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GENERALIZATION OF AN INEQUALITY CONCERNING THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n and having no zeros in $|z| < 1$, then Aziz [2] proved that for every real α,

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{2}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}},$$

where

$$M_\alpha = \max_{1\leq\rho\leq n} \left|p(e^{i(\alpha+2\rho\pi)/n})\right|.$$

In this paper, we consider a class of polynomial P_n^μ and $P_{n,\mu}$ of degree n with restricted zeros and present certain generalizations of above inequality in terms of polar derivatives of polynomials.

1. Introduction

Let P_n denote the space of all complex polynomials of degree n, defined as $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$. Then

$$\max_{|z|=1} \left|p'(z)\right| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

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Inequality (1.1) is well-known result of Bernstein (see [12]) and the equality in above holds if $p(z) = az^n$.

Frappier, Rahman and Ruscheweyh [5, Theorem 8] was proved that

$$\max_{|z|=1} \left|p'(z)\right| \leq n \max_{1\leq \rho \leq 2n} \left|p(e^{\frac{i\rho\pi}{n}})\right|. \tag{1.2}$$

Clearly (1.2) represent the refinement of inequality (1.1) because the maximum of $|p(z)|$ on $|z| = 1$ may be larger than the maximum of $|p(z)|$ taken over the $2n^{th}$ roots of unity, as one can shown by taking an example $p(z) = z^n + ia, a > 0$.

In this connection, Aziz [2] improved the inequality (1.2) by showing that if $p \in P_n$, then for every real α,

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{1.3}$$

where

$$M_\alpha = \max_{1\leq \rho \leq n} \left|p(e^{i(\alpha+2\rho\pi)/n})\right|. \tag{1.4}$$

If we restrict ourselves on the class of polynomial $p \in P_n$ having all zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| \tag{1.5}$$

and if $p(z) \neq 0$ in $|z| \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.6}$$

Inequality (1.5) proved by Turán [13] and inequality (1.6) was conjectured by Erdös and proved by Lax [8].

Chan and Malik [4] improved inequalities (1.5) and (1.6) by considering the class of n^{th} degree polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 0 \leq \mu \leq n$, denoted as P_n^μ, and proved that for $p(z) \neq 0$ in $|z| < k, k \geq 1$,

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{1.7}$$

Also, for the class of polynomials $p(z) = a_n z^n + \sum_{\nu=0}^{n-\mu} a_\nu z^\nu, 0 \leq \mu \leq n$ of degree n, denoted as $P_{n,\mu}$, and having all its zeros in $|z| \leq k, k \leq 1$,

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{1.8}$$

Hans, Tripathi and Tyagi [6] proved the following generalization of inequality (1.3) due to Aziz [2].

Theorem 1.1. ([2]) *If $p \in P_n^\mu$ is polynomial and $p(z) \neq 0$ in $|z| < k, k \geq 1$, then for every real α and $m = \min_{|z|=k} |p(z)|$,*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2 \right\}^{\frac{1}{2}}, \tag{1.9}$$

where M_α is defined in (1.4)

Let $D_\delta p(z)$ denote the polar derivative of polynomials $p(z)$ of degree n with respect to δ with $|\delta| > 1$ and defined as

$$D_\delta p(z) = np(z) + (\delta - z)p'(z).$$

The polynomial $D_\delta p(z)$ is of degree $n-1$ and it generalize the ordinary derivative by dividing $D_\delta p(z)$ to δ and taking $\delta \to \infty$, that is,

$$\lim_{\delta \to \infty} \left[\frac{D_\delta p(z)}{\delta} \right] = p'(z).$$

In 1988, Aziz [1] was first proved the inequalities (1.1), (1.6) and other related inequality in terms of polar derivative. Latter on, Aziz and Shah [3] proved inequality (1.5) for the polar derivative of polynomials.

2. Lemmas

For the proofs of results, following lemmas are required.

