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ON THE LOCATION OF ZEROS OF A CERTAIN CLASS OF POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. In this paper, the authors employ a new proof method to establish some results on the location of zeros of a certain class of polynomials $P_\mu(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ with restricted coefficients and thereby obtain a variety of interesting extensions and generalizations of Eneström-Kakeya's Theorem.

1. **Introduction**

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. The following famous and interesting result on the location of zeros of a polynomial with restricted coefficients is known as the Eneström-Kakeya theorem [1, 2].

Theorem 1.1. ([1]) *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients a_j satisfy*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ lie in the closed unit disk $|z| \leq 1$.

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There are several extensions and generalizations of the Eneström-Kakeya theorem in the literature [1-13].

In this paper, we prove some extensions of the above result by considering a certain class of polynomials

$$P_\mu(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$$

with a gap between the leading and the preceding real coeffiencients, which has an index of μ. This polynomial is greatly studied in connection with Berstein type inequalities [12]. It easy to see that

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

is a special case of $P_\mu(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ and thereby obtain a variety of interesting extensions and generalizations of Eneström-Kakeya's Theorem.

2. Main results

Theorem 2.1. *Let* $P_\mu(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$*, where* $a_0 \neq 0$ *and for some reals* $\rho_1, \rho_2, t > 0$*,* $1 \leq \mu \leq \lambda \leq n$ *and*

$$\rho_1 + t^n a_n \leq t^{n-1} a_{n-1} \leq \cdots \leq t^{\lambda+1} a_{\lambda+1} \leq t^\lambda a_\lambda,$$
$$t^\lambda a_\lambda \geq t^{\lambda-1} a_{\lambda-1} \geq \cdots \geq t^\mu a_\mu \geq a_0 - \rho_2.$$

Then all the zeros of $P_\mu(z)$ *lie in the union of the disks* $|z| \leq t$ *and*

$$|z + \tfrac{\rho_1}{a_n}| \leq \tfrac{1}{|a_n|}(-a_n t - \rho_1 - \tfrac{a_{\mu-1}}{t^{n-1}} + \tfrac{2\rho_2}{t^{n-1}} + \tfrac{|a_{\mu-1}|}{t^{n-1}} + \tfrac{2a_\lambda}{t^{n-\lambda-1}}).$$

Proof. Consider the polynomial

$$\begin{aligned}
F(z) &= (t-z)P_\mu(z) \\
&= a_0 t + \textstyle\sum_{j=\mu}^{n} a_j t z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^j \\
&= a_0 t + \textstyle\sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\
&= a_0(t-z) + \textstyle\sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\
&= a_0 t + (a_{\mu-1} - a_0 + \rho_2) z - \rho_2 z - a_{\mu-1} z + a_\mu t z^\mu \\
&\quad + (a_n t - a_{n-1}) z^n + \textstyle\sum_{j=\mu+1}^{n-1} (a_j t - a_{j-1}) z^j - a_n z^{n+1} \\
&= -a_n z^{n+1} + (\rho_1 - a_n t) z^n + a_\mu t z^\mu + (a_n t - \rho_1 - a_{n-1}) z^n \\
&\quad + a_n t z^n + a_0 t + (a_{\mu-1} - a_0 + \rho_2) z - \rho_2 z - a_{\mu-1} z \\
&\quad + (a_{\mu+1} t - a_\mu) z^{\mu+1} + \textstyle\sum_{j=\mu+1}^{n-1} (a_j t - a_{j-1}) z^j.
\end{aligned}$$

