Nonlinear Functional Analysis and Applications
Vol. 23, No. 2 (2018), pp. 323-352
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES VIA CONFORMABLE FRACTIONAL INTEGRALS CONCERNING TWICE DIFFERENTIABLE GENERALIZED RELATIVE SEMI-$(r; m, p, q, h_1, h_2)$-PREINVEX MAPPINGS

Artion Kashuri[1], Rozana Liko[1], Muhammad Adil Khan[2] and Arshad Iqbal[2]

[1]Department of Mathematics, Faculty of Technical Science,
University Ismail Qemali, Vlora, Albania
e-mail: artionkashuri@gmail.com, rozanaliko86@gmail.com

[2]Department of Mathematics,
University of Peshawar, Pakistan
e-mail: adilswati@gmail.com, arshadiqbal80@gmail.com

Abstract. In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set via conformable fractional integrals is derived. By using the notion of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via conformable fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I.

[0]Received October 18, 2017. Revised March 7, 2018.
[0]2010 Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15.
[0]Keywords: Hermite-Hadamard type inequality, MT-convex function, Hölder's inequality, Minkowski inequality, power mean inequality, conformable fractional integral, m-invex, P-function.
[0]Corresponding author: M. Adil Khan(adilswati@gmail.com).

For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of integrable functions on the interval $[a,b]$ is denoted by $L_1[a,b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [6], [21], Tunç and Yildirim defined the following so-called MT-convex function:

Definition 1.2. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of $\mathrm{MT}(I)$, if it is non-negative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.2}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions the reader are refer to [3]-[14], [16], [17], [19], [23], [27], [30], [36], [42], [43]. Fractional calculus [22], was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.4. The Euler beta function is defined for $a, b > 0$ as

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 1.5. The incomplete beta function is defined for $a, b > 0$ as

$$\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 < x \leq 1.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

Definition 1.6. ([41]) A set $M_\varphi \subseteq \mathbb{R}^n$ is said to be a relative convex (φ-convex) set, if there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \tag{1.3}$$

for all $x, y \in \mathbb{R}^n, \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]$.

Definition 1.7. ([41]) A function f is said to be a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{1.4}$$

for all $x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]$.

Definition 1.8. ([8]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

Definition 1.9. ([2]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true ([2], [40]).

Definition 1.10. ([29]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η, if for every $x, y \in K$ and $t \in [0,1]$, we have

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.11. ([24]) Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.5}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Definition 1.12. ([39]) Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function, $h \not\equiv 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex, if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$, one has

$$f\big(tx + (1-t)y\big) \leq h(t)f(x) + h(1-t)f(y). \tag{1.6}$$

Definition 1.13. ([38]) Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, we say that $f : K \longrightarrow \mathbb{R}$ is a tgs-convex function on K if the inequality

$$f\Big((1-t)x + ty\Big) \leq t(1-t)[f(x) + f(y)] \tag{1.7}$$

holds for all $x, y \in K$ and $t \in (0,1)$. We say that f is tgs-concave if $(-f)$ is tgs-convex.

Definition 1.14. ([25]) A function: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for all $x, y \in I$, and $t \in (0,1)$, with $m \in [0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.8}$$

Definition 1.15. ([28]) Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex function with respect to η, if

$$f\big(mx + t\eta(y,x,m)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.9}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, with $m \in (0,1]$. If the inequality (1.9) reverses, then f is said to be (m, h_1, h_2)-preincave on K.

In the following, we recall some definitions of conformable fractional integrals which help to obtain main identity and results. Recently, some authors, started to study on conformable fractional integrals [1], [18], [31]-[35]. In [15], Khalil *et al.* defined the fractional integral of order $0 < \alpha \leq 1$ only. In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 1.16. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$. Then the left conformable fractional integral starting at a is defined by

$$\left(I_\alpha^a f\right)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left({}^{b}I_{\alpha}f\right)(t) = \frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, where $n = 0, 1, 2, \ldots$, and hence $\left(I_{\alpha}^{a}f\right)(t) = \left(J_{n+1}^{a}f\right)(t)$.

In [31], Set *et al.* established a generalization of Hermite-Hadamard type inequality for s-convex functions and gave some remarks to show the relationships with the classical and Riemann-Liouville fractional integrals inequality by using the given properties of conformable fractional integrals.

Theorem 1.17. ([31]) *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function with $0 \leq a < b$, $s \in (0, 1]$, and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for conformable fractional integrals holds:*

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s}(b-a)^{\alpha}}\Big[\left(I_{\alpha}^{a}f\right)(b) + \left({}^{b}I_{\alpha}f\right)(a)\Big]$$
$$\leq \Big[\frac{\beta(n+s+1, \alpha-n) + \beta(n+1, \alpha-n+s)}{n!}\Big]\frac{f(a)+f(b)}{2^{s}},$$

with $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2, \ldots,$ where Γ is Euler gamma function.

Also Set *et al.* established some results for some kind of inequalities via conformable fractional integrals ([32]-[35]).

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes ([22]-[30]).

The Gauss-Jacobi type quadrature formula has the following

$$\int_{a}^{b}(x-a)^{p}(b-x)^{q}f(x)dx = \sum_{k=0}^{+\infty}B_{m,k}f(\gamma_{k}) + R_{m}^{\star}|f|, \qquad (1.10)$$

for certain $B_{m,k}, \gamma_k$ and rest $R_{m}^{\star}|f|$ [37].