Lemma 2.1. ([4]) *If $p \in P_n^\mu$ and having no zero in $|z| < k, k \geq 1$, then*

$$k^\mu |p'(z)| \leq |q'(z)|, \tag{2.1}$$

where $q(z) = z^n \overline{p(1/\overline{z})}$.

Lemma 2.2. ([4]) *If $p \in P_{n,\mu}$ and having all it's zeros in $|z| \leq k, k \leq 1$, then*

$$|q'(z)| \leq k^\mu |p'(z)|, \tag{2.2}$$

where $q(z) = z^n \overline{p(1/\overline{z})}$.

Lemma 2.3. ([7]) *If $p \in P_n$, then for $1 \leq s < n$ and $|z| = 1$*

$$|p^s(z)| + |q^s(z)| \leq n(n-1)\cdots(n-s+1) \max_{|z|=1} |p(z)|, \tag{2.3}$$

where $q(z) = z^n \overline{p(1/\overline{z})}$.

Next lemma is implicit in [2].

Lemma 2.4. ([2]) *If $p \in P_n$, then for $|z| = 1$ and for real α*

$$|p'(z)|^2 + |np(z) - zp'(z)|^2 \leq \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2), \tag{2.4}$$

where M_α is defined same as in (1.4).

Lemma 2.5. ([9]) *If $p \in P_n^\mu$ and having no zeros in $|z| < k, k \geq 1$, then for $|z| = 1$,*

$$k^{\mu+1}\left\{\frac{\mu|a_\mu|k^{\mu-1} + n|a_0 - m|}{n|a_0 - m| + \mu|a_\mu|k^{\mu+1}}\right\}|p'(z)| \leq |q'(z)| - nm, \tag{2.5}$$

where $m = \min_{|z|=1}|p(z)|$ and $q(z) = z^n\overline{p(1/\overline{z})}$.

3. Main results

In this article, we first prove the following results concerning the polar derivative of a polynomial.

Theorem 3.1. *If $p \in P_n^\mu$ and having no zeros in $|z| < k, k \geq 1$, then for every real α and $|\delta| > k^\mu$,*

$$\max_{|z|=1}|D_\delta p(z)| \tag{3.1}$$
$$\leq n\left[k^\mu \max_{|z|=1}|p(z)| + \frac{(|\delta| - k^\mu)}{\sqrt{2(1+k^{2\mu})}}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}}\right\}^{\frac{1}{2}}\right],$$

where M_α is defined in (1.4) *and $m = \min_{|z|=k}|p(z)|$.*

Proof. Let $p(z) \in P_n^\mu$ have no zeros in $|z| < k, k \geq 1$ and $q(z) = z^n\overline{p(\frac{1}{\overline{z}})}$. Since

$$\begin{aligned} |D_\delta p(z)| &= |np(z) + (\delta - z)p'(z)| \\ &\leq |\delta||p'(z)| + |q'(z)| \end{aligned} \tag{3.2}$$

and since for $k \geq 1$, $k^\mu \geq 1, \mu \geq 0$, from (3.1), we have

$$\begin{aligned} |D_\delta p(z)| &\leq |\delta||p'(z)| + k^\mu|q'(z)| \\ &= (|\delta| - k^\mu)|p'(z)| + k^\mu\left(|p'(z)| + |q'(z)|\right). \end{aligned} \tag{3.3}$$

Now, using Lemma 2.3 for $s = 1$ in inequality (3.2), we have

$$\max_{|z|=1}|D_\delta p(z)| \leq nk^\mu \max_{|z|=1}|p(z)| + (|\delta| - k^\mu)\max_{|z|=1}|p'(z)|. \tag{3.4}$$

From inequality (1.9) of Theorem 1.1, inequality (3.4) become

$$\max_{|z|=1}|D_\delta p(z)| \leq nk^\mu \max_{|z|=1}|p(z)| + \frac{n(|\delta| - k^\mu)}{\sqrt{2(1+k^{2\mu})}}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}}\right\}^{\frac{1}{2}}. \tag{3.5}$$

This completes the proof of Theorem 3.1. □

If we consider $k = 1$ in above Theorem 3.1, then the following result has been obtained.