We have

$$\begin{aligned}
|F(z)| &= |-a_nz^{n+1}+(\rho_1-a_nt)z^n+a_\mu tz^\mu+(a_nt-\rho_1-a_{n-1})z^n \\
&\quad +a_ntz^n+a_0t+(a_{\mu-1}-a_0+\rho_2)z-\rho_2z-a_{\mu-1}z \\
&\quad +(a_{\mu+1}t-a_\mu)z^{\mu+1}+\textstyle\sum_{j=\mu+1}^{n-1}(a_jt-a_{j-1})z^j| \\
&= |-a_nz^{n+1}+(\rho_1-a_nt)z^n+a_\mu tz^\mu+(a_nt-\rho_1-a_{n-1})z^n \\
&\quad +a_ntz^n+a_0t+(a_{\mu-1}-a_0+\rho_2)z-\rho_2z-a_{\mu-1}z \\
&\quad +(a_{\mu+1}t-a_\mu)z^{\mu+1}+\textstyle\sum_{j=\mu+2}^{\lambda}(a_jt-a_{j-1})z^j \\
&\quad +\textstyle\sum_{j=1+\lambda}^{n-1}(a_jt-a_{j-1})z^j| \\
&= |z^n(-a_nz-\rho_1+(a_nt+\rho_1-a_{n-1})+\tfrac{(a_{\mu+1}t-a_\mu)z^{\mu+1}}{z^n}+\tfrac{a_\mu tz^\mu}{z^n} \\
&\quad +\tfrac{a_0t+(a_{\mu-1}-a_0+\rho_2)z-\rho_2z-a_{\mu-1}z}{z^n}+\textstyle\sum_{j=\mu+2}^{\lambda}\tfrac{(a_jt-a_{j-1})z^j}{z^n} \\
&\quad +\textstyle\sum_{j=1+\lambda}^{n-1}\tfrac{(a_jt-a_{j-1})z^j}{z^n})| \\
&\geq |z|^n|a_nz+\rho_1| \\
&\quad -|z|^n[|a_nt+\rho_1-a_{n-1}|+\tfrac{|a_{\mu+1}t-a_\mu|}{|z|^{n-\mu-1}}+\tfrac{|a_\mu|}{|z|^{n-\mu}} \\
&\quad +\tfrac{|a_0|t}{|z|^n}+\tfrac{|a_{\mu-1}-a_0+\rho_2|}{|z|^{n-1}}+\tfrac{\rho_2}{|z|^{n-1}} \\
&\quad +\tfrac{|a_{\mu-1}|}{|z|^{n-1}}+\textstyle\sum_{j=\mu+2}^{\lambda}\tfrac{|a_jt-a_{j-1}|}{|z|^{n-j}}+\textstyle\sum_{j=1+\lambda}^{n-1}\tfrac{|a_jt-a_{j-1}|}{|z|^{n-j}}].
\end{aligned}$$

Now, let $|z|\geq t$, so that $\frac{1}{|z|^{n-j}}\leq\frac{1}{|t|^{n-j}}$ for $1\leq\mu\leq j\leq n$. Then, we have

