guarantees the existence of a unique global classical solution of IVP (4.1)-(4.3).

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