Corollary 3.2. *If $p \in P_n$ and having no zero in $|z| \leq 1$, then for every real α and $|\delta| > 1$,*

$$\max_{|z|=1} |D_\delta p(z)| \leq n \left\{ \max_{|z|=1} |p(z)| + \frac{(|\delta| - 1)}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} \right\}, \quad (3.6)$$

where M_α is defined in (1.4) *and $m = \min_{|z|=1} |p(z)|$.*

On taking $\mu = 1$ in inequality (3.1), we have the following result.

Corollary 3.3. *If $p \in P_n$ and having no zero in $|z| < k, k \geq 1$, then for every real α and $|\delta| > k$,*

$$\max_{|z|=1} |D_\delta p(z)| \leq n \left[k \max_{|z|=1} |p(z)| + \frac{(|\delta| - k)}{\sqrt{2(1+k^2)}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}} \right\}^{\frac{1}{2}} \right], (3.7)$$

where M_α is defined in (1.4) *and $m = \min_{|z|=k} |p(z)|$.*

Remark 3.4. On dividing inequality (3.6) and (3.7) by δ and taking $\delta \to \infty$, we have some other generalization of inequality (1.9). Which was proved by Rather and Shah [9].

If we consider some zeros of $p \in P_n^\mu$ are on $|z| = k$, i.e. $m = 0$, then following result has been obtained form Theorem 3.1.

Corollary 3.5. *If $p \in P_n^\mu$ and having no zeros in $|z| < k, k \geq 1$, then for every real α and $|\delta| > k^\mu$,*

$$\max_{|z|=1} |D_\delta p(z)| \leq n \left[k^\mu \max_{|z|=1} |p(z)| + \frac{(|\delta| - k^\mu)}{\sqrt{2(1+k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 \right)^{\frac{1}{2}} \right], \quad (3.8)$$

where M_α is defined in (1.4).

On dividing inequality (3.8) by δ and taking $\delta \to \infty$, we have following inequality.

Corollary 3.6. *If $p \in P_n^\mu$ and having no zeros in $|z| < k, k \geq 1$, then for every real α,*

$$\max_{|z|=1} \left| p'(z) \right| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 \right)^{\frac{1}{2}}, \quad (3.9)$$

where M_α is defined in (1.4).

Remark 3.7. Corollary 3.6 was proved by Hans, Tripathi and Tyagi [9]. If we take $k = 1$ in above inequality (3.9), inequality (1.3) due to Aziz [2] has been obtained.

Now, we proved a generalization of inequality (3.1) of Theorem 3.1 in following manner.

Theorem 3.8. *If $p \in P_n^\mu$ and having no zeros in $|z| < k, k \geq 1$, then for every real α and $|\delta| > k^\mu$,*

$$\max_{|z|=1} |D_\delta p(z)| \leq n \left\{ k^\mu |p(z)| + \frac{(|\delta| - k^\mu)}{\sqrt{2\left(1 + A_\mu^2\right)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} \right\}, \quad (3.10)$$

where M_α is defined in (1.4)*, $m = \min_{|z|=k} |p(z)|$ and*

$$A_\mu = k^{(\mu+1)} \left\{ \frac{\mu |a_\mu| k^{\mu-1} + n(|a_0| - m)}{n(|a_0| - m) + \mu |a_\mu| k^{\mu+1}} \right\}.$$

Proof. Since $p \in P_n^\mu$ having no zeros in $|z| < k, k \geq 1$, we get from inequality (3.3) of proof of Theorem 3.1

$$\max_{|z|=1} |D_\delta p(z)| \leq n k^\mu \max_{|z|=1} |p(z)| + (|\delta| - k^\mu) \max_{|z|=1} |p'(z)|. \quad (3.11)$$