$$\begin{aligned}
|F(z)| &\geq |z|^n[|a_nz+\rho_1| \\
&\quad -(|a_nt+\rho_1-a_{n-1}|+\tfrac{|a_{\mu+1}t-a_\mu|}{|t|^{n-\mu-1}}+\tfrac{|a_\mu|}{|z|^{n-\mu}} \\
&\quad +\tfrac{|a_0|t}{|t|^n}+\tfrac{|a_{\mu-1}-a_0+\rho_2|}{|t|^{n-1}}+\tfrac{\rho_2}{|t|^{n-1}} \\
&\quad +\tfrac{|a_{\mu-1}|}{|t|^{n-1}}+\textstyle\sum_{j=\mu+2}^{\lambda}\tfrac{|a_jt-a_{j-1}|}{|t|^{n-j}}+\textstyle\sum_{j=\lambda+1}^{n-1}\tfrac{|a_jt-a_{j-1}|}{|t|^{n-j}})] \\
&= |z|^n[|a_nz+\rho_1| \\
&\quad -(-a_nt-\rho_1+a_{n-1}+\tfrac{-a_{\mu+1}}{t^{n-j-2}}-\tfrac{a_\mu}{t^{n-\mu-1}}+\tfrac{a_\mu}{|t|^{n-\mu}} \\
&\quad +\tfrac{a_0}{t^{n-1}}-\tfrac{a_{\mu-1}}{t^{n-1}}-\tfrac{a_0}{t^{n-1}}+\tfrac{\rho_2}{t^{n-1}}+\tfrac{\rho_2}{t^{n-1}} \\
&\quad +\tfrac{|a_{\mu-1}|}{t^{n-1}}+\tfrac{a_\lambda}{t^{n-\lambda-1}}-\textstyle\sum_{j=\mu+2}^{\lambda-1}-\tfrac{a_\mu}{t^{n-\mu-1}}-\textstyle\sum_{j=\mu+2}^{\lambda-1}\tfrac{a_j}{t^{n-j-1}} \\
&\quad +\tfrac{a_\lambda}{t^{n-\lambda-1}}+\textstyle\sum_{j=\lambda+1}^{n-2}\tfrac{a_j}{t^{n-\lambda-1}}-a_{n-1}-\textstyle\sum_{j=\lambda+1}^{n-2}\tfrac{a_j}{t^{n-\lambda-1}})] \\
&\geq |z|^n[|a_nz+\rho_1|-(-a_nt-\rho_1+a_{n-1}+\tfrac{a_\mu}{t^{n-\mu-1}} \\
&\quad +\tfrac{a_0}{t^{n-1}}-\tfrac{a_{\mu-1}}{t^{n-1}}-\tfrac{a_0}{t^{n-1}}+\tfrac{\rho_2}{t^{n-1}}+\tfrac{\rho_2}{t^{n-1}} \\
&\quad +\tfrac{|a_{\mu-1}|}{t^{n-1}}+\tfrac{a_\lambda}{t^{n-\lambda-1}}-\textstyle\sum_{j=\mu+2}^{\lambda-1}-\tfrac{a_\mu}{t^{n-\mu-1}}-\textstyle\sum_{j=\mu+2}^{\lambda-1}\tfrac{a_j}{t^{n-j-1}} \\
&\quad +\tfrac{a_\lambda}{t^{n-\lambda-1}}+\textstyle\sum_{j=\lambda+1}^{n-2}\tfrac{a_j}{t^{n-\lambda-1}}-a_{n-1}-\textstyle\sum_{j=\lambda+1}^{n-2}\tfrac{a_j}{t^{n-\lambda-1}})] \\
&> 0.
\end{aligned}$$

It is easy to see that,

$$|a_nz+\rho_1| \quad > \quad (-a_nt-\rho_1-\tfrac{a_{\mu-1}}{t^{n-1}}+\tfrac{2\rho_2}{t^{n-1}}+\tfrac{|a_{\mu-1}|}{t^{n-1}}+\tfrac{2a_\lambda}{t^{n-\lambda-1}}),$$

that is,

$$|z+\frac{\rho_1}{a_n}| \;>\; \frac{1}{|a_n|}(-a_nt-\rho_1-\frac{a_{\mu-1}}{t^{n-1}}+\frac{2\rho_2}{t^{n-1}}+\frac{|a_{\mu-1}|}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}),$$

then all the zeros of $F(z)$ whose modulus is greater than or equal to t lie in

$$|z+\frac{\rho_1}{a_n}| \;\leq\; \frac{1}{|a_n|}(-a_nt-\rho_1-\frac{a_{\mu-1}}{t^{n-1}}+\frac{2\rho_2}{t^{n-1}}+\frac{|a_{\mu-1}|}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}).$$

But observe that all the zeros of $P_\mu(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $F(z)$ and hence of $P_\mu(z)$ lie in the union of the disks $|z|\leq t$ and

$$|z+\frac{\rho_1}{a_n}| \;\leq\; \frac{1}{|a_n|}(-a_nt-\rho_1-\frac{a_{\mu-1}}{t^{n-1}}+\frac{2\rho_2}{t^{n-1}}+\frac{|a_{\mu-1}|}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}).$$

This completes the proof. □

For $\rho_1=-(1-k)t^na_n$, we have the following corollary.