Recently, Liu [20] obtained several integral inequalities for the left-hand side of (1.10) under the Definition 1.8 of P-function. Also in [26], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.10) via some kinds of convexity.

Motivated by the above literatures, the main objective of this article is to establish integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula and some new estimates on generalizations to Hermite-Hadamard type inequalities via conformable fractional integrals associated with twice differentiable generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings on m-invex set. It is pointed out that some new special cases will be deduced from main results of the article.

2. Main results involving Gauss-Jacobi type quadrature formula

Definition 2.1. ([9]) A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ grips for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Suppose that $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. A mapping $f : K \longrightarrow (0,+\infty)$ is said to be generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\big) \ leqM_r(h_1(t), h_2(t); mf(x), f(y), p, q) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$, for $p, q > -1$ and some fixed $m \in (0,1]$, where

$$M_r(h_1(t), h_2(t); mf(x), f(y), p, q) = \begin{cases} \left[mh_1^p(t)f^r(x) + h_2^q(t)f^r(y)\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \left[mf(x)\right]^{h_1^p(t)}\left[f(y)\right]^{h_2^q(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2.4. In Definition 2.3, if we choose $r = p = q = 1$ and $\varphi(x) = x$, then we get Definition 1.15.

Remark 2.5. For $r = p = q = 1$, let us discuss some special cases in Definition 2.3 as follows:

(i) If taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)-$ Breckner-preinvex mappings.
(ii) If taking $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m,P)-preinvex mappings.
(iii) If taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings.

(iv) If taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m,h)-preinvex mappings.

(v) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex mappings.

(vi) If taking $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

We claim the following integral identity.

Lemma 2.6. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0,1]$ and $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt. \tag{2.2}$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

This completes the proof of the lemma. □

With the help of Lemma 2.6, we have the following results.

Theorem 2.7. *Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0,+\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. Let $k > 1$ and $0 < r \le 1$. If $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r; m, \overline{p}, \overline{q}, h_1, h_2)$-preinvex mappings on an open m-invex set K for* *some fixed $m \in (0,1]$, where $\overline{p}, \overline{q} > -1$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \sqrt[k]{\beta(kp+1, kq+1)}$$
$$\times \left[m f^{\frac{rk}{k-1}}(a) \Psi^r(h_1(t); r, \overline{p}) + f^{\frac{rk}{k-1}}(b) \Psi^r(h_2(t); r, \overline{q}) \right]^{\frac{k-1}{rk}}, \tag{2.3}$$

where

$$\Psi(h_1(t); r, \overline{p}) := \int_0^1 h_1^{\frac{\overline{p}}{r}}(t)dt, \quad \Psi(h_2(t); r, \overline{q}) := \int_0^1 h_2^{\frac{\overline{q}}{r}}(t)dt.$$

Proof. Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is a generalized relative semi-$(r; m, \overline{p}, \overline{q}, h_1, h_2)$-preinvex function on K, combining with Lemma 2.6, Hölder inequality and Minkowski inequality, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \left[\int_0^1 t^{kp}(1-t)^{kq} dt \right]^{\frac{1}{k}}$$
$$\times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a) + t\eta(\varphi(b),\varphi(a),m)) dt \right]^{\frac{k-1}{k}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \sqrt[k]{\beta(kp+1, kq+1)}$$
$$\times \left[\int_0^1 \left[m h_1^{\overline{p}}(t) f^{\frac{rk}{k-1}}(a) + h_2^{\overline{q}}(t) f^{\frac{rk}{k-1}}(b) \right]^{\frac{1}{r}} dt \right]^{\frac{k-1}{k}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \sqrt[k]{\beta(kp+1, kq+1)}$$
$$\times \left\{ \left(\int_0^1 m^{\frac{1}{r}} h_1^{\frac{\overline{p}}{r}}(t) f^{\frac{k}{k-1}}(a) dt \right)^r + \left(\int_0^1 h_2^{\frac{\overline{q}}{r}}(t) f^{\frac{k}{k-1}}(b) dt \right)^r \right\}^{\frac{k-1}{rk}}$$
$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m) \sqrt[k]{\beta(kp+1, kq+1)}$$
$$\times \left[m f^{\frac{rk}{k-1}}(a) \Psi^r(h_1(t); r, \overline{p}) + f^{\frac{rk}{k-1}}(b) \Psi^r(h_2(t); r, \overline{q}) \right]^{\frac{k-1}{rk}}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 2.7.

Corollary 2.8. *In Theorem 2.7 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, we have the following inequality for generalized relative semi-(m,h)-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\sqrt[k]{\beta(kp+1,kq+1)}\Psi^{\frac{k-1}{k}}(h(t);1,1)\\&\quad\times\left[mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}.\end{aligned} \tag{2.4}$$

Corollary 2.9. *In Theorem 2.7 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\sqrt[k]{\beta(kp+1,kq+1)}\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{s+1}\right]^{\frac{k-1}{k}}.\end{aligned} \tag{2.5}$$

Corollary 2.10. *In Theorem 2.7 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ where $s \in (0,1)$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\sqrt[k]{\beta(kp+1,kq+1)}\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{1-s}\right]^{\frac{k-1}{k}}.\end{aligned} \tag{2.6}$$

Corollary 2.11. *In Theorem 2.7 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = h_2(t) = t(1-t)$, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\&\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\sqrt[k]{\beta(kp+1,kq+1)}\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{6}\right]^{\frac{k-1}{k}}.\end{aligned} \tag{2.7}$$