On combining inequality (3.11) and (3.13) of Corollary 3.9, we have

$$\max_{|z|=1} |D_\delta p(z)| \quad (3.12)$$
$$\leq n \left\{ k^\mu \max_{|z|=1} |p(z)| + \frac{(|\delta| - k^\mu)}{\sqrt{2(1 + A_\mu^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} \right\},$$

which follows the Theorem 3.8. □

The following result has been obtained from Theorem 3.8 by dividing inequality (3.16) to δ and taking $\delta \to \infty$, which was proved by Rather, Ahangar and Shah [11].

Corollary 3.9. *If $p \in P_n^\mu$ and having no zeros in $|z| < k, k \geq 1$, then for every real α,*

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{\sqrt{2\left(1 + A_\mu^2\right)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{\frac{1}{2}}, \quad (3.13)$$

where M_α is defined in (1.4)*, $m = \min_{|z|=k} |p(z)|$ and A_μ is defined in Theorem* 3.8*.*

Remark 3.10. We also find some other generalization of Theorem 3.8 by applying same condition as on Theorem 3.1 and their respective corollaries.

Next, we prove following result by considering the class of polynomial $P_{n,\mu}$ and all of its zeros lies in $|z| \leq k, k \leq 1$.

Theorem 3.11. *If $p \in P_{n,\mu}$ and having all its zeros in $|z| \le k, k \le 1$, then for every real α and $|\delta| > 1$,*

$$\max_{|z|=1} |D_\delta p(z)| \tag{3.14}$$

$$\ge n(|\delta| - k^\mu) \left[\max_{|z|=1} |p(z)| - \frac{k^\mu}{\sqrt{2(1+k^{2\mu})}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}} \right\}^{\frac{1}{2}} \right],$$

where M_α is define in (1.4) *and* $m = \min_{|z|=k} |p(z)|$.

Proof. Since $p(z) \in P_{n,\mu}$ having all its zero in $|z| \le k \le 1$, $q(z) = z^n \overline{p(\frac{1}{\bar z})} \in P_n^\mu$ having no zeros in $|z| < \frac{1}{k}$. Now, we know that

$$\begin{aligned} m^* = \min_{|z|=\frac{1}{k}} |q'(z)| &= \min_{|z|=1/k} |z^n \overline{p(1/\overline{z})}| \\ &= \min_{|z|=1} \left| \frac{z^n}{k^n} \overline{p(k/\overline{z})} \right| = \frac{1}{k^n} \min_{|z|=1} |p(kz)| \\ &= \frac{1}{k^n} \min_{|z|=k} |p(z)| = \frac{1}{k^n} m, \end{aligned} \tag{3.15}$$

hence, for $q(z)$ inequality (1.9) of Theorem 1.1 becomes

$$|q'(z)| \le \frac{nk^\mu}{\sqrt{2(1+k^{2\mu})}} \left\{ \frac{1}{2}(M_\alpha^2 + M_{\alpha+\pi}^2) - m^{(*2)} \right\}^{\frac{1}{2}}. \tag{3.16}$$

Also it is simple to obtain that $|q'(z)| = |np(z) - p'(z)| \ge n|p(z)| - |p'(z)|$ for $|z| = 1$, then inequality (3.16) follows for $|z| = 1$,

$$|p'(z)| \ge n|p(z)| - \frac{nk^\mu}{\sqrt{2(1+k^{2\mu})}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2m^2}{k^{2n}} \right\}^{\frac{1}{2}}. \tag{3.17}$$