Corollary 2.2. *Let $P_\mu(z)=a_0+\sum_{j=\mu}^n a_jz^j$, where $a_0\neq 0$ and for some real $\rho_2, t>0, 0<k\leq 1$, $1\leq\mu\leq\lambda\leq n$ and*

$$kt^na_n\leq t^{n-1}a_{n-1}\leq\cdots\leq t^{\lambda+1}a_{\lambda+1}\leq t^\lambda a_\lambda,$$
$$t^\lambda a_\lambda\geq t^{\lambda-1}a_{\lambda-1}\geq\cdots\geq t^\mu a_\mu\geq a_0-\rho_2.$$

Then all the zeros of $P_\mu(z)$ lie in the union of the disks $|z|\leq t$ and

$$|z+(k-1)t^n| \;\leq\; \frac{1}{|a_n|}(-kt^na_n+\frac{a_{\mu-1}}{t^{n-1}}-\frac{2\rho_2}{t^{n-1}}+\frac{|a_{\mu-1}|}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}).$$

For $a_0>0$, we obtain the following result.

Corollary 2.3. *Let $P_\mu(z)=a_0+\sum_{j=\mu}^n a_jz^j$, where $a_0\neq 0$ and for some reals $\rho_1, t>0$, $1\leq\mu\leq\lambda\leq n$ and*

$$\rho_1+t^na_n\leq t^{n-1}a_{n-1}\leq\cdots\leq t^{\lambda+1}a_{\lambda+1}\leq t^\lambda a_\lambda,$$
$$t^\lambda a_\lambda\geq t^{\lambda-1}a_{\lambda-1}\geq\cdots\geq t^\mu a_\mu\geq a_0>0.$$

Then all the zeros of $P_\mu(z)$ lie in the union of the disks $|z|\leq t$ and

$$|z+\frac{\rho_1}{a_n}| \;\leq\; \frac{1}{|a_n|}(-a_nt-\rho_1+\frac{2a_\lambda}{t^{n-\lambda-1}}).$$

Theorem 2.4. *Let $P_\mu(z)=a_0+\sum_{j=\mu}^n a_jz^j$, where $a_0\neq 0$ and for some reals ρ, $t>0$, $1\leq\mu\leq\lambda\leq n$ and*

$$t^\lambda a_\lambda\geq t^{\lambda-1}a_{\lambda-1}\geq\cdots\geq t^\mu a_\mu\geq a_0-\rho\geq 0.$$

Then all the zeros of $P_\mu(z)$ lie in the union of the disks $|z|\leq t$ and

$$|z| \;\leq\; \frac{1}{|a_\lambda|}(-a_\lambda t+\frac{2\rho}{t^{n-1}}+\frac{2a_\lambda}{t^{n-\lambda-1}}).$$

Proof. The proof follows directly from the proof of Theorem 2.1 if we set $\rho_1 = 0, \rho_2 = \rho,\ n = \lambda$ and so the details are omitted. □

Remark 2.5. Theorem 2.1 remains true for complex coeffients a_j with a slight modification.

Remark 2.6. By setting $\rho = 0$ and $t = 1$, we recapture the famous Eneström-Kakeya theorem.

3. Applications

Finding the zeros of a polynomial is long standing classifical problem which has emerged as an interesting and fascinating area of research for Mathematicians and Engineers (see [7, 10]). Eneström-Kakeya result serves as a very powerful tool for obtaining the region in the complex plane having all the zeros of a class of polynomials. This result has been employed to : analyze overflow oscillation of discrete-time dynamical system [10], investigate the properties of orthogonal wavelets [10], determine the asymptotic behavior of zeros of the Daubechies filter [10, 12], model high energy collision [10, 12].

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