Corollary 2.12. *In Theorem 2.7 for $r=\overline{p}=\overline{q}=1$ and $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq\left(\frac{\pi}{4}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\sqrt[k]{\beta(kp+1,kq+1)}$$
$$\times\left[mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}. \tag{2.8}$$

Theorem 2.13. *Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ is a continuous function on K° with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$, for $\eta(\varphi(b),\varphi(a),m)>0$. Let $l\geq 1$ and $0<r\leq 1$. If f^l is generalized relative semi-$(r;m,\overline{p},\overline{q},h_1,h_2)$-preinvex functions on an open m-invex set K for some fixed $m\in(0,1]$, where $\overline{p},\overline{q}>-1$, then for any fixed $p,q>0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\sqrt[rl]{mf^{rl}(a)I^r(h_1(t);r,p,q,\overline{p})+f^{rl}(b)I^r(h_2(t);r,p,q,\overline{q})}, \tag{2.9}$$

where

$$I(h_1(t);r,p,q,\overline{p})=\int_0^1 t^p(1-t)^q h_1^{\frac{\overline{p}}{r}}(t)dt$$

and

$$I(h_2(t);r,p,q,\overline{q})=\int_0^1 t^p(1-t)^q h_2^{\frac{\overline{q}}{r}}(t)dt.$$

Proof. Let $l\geq 1$ and $0<r\leq 1$. Since f^l is generalized relative semi-$(r;m,\overline{p},\overline{q},h_1,h_2)$-preinvex functions on K, combining with Lemma 2.6, the

well-known power mean inequality and Minkowski inequality, we get

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\int_0^1\Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}\Big[t^p(1-t)^q\Big]^{\frac{1}{l}}\\
&\quad\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\\
&\quad\times\left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{1}{l}}\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\quad\times\left[\int_0^1 t^p(1-t)^q\Big[mh_1^{\overline{p}}(t)f^{rl}(a)+h_2^{\overline{q}}(t)f^{rl}(b)\Big]^{\frac{1}{r}}dt\right]^{\frac{1}{l}}\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\quad\times\left\{\left(\int_0^1 m^{\frac{1}{r}}t^p(1-t)^q h_1^{\frac{\overline{p}}{r}}(t)f^l(a)dt\right)^r+\left(\int_0^1 t^p(1-t)^q h_2^{\frac{\overline{q}}{r}}(t)f^l(b)dt\right)^r\right\}^{\frac{1}{rl}}\\
&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\quad\times\sqrt[rl]{mf^{rl}(a)I^r(h_1(t);r,p,q,\overline{p})+f^{rl}(b)I^r(h_2(t);r,p,q,\overline{q})}.
\end{aligned}$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 2.13.

Corollary 2.14. *In Theorem* 2.13 *for* $r=\overline{p}=\overline{q}=1$ *and* $h_1(t)=h(1-t)$, $h_2(t)=h(t)$, *we have the following inequality for generalized relative semi-*(m,h)*-preinvex mappings:*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&\le\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\quad\times\sqrt[l]{mf^l(a)I(h(t);1,p,q,1)+f^l(b)I(h(t);1,q,p,1)}.
\end{aligned}\tag{2.10}$$

Corollary 2.15. *In Theorem* 2.13 *for* $r = \overline{p} = \overline{q} = 1$ *and* $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, *we have the following inequality for generalized relative semi-*(m, s)*-Breckner-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1) \\ &\quad \times \sqrt[l]{mf^l(a)\beta(p + 1, q + s + 1) + f^l(b)\beta(q + 1, p + s + 1)}.\end{aligned} \tag{2.11}$$

Corollary 2.16. *In Theorem* 2.13 *for* $r = \overline{p} = \overline{q} = 1$ *and* $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, *we get the following inequality for generalized relative semi-*(m, s)*-Godunova-Levin-Dragomir preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1) \\ &\quad \times \sqrt[l]{mf^l(a)\beta(p + 1, q - s + 1) + f^l(b)\beta(q + 1, p - s + 1)}.\end{aligned} \tag{2.12}$$

Corollary 2.17. *In Theorem* 2.13 *for* $r = \overline{p} = \overline{q} = 1$ *and* $h_1(t) = h_2(t) = t(1-t)$, *we obtain the following inequality for generalized relative semi-*(m, tgs)*-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\sqrt[l]{\beta(p + 2, q + 2)}\beta^{\frac{l-1}{l}}(p + 1, q + 1) \\ &\quad \times \sqrt[l]{mf^l(a) + f^l(b)}.\end{aligned} \tag{2.13}$$

Corollary 2.18. *In Theorem* 2.13 *for* $r = \overline{p} = \overline{q} = 1$ *and* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *we deduce the following inequality for generalized relative* *semi-m-MT-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \sqrt[l]{\frac{1}{2}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times \sqrt[l]{mf^l(a)\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+f^l(b)\beta\left(q+\frac{1}{2},p+\frac{3}{2}\right)}. \tag{2.14}$$

3. Other results involving conformable fractional integrals

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for twice differentiable generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvex functions via conformable fractional integrals, we need the following new integral identity:

Lemma 3.1. *Let $\varphi: I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1] \longrightarrow \mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f: K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ be a twice differentiable mapping on K° and $f'' \in L_1[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for r, $\alpha>0$, the following integral identity holds:*

$$\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(r+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\Bigg\{\frac{-(r+1)\beta(n+2,\alpha-n)f'(m\varphi(a))}{\eta(\varphi(x),\varphi(a),m)}$$
$$+\frac{(r+1)^{n+3}}{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}\Bigg[-(n+1)\int_{m\varphi(a)}^{m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{r+1}}(t-m\varphi(a))^n$$
$$\times\big[\eta(\varphi(x),\varphi(a),m)-(r+1)(t-m\varphi(a))\big]^{\alpha-n-1}f(t)dt$$
$$-(\alpha-n-1)\int_{m\varphi(a)}^{m\varphi(a)+\frac{\eta(\varphi(x),\varphi(a),m)}{r+1}}(t-m\varphi(a))^{n+1} \tag{3.1}$$
$$\times\big[\eta(\varphi(x),\varphi(a),m)-(r+1)(t-m\varphi(a))\big]^{\alpha-n-2}f(t)dt\Bigg]\Bigg\}$$
$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(r+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\Bigg\{\frac{-(r+1)\beta(n+2,\alpha-n)f'(m\varphi(b))}{\eta(\varphi(x),\varphi(b),m)}$$

$$
\begin{aligned}
&+\frac{(r+1)^{n+3}}{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}\Bigg[-(n+1)\int_{m\varphi(b)}^{m\varphi(b)+\frac{\eta(\varphi(x),\varphi(b),m)}{r+1}}(t-m\varphi(b))^{n}\\
&\times\left[\eta(\varphi(x),\varphi(b),m)-(r+1)(t-m\varphi(b))\right]^{\alpha-n-1}f(t)dt\\
&-(\alpha-n-1)\int_{m\varphi(b)}^{m\varphi(b)+\frac{\eta(\varphi(x),\varphi(b),m)}{r+1}}(t-m\varphi(b))^{n+1}\\
&\times\left[\eta(\varphi(x),\varphi(b),m)-(r+1)(t-m\varphi(b))\right]^{\alpha-n-2}f(t)dt\Bigg]\Bigg\}\\
=&\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(r+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\\
&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))\\
&\times f''\left(m\varphi(a)+\frac{t}{r+1}\eta(\varphi(x),\varphi(a),m)\right)dt\\
&+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(r+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\\
&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))\\
&\times f''\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(x),\varphi(b),m)\right)dt.
\end{aligned}
$$

We denote

$$
\begin{aligned}
I_{f,\eta,\varphi}(x;\alpha,r,n,m,a,b)=&\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(r+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\\
&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))\\
&\times f''\left(m\varphi(a)+\left(\frac{t}{r+1}\right)\eta(\varphi(x),\varphi(a),m)\right)dt\\
&+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(r+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\\
&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))\\
&\times f''\left(m\varphi(b)+\left(\frac{t}{r+1}\right)\eta(\varphi(x),\varphi(b),m)\right)dt.
\end{aligned}
\tag{3.2}
$$

Proof. A simple proof of equality (3.1) can be done by performing two integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Using relation (3.2), the following results can be obtained for the corresponding version for power of the second derivative.

Theorem 3.2. *Let $\alpha > 0$, $r_1 \in [0,1]$, $0 < r \leq 1$ and $p_1, p_2 > -1$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Suppose $K \subseteq \mathbb{R}$ is an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$. Assume that $f : K \longrightarrow (0,+\infty)$ is a twice differentiable mapping on K°, where $\eta(\varphi(b),\varphi(a),m) > 0$. If $(f''(x))^q$ is generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex mapping, $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,r_1,n,m,a,b)\right| \\
&\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} \Big[m\left(f''(a)\right)^{rq} I^r(h_1(t);r,r_1,p_1) \\
&\quad + \left(f''(x)\right)^{rq} I^r(h_2(t);r,r_1,p_2) \Big]^{\frac{1}{rq}} \\
&\quad + |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2} \Big[m\left(f''(b)\right)^{rq} I^r(h_1(t);r,r_1,p_1) \\
&\quad + \left(f''(x)\right)^{rq} I^r(h_2(t);r,r_1,p_2) \Big]^{\frac{1}{rq}} \Bigg\},
\end{aligned} \tag{3.3}$$

where

$$\delta(p,\alpha,n) = \int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right]^p dt$$

and

$$I(h_i(t);r,r_1,p_i) = \int_0^1 h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right) dt, \ \forall\, i = 1,2.$$

Proof. Suppose that $q > 1$, $r_1 \in [0,1]$ and $0 < r \leq 1$. From Lemma 3.1, generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvexity of $(f''(x))^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,r_1,n,m,a,b)\right| \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}|\eta(\varphi(b),\varphi(a),m)|}
\end{aligned}$$

$$\times \int_0^1 \left|\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right|$$

$$\times \left|\left|f''\left(m\varphi(a) + \left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt$$

$$+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times \int_0^1 \left|\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right|$$

$$\times \left|\left|f''\left(m\varphi(b) + \left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right|dt$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left(f''\left(m\varphi(a) + \left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left(f''\left(m\varphi(b) + \left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)\left(f''(a)\right)^{rq} + h_2^{p_2}\left(\frac{t}{r_1+1}\right)\left(f''(x)\right)^{rq}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left[mh_1^{p_1}\left(\frac{t}{r_1+1}\right)\left(f''(b)\right)^{rq} + h_2^{p_2}\left(\frac{t}{r_1+1}\right)\left(f''(x)\right)^{rq}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times \left\{\left(\int_0^1 m^{\frac{1}{r}}\left(f''(a)\right)^q h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\right.$$

$$+\left(\int_0^1 \left(f''(x)\right)^q h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\Bigg\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}}$$

$$\times\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}\left(f''(b)\right)^q h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1 \left(f''(x)\right)^q h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right)dt\right)^r\Bigg\}^{\frac{1}{rq}}$$

$$=\frac{\sqrt[p]{\delta(p,\alpha,n)}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq} I^r(h_1(t);r,r_1,p_1)$$

$$+\left(f''(x)\right)^{rq} I^r(h_2(t);r,r_1,p_2)\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq} I^r(h_1(t);r,r_1,p_1)$$

$$+\left(f''(x)\right)^{rq} I^r(h_2(t);r,r_1,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 3.2.