Since

$$\begin{aligned} |D_\delta p(z)| &= |np(z) + (\delta - z)p'(z)| \\ &\ge |\delta||p'(z)| - |q'(z)|, \end{aligned} \tag{3.18}$$

from Lemma 2.2 for $p \in P_{n,\mu}$ in inequality (3.18), we have for $|z| = 1$

$$|D_\delta p(z)| \ge (|\delta| - k^\mu)\, |p'(z)|. \tag{3.19}$$

On combining (3.17) and (3.19), we get

$$|D_\delta p(z)| \tag{3.20}$$

$$\ge n\,(|\delta| - k^\mu) \left[|p(z)| - \frac{k^\mu}{\sqrt{2(1+k^{2\mu})}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2m^2}{k^{2n}} \right\}^{\frac{1}{2}} \right].$$

Theorem 3.11 is completed. □

By taking $k = 1$ in inequality (3.14) of Theorem 3.11, we get the following result.

Corollary 3.12. *If $p \in P_n$ and having all its zeros in $|z| \leq 1$, then for every real α and $|\delta| > 1$,*

$$\max_{|z|=1} |D_\delta p(z)| \geq n(|\delta| - 1) \left[\max_{|z|=1} |p(z)| - \frac{1}{2} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2 \right\}^{\frac{1}{2}} \right], \quad (3.21)$$

where M_α is define in (1.4) and $m = \min_{|z|=1} |p(z)|$.

By considering $\mu = 1$ in inequality (3.14) of Theorem 3.11, following result has been obtained.

Corollary 3.13. *If $p \in P_n$ and having all its zeros in $|z| \leq k, k \leq 1$, then for every real α and $|\delta| > 1$,*

$$\max_{|z|=1} |D_\delta p(z)| \quad (3.22)$$

$$\geq n(|\delta| - k) \left[\max_{|z|=1} |p(z)| - \frac{k}{\sqrt{2(1+k^2)}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}} \right\}^{\frac{1}{2}} \right],$$

where M_α is defined in (1.4) and $m = \min_{|z|=k} |p(z)|$.

By assuming some zeros of the polynomial $p(z) \in P_{n,\mu}$ are on $|z| = k$, i.e. $m = 0$, then from inequality (3.14) following result has been obtained.

Corollary 3.14. *If $p \in P_{n,\mu}$ and having all its zeros in $|z| \leq k, k \leq 1$, then for every real α and $|\delta| > 1$,*

$$\max_{|z|=1} |D_\delta p(z)| \quad (3.23)$$

$$\geq n(|\delta| - k^\mu) \left[\max_{|z|=1} |p(z)| - \frac{k^\mu}{\sqrt{2(1+k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 \right)^{\frac{1}{2}} \right],$$

where M_α is defined in (1.4).

If we divide inequality (3.14) by δ and taking $\delta \to \infty$, we get following result.

Corollary 3.15. *If $p \in P_{n,\mu}$ and having all its zeros in $|z| \leq k, k \leq 1$, then for every real α,*

$$\max_{|z|=1} |p'(z)| \geq n \left[\max_{|z|=1} |p(z)| - \frac{k^\mu}{\sqrt{2(1+k^{2\mu})}} \left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}} \right\}^{\frac{1}{2}} \right], \quad (3.24)$$

where M_α is define in (1.4) and $m = \min_{|z|=k} |p(z)|$.

Remark 3.16. We also have some other generalizations of inequalities (1.18), (1.19) and (1.20) by dividing it to δ and making $\delta \to \infty$.

In accordance with Theorem 3.8, we also generalized Theorem 3.11 by proving the following result.

Theorem 3.17. *If $p \in P_{n,\mu}$ and having all its zeros in $|z| \leq k, k \leq 1$, then for every real α and $|\delta| > 1$,*

$$\max_{|z|=1} |D_\delta p(z)| \geq n\left(|\delta| - k^\mu\right)\left\{|p(z)| - \frac{n}{\sqrt{2(1+B_\mu^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}})^{1/2}\right\}, \tag{3.25}$$

where M_α is defined in (1.4), $m = \min_{|z|=k} |p(z)|$ and

$$B_\mu = \frac{\mu|a_\mu|k^{n-\mu+1} + n|k^n a_n - m|}{nk^{\mu+1}|k^n a_n - m| + \mu|a_\mu|k^n}.$$