Corollary 3.3. *In Theorem* 3.2 *for* $h_1(t)=h(1-t)$, $h_2(t)=h(t)$, $p_1=p_2=r=1, r_1=0$ *and* $\alpha\in(n,n+1]$ *where* $n=0,1,2,\dots,$ *we get the following inequality for conformable fractional integrals:*

$$\Bigg|-\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))+\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times \left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))} I_\alpha f \right) (m\varphi(a)) \right.$$
$$\left. + \left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))} I_\alpha f \right) (m\varphi(b)) \right] \Bigg|$$
$$\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 h(t)dt \right)^{\frac{1}{q}} \tag{3.4}$$
$$\times \left\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} \sqrt[q]{m\left(f''(a)\right)^q + \left(f''(x)\right)^q} \right.$$
$$\left. + |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2} \sqrt[q]{m\left(f''(b)\right)^q + \left(f''(x)\right)^q} \right\}.$$

Corollary 3.4. *In Corollary* 3.3*, if we choose* $\alpha = n+1$ *where* $n = 0, 1, 2, \ldots$ *and* $f''(x) \leq L$*, for all* $x \in I$*, we get the following inequality for fractional integrals:*

$$\Bigg| - \frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m) f'(m\varphi(a)) + \eta^{\alpha+1}(\varphi(x),\varphi(b),m) f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$
$$+ \frac{\eta^{\alpha}(\varphi(x),\varphi(a),m) f(m\varphi(a) + \eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$+ \frac{\eta^{\alpha}(\varphi(x),\varphi(b),m) f(m\varphi(b) + \eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)} - \frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times \left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \right] \Bigg|$$
$$\leq L \sqrt[q]{m+1} \sqrt[p]{\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}} \left(\int_0^1 h(t)dt \right)^{\frac{1}{q}}$$
$$\times \left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} + |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \right]. \tag{3.5}$$

Corollary 3.5. *In Theorem* 3.2 *for* $r_1 = 0$, $h_1(t) = h(1-t)$ *and* $h_2(t) = h(t)$*, we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, h)$*-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq} I^r(h(1-t);r,0,p_1) \\
&\quad+\left(f''(x)\right)^{rq} I^r(h(t);r,0,p_2)\Big]^{\frac{1}{rq}} \\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq} I^r(h(1-t);r,0,p_1) \\
&\quad+\left(f''(x)\right)^{rq} I^r(h(t);r,0,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.
\end{aligned} \tag{3.6}$$

Corollary 3.6. *In Theorem* 3.2 *for* $r_1 = 0$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, *we have the following inequality for generalized relative semi-*$(r;m,p_1,p_2,s)$*-Breckner-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{\eta(\varphi(b),\varphi(a),m)} \\
&\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\sqrt[rq]{m\left(f''(a)\right)^{rq}\left(\frac{r}{r+sp_1}\right)^r+\left(f''(x)\right)^{rq}\left(\frac{r}{r+sp_2}\right)^r} \\
&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\sqrt[rq]{m\left(f''(b)\right)^{rq}\left(\frac{r}{r+sp_1}\right)^r+\left(f''(x)\right)^{rq}\left(\frac{r}{r+sp_2}\right)^r}\Bigg\}.
\end{aligned} \tag{3.7}$$

Corollary 3.7. *In Theorem* 3.2 *for* $r_1 = 0$, $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *where* $r > \max\{sp_1, sp_2\}$, *we have the following inequality for generalized relative semi-*$(r;m,p_1,p_2,s)$*-Godunova-Levin-Dragomir-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\sqrt[rq]{m\left(f''(a)\right)^{rq}\left(\frac{r}{r-sp_1}\right)^r+\left(f''(x)\right)^{rq}\left(\frac{r}{r-sp_2}\right)^r}
\end{aligned}$$

$$+\left|\eta(\varphi(x),\varphi(b),m)\right|^{\alpha+2}\sqrt[rq]{m\left(f''(b)\right)^{rq}\left(\frac{r}{r-sp_1}\right)^{r}+\left(f''(x)\right)^{rq}\left(\frac{r}{r-sp_2}\right)^{r}}\Bigg\}. \tag{3.8}$$

Corollary 3.8. *In Theorem* 3.2 *for* $r_1 = 0$, $h_1(t) = h_2(t) = t(1-t)$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, tgs)$*-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{\left|\eta(\varphi(x),\varphi(a),m)\right|^{\alpha+2}\Bigg[m\left(f''(a)\right)^{rq}\beta^{r}\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right) \\
&\quad+\left(f''(x)\right)^{rq}\beta^{r}\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}} \\
&\quad+\left|\eta(\varphi(x),\varphi(b),m)\right|^{\alpha+2}\Bigg[m\left(f''(b)\right)^{rq}\beta^{r}\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right) \\
&\quad+\left(f''(x)\right)^{rq}\beta^{r}\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.
\end{aligned} \tag{3.9}$$

Corollary 3.9. *In Theorem* 3.2 *for* $r_1 = 0$, $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ *where* $r > \frac{\max\{p_1,p_2\}}{2}$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2)$*-*MT*-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\sqrt[p]{\delta(p,\alpha,n)}}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{\left|\eta(\varphi(x),\varphi(a),m)\right|^{\alpha+2}\Bigg[m\left(f''(a)\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^{r}\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right) \\
&\quad+\left(f''(x)\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^{r}\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}
\end{aligned}$$

$$
\begin{aligned}
&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\bigg[m\left(f''(b)\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)\\
&+\left(f''(x)\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\bigg]^{\frac{1}{rq}}\Bigg\}.
\end{aligned}
\tag{3.10}
$$

Theorem 3.10. *Let $\alpha>0$, $r_1\in[0,1]$, $0<r\le 1$ and $p_1,p_2>-1$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Suppose $K\subseteq\mathbb{R}$ is an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$. Assume that $f:K\longrightarrow(0,+\infty)$ is a twice differentiable mapping on K°, where $\eta(\varphi(b),\varphi(a),m)>0$. If $(f''(x))^q$ is generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex mapping, $q\ge 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,r_1,n,m,a,b)\right|\\
&\le\frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}\\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq}A^r(h_1(t);r,r_1,\alpha,n,p_1)\\
&\quad+\left(f''(x)\right)^{rq}A^r(h_2(t);r,r_1,\alpha,n,p_2)\Big]^{\frac{1}{rq}}\\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq}A^r(h_1(t);r,r_1,\alpha,n,p_1)\\
&\quad+\left(f''(x)\right)^{rq}A^r(h_2(t);r,r_1,\alpha,n,p_2)\Big]^{\frac{1}{rq}}\Bigg\},
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
&A(h_i(t);r,r_1,\alpha,n,p_i)\\
&=\int_0^1\left(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right)h_i^{\frac{p_i}{r}}\left(\frac{t}{r_1+1}\right)dt,\ \forall\, i=1,2.
\end{aligned}
$$

Proof. Suppose that $q\ge 1$, $r_1\in[0,1]$ and $0<r\le 1$. From Lemma 3.1, generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvexity of $(f''(x))^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,r_1,n,m,a,b)\right|\\
&\le\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}|\eta(\varphi(b),\varphi(a),m)|}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^1 \left|\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right| \\
&\times \left|f''\left(m\varphi(a)+\left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}|\eta(\varphi(b),\varphi(a),m)|} \\
&\times \int_0^1 \left|\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right| \\
&\times \left|f''\left(m\varphi(b)+\left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right|dt \\
\leq\ &\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \\
&\times \left(\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]dt\right)^{1-\frac{1}{q}} \\
&\times \Bigg[\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right] \\
&\times \left(f''\left(m\varphi(a)+\left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right)^q dt\Bigg]^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \\
&\times \left(\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]dt\right)^{1-\frac{1}{q}} \\
&\times \Bigg[\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right] \\
&\times \left(f''\left(m\varphi(b)+\left(\frac{t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right)^q dt\Bigg]^{\frac{1}{q}} \\
\leq\ &\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \\
&\times \left(\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]dt\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] \right. \\
&\times \left. \left[m h_1^{p_1}\left(\frac{t}{r_1+1}\right)\left(f''(a)\right)^{rq} + h_2^{p_2}\left(\frac{t}{r_1+1}\right)\left(f''(x)\right)^{rq} \right]^{\frac{1}{r}} dt \right]^{\frac{1}{q}} \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \\
&\times \left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] dt \right)^{1-\frac{1}{q}} \\
&\times \left[\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] \right. \\
&\times \left. \left[m h_1^{p_1}\left(\frac{t}{r_1+1}\right)\left(f''(b)\right)^{rq} + h_2^{p_2}\left(\frac{t}{r_1+1}\right)\left(f''(x)\right)^{rq} \right]^{\frac{1}{r}} dt \right]^{\frac{1}{q}} \\
\leq\ & \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] dt \right)^{1-\frac{1}{q}} \\
&\times \left\{ \left(\int_0^1 m^{\frac{1}{r}} \left(f''(a)\right)^q \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right) dt \right)^r \right. \\
&+ \left. \left(\int_0^1 \left(f''(x)\right)^q \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right) dt \right)^r \right\}^{\frac{1}{rq}} \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)} \\
&\times \left(\int_0^1 \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] dt \right)^{1-\frac{1}{q}} \\
&\times \left\{ \left(\int_0^1 m^{\frac{1}{r}} \left(f''(b)\right)^q \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] h_1^{\frac{p_1}{r}}\left(\frac{t}{r_1+1}\right) dt \right)^r \right. \\
&+ \left. \left(\int_0^1 \left(f''(x)\right)^q \left[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\right] h_2^{\frac{p_2}{r}}\left(\frac{t}{r_1+1}\right) dt \right)^r \right\}^{\frac{1}{rq}} \\
=\ & \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{(r_1+1)^{n+3}\eta(\varphi(b),\varphi(a),m)}
\end{aligned}
$$

$$\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} \Big[m\left(f''(a)\right)^{rq} A^r(h_1(t);r,r_1,\alpha,n,p_1)$$
$$+\left(f''(x)\right)^{rq} A^r(h_2(t);r,r_1,\alpha,n,p_2)\Big]^{\frac{1}{rq}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2} \Big[m\left(f''(b)\right)^{rq} A^r(h_1(t);r,r_1,\alpha,n,p_1)$$
$$+\left(f''(x)\right)^{rq} A^r(h_2(t);r,r_1,\alpha,n,p_2)\Big]^{\frac{1}{rq}} \Bigg\}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 3.10.

Corollary 3.11. *In Theorem* 3.10 *for* $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, $p_1 = p_2 = r = 1, r_1 = 0$ *and* $\alpha \in (n, n+1]$ *where* $n = 0, 1, 2, \dots,$ *we get the following inequality for conformable fractional integrals:*

$$\Bigg| -\frac{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a)) + \eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times \Bigg[\left(^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f \right)(m\varphi(a))$$
$$+\left(^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f \right)(m\varphi(b)) \Bigg] \Bigg|$$
$$\le \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)} \tag{3.12}$$
$$\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} \Big[m\left(f''(a)\right)^{q} A(h(1-t);1,0,\alpha,n,1)$$
$$+\left(f''(x)\right)^{q} A(h(t);1,0,\alpha,n,1)\Big]^{\frac{1}{q}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2} \Big[m\left(f''(b)\right)^{q} A(h(1-t);1,0,\alpha,n,1)$$
$$+\left(f''(x)\right)^{q} A(h(t);1,0,\alpha,n,1)\Big]^{\frac{1}{q}} \Bigg\}.$$

Corollary 3.12. *In Corollary* 3.11*, if we choose* $\alpha = n+1$ *where* $n = 0, 1, 2, \ldots$ *and* $f''(x) \leq L$ *for all* $x \in I$*, we get the following inequality for fractional integrals:*

$$\begin{aligned}
&\Bigg| - \frac{\eta^{\alpha+1}(\varphi(x), \varphi(a), m) f'(m\varphi(a)) + \eta^{\alpha+1}(\varphi(x), \varphi(b), m) f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b), \varphi(a), m)} \\
&\quad + \frac{\eta^{\alpha}(\varphi(x), \varphi(a), m) f(m\varphi(a) + \eta(\varphi(x), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\
&\quad + \frac{\eta^{\alpha}(\varphi(x), \varphi(b), m) f(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(b), \varphi(a), m)} - \frac{\Gamma(\alpha+1)}{\eta(\varphi(b), \varphi(a), m)} \\
&\quad \times \Big[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \Big] \Bigg| \\
&\leq L \beta^{1-\frac{1}{q}}(n+3, \alpha-n) \sqrt[q]{m A(h(1-t); 1, 0, \alpha, n, 1) + A(h(t); 1, 0, \alpha, n, 1)} \\
&\quad \times \left[\frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{\eta(\varphi(b), \varphi(a), m)} \right].
\end{aligned} \tag{3.13}$$

Corollary 3.13. *In Theorem* 3.10 *for* $r_1 = 0$*,* $h_1(t) = h(1-t)$ *and* $h_2(t) = h(t)$*, we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, h)$*-preinvex mappings:*

$$\begin{aligned}
&\left| I_{f,\eta,\varphi}(x; \alpha, 0, n, m, a, b) \right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\eta(\varphi(b), \varphi(a), m)} \\
&\quad \times \Bigg\{ |\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} \Big[m \left(f''(a)\right)^{rq} A^{r}(h(1-t); r, 0, \alpha, n, p_1) \\
&\quad + \left(f''(x)\right)^{rq} A^{r}(h(t); r, 0, \alpha, n, p_2) \Big]^{\frac{1}{rq}} \\
&\quad + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2} \Big[m \left(f''(b)\right)^{rq} A^{r}(h(1-t); r, 0, \alpha, n, p_1) \\
&\quad + \left(f''(x)\right)^{rq} A^{r}(h(t); r, 0, \alpha, n, p_2) \Big]^{\frac{1}{rq}} \Bigg\}.
\end{aligned} \tag{3.14}$$

Corollary 3.14. *In Theorem* 3.10 *for* $r_1 = 0$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, s)$*-Breckner-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq}A^r((1-t)^s;r,0,\alpha,n,p_1) \\
&\quad+\left(f''(x)\right)^{rq}A^r(t^s;r,0,\alpha,n,p_2)\Big]^{\frac{1}{rq}} \\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq}A^r((1-t)^s;r,0,\alpha,n,p_1) \\
&\quad+\left(f''(x)\right)^{rq}A^r(t^s;r,0,\alpha,n,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.
\end{aligned} \tag{3.15}$$

Corollary 3.15. *In Theorem* 3.10 *for* $r_1 = 0$, $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, s)$*-Godunova-Levin-Dragomir-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq}A^r((1-t)^{-s};r,0,\alpha,n,p_1) \\
&\quad+\left(f''(x)\right)^{rq}A^r(t^{-s};r,0,\alpha,n,p_2)\Big]^{\frac{1}{rq}} \\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq}A^r((1-t)^{-s};r,0,\alpha,n,p_1) \\
&\quad+\left(f''(x)\right)^{rq}A^r(t^{-s};r,0,\alpha,n,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.
\end{aligned} \tag{3.16}$$

Corollary 3.16. *In Theorem* 3.10 *for* $r_1 = 0$, $h_1(t) = h_2(t) = t(1-t)$, *we have the following inequality for generalized relative semi-*$(r; m, p_1, p_2, tgs)$*-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq}A^{r}(t(1-t);r,0,\alpha,n,p_1) \\
&\quad+\left(f''(x)\right)^{rq}A^{r}(t(1-t);r,0,\alpha,n,p_2)\Big]^{\frac{1}{rq}} \\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq}A^{r}(t(1-t);r,0,\alpha,n,p_1) \\
&\quad+\left(f''(x)\right)^{rq}A^{r}(t(1-t);r,0,\alpha,n,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.
\end{aligned} \tag{3.17}$$

Corollary 3.17. *In Theorem* 3.10 *for* $r_1=0$, $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *we have the following inequality for generalized relative semi-*$(r;m,p_1,p_2)$*-*MT*-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(x;\alpha,0,n,m,a,b)\right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Bigg[m\left(f''(a)\right)^{rq}A^{r}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,0,\alpha,n,p_1\right) \\
&\quad+\left(f''(x)\right)^{rq}A^{r}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,0,\alpha,n,p_2\right)\Bigg]^{\frac{1}{rq}} \\
&\quad+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Bigg[m\left(f''(b)\right)^{rq}A^{r}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,0,\alpha,n,p_1\right) \\
&\quad+\left(f''(x)\right)^{rq}A^{r}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,0,\alpha,n,p_2\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.
\end{aligned} \tag{3.18}$$

Remark 3.18. Applying our Theorems 3.2 and 3.10, we can deduce some new inequalities using special means associated with generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex mappings.

4. Conclusions

In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set via conformable fractional integrals is derived. By using the notion of generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via conformable fractional integrals are established. It is pointed out that some new special cases are deduced from main results of the article. Motivated by this new interesting class of generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57-66.

[2] T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal., **60** (2005), 1473-1484.

[3] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., **33** (2014), 299-306.

[4] Y.M. Chu, A. Kashuri, R. Liko and M. Adil Khan, *Hermite-Hadamard type fractional integral inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions*, J. Comput. Anal. Appl., **168** (2019), 1-19.

[5] F.X. Chen and S.H. Wu, *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., **9**(**2**) (2016), 705-716.

[6] Y.M. Chu, M.A. Khan, T.U. Khan and T. Ali, *Generalizations of Hermite-Hadamard type inequalities for MT-convex functions*, J. Nonlinear Sci. Appl., **9**(**5**) (2016), 4305-4316.

[7] Y.M. Chu, G.D. Wang and X.H. Zhang, *Schur convexity and Hadamard's inequality*, Math. Inequal. Appl., **13**(**4**) (2010), 725-731.

[8] S.S. Dragomir, J. Pečarić and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (1995), 335-341.

[9] T.S. Du, J.G. Liao and Y.J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions*, J. Nonlinear Sci. Appl., **9** (2016), 3112-3126.

[10] T.S. Du, J.G. Liao, L.Z. Chen and M.U. Awan, *Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α,m)-preinvex functions*, J. Inequal. Appl., **2016**, (2016), Article ID 306, 24 pages.

[11] A. Kashuri and R. Liko, *Ostrowski type fractional integral inequalities for generalized (s,m,φ)-preinvex functions*, Aust. J. Math. Anal. Appl., **13**(**1**) (2016), Article 16, 1-11.
[12] A. Kashuri and R. Liko, *Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions*, Proyecciones, **36**(**1**) (2017), 45-80.
[13] A. Kashuri and R. Liko, *Hermite-Hadamard type fractional integral inequalities for generalized $(r;s,m,\varphi)$-preinvex functions*, Eur. J. Pure Appl. Math., **10**(**3**) (2017), 495-505.
[14] H. Kavurmaci, M. Avci and M.E. Özdemir, *New inequalities of Hermite-Hadamard type for convex functions with applications*, arXiv:1006.1593v1 [math. CA], (2010), 1-10.
[15] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65-70.
[16] M. Adil Khan, Y. Khurshid and T. Ali, *Hermite-Hadamard inequality for fractional integrals via η-convex functions*, Acta Math. Univ. Comenianae, **79**(**1**) (2017), 153-164.
[17] M. Adil Khan, Y. Khurshid, T. Ali and N. Rehman, *Inequalities for three times differentiable functions*, J. Math., Punjab Univ., **48**(**2**) (2016), 35-48.
[18] M. Adil Khan, T. Ali, S.S. Dragomir and M.Z. Sarikaya, *Hermite- Hadamard type inequalities for conformable fractional integrals*, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matematicas, (2017), doi:10.1007/s13398-017-0408-5.
[19] M. Adil Khan, Y.M. Chu, A. Kashuri, R. Liko and G. Ali, *New Hermite-Hadamard inequalities for conformable fractional integrals*, J. Funct. Spaces, In press.
[20] W. Liu, *New integral inequalities involving beta function via P-convexity*, Miskolc Math. Notes, **15**(**2**) (2014), 585-591.
[21] W. Liu, W. Wen and J. Park, *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes, **16**(**1**) (2015), 249-256.
[22] W. Liu, W. Wen and J. Park, *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9** (2016), 766-777.
[23] C. Luo, T.S. Du, M. Adil Khan, A. Kashuri and Y. Shen, *Some k-fractional integrals inequalities through generalized φ-m-MT-preinvexity*, J. Comput. Anal. Appl., **240** (2019), 1-21.
[24] M. Matłoka, *Inequalities for h-preinvex functions*, Appl. Math. Comput., **234** (2014), 52-57.
[25] O. Omotoyinbo and A. Mogbodemu, *Some new Hermite-Hadamard integral inequalities for convex functions*, Int. J