Proof. If $p \in P_{n,\mu}$ and having all its zero in $|z| \leq k, k \leq 1$, then no zeros of $q(z) \in P_n^\mu$ lies in $|z| < \frac{1}{k}$. Therefore from inequality (2.5) of Lemma 2.5 for $q(z)$, we have

$$\frac{1}{k^{\mu+1}}\left\{\frac{\mu|a_{n-\mu}|k^{1-\mu} + n|a_n - m^*|}{n|a_n - m^*| + \mu|a_{n-\mu}|k^{-(\mu+1)}}\right\}|q'(z)| \leq |p'(z)| - nm^*, \tag{3.26}$$

where m^* is defined in (3.15).
Equivalently,

$$(B_\mu|q'(z)| + nm^*)^2 \leq |p'(z)|^2,$$

that is, for $|z| = 1$

$$B_\mu^2|q'(z)|^2 + n^2 m^{*2} \leq |p'(z)|^2$$

or

$$(B_\mu^2 + 1)|q'(z)|^2 + n^2 m^{*2} \leq |p'(z)|^2 + |q'(z)|^2, \tag{3.27}$$

where $B_\mu = \frac{1}{k^{\mu+1}}\left\{\frac{\mu|a_{n-\mu}|k^{1-\mu}+n|a_n-m^*|}{n|a_n-m^*|+\mu|a_{n-\mu}|k^{-(\mu+1)}}\right\}$.

Using Lemma 2.4 in inequality (3.27), we get for $|z| = 1$

$$(B_\mu^2 + 1)|q'(z)|^2 + n^2 m^{*2} \leq \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2),$$

that is,

$$|q'(z)| \leq \frac{n}{\sqrt{2(1+B_\mu^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^{*2})^{1/2}. \tag{3.28}$$

Since $|q'(z)| = |np(z) - zp'(z)| \geq n|p(z)| - |p'(z)|$ for $|z| = 1$, inequality (3.28) follows for $|z| = 1$,

$$|p'(z)| \geq n|p(z)| - \frac{n}{\sqrt{2(1+B_\mu^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^{*2})^{1/2}. \tag{3.29}$$

Now, on combining inequality (3.19) with above inequality (3.29) and using (3.15), we have for $|z| = 1$,

$$\begin{aligned} &|D_\delta p(z)| \\ &\geq n\left(|\delta| - k^\mu\right)\left\{|p(z)| - \frac{n}{\sqrt{2(1+B_\mu^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}})^{1/2}\right\}, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} B_\mu &= \frac{1}{k^{\mu+1}}\left\{\frac{\mu|a_\mu|k^{1-\mu} + n|a_n - m^*|}{n|a_n - m^*| + \mu|a_\mu|k^{-(\mu+1)}}\right\} \\ &= \frac{\mu|a_\mu|k^{n-\mu+1} + n|k^n a_n - m|}{nk^{\mu+1}|k^n a_n - m| + \mu|a_\mu|k^n}. \end{aligned}$$

This completes the proof. □

By dividing inequality (3.25) to δ and letting $\delta \to \infty$, we have following generalization of Corollary 3.15.

Corollary 3.18. *If $p \in P_{n,\mu}$ and having all its zeros in $|z| \leq k, k \leq 1$, then for every real α,*

$$\max_{|z|=1}|p'(z)| \geq n\left\{|p(z)| - \frac{n}{\sqrt{2(1+B_\mu^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2\frac{m^2}{k^{2n}})^{1/2}\right\}, \tag{3.31}$$

where M_α is defined in (1.4), $m = \min_{|z|=k}|p(z)|$ and B_μ is defined in Theorem 3.17.

Remark 3.19. By applying same conditions as on Theorem 3.11 and their respective corollaries, we have been obtained some other generalization of Theorem 3.17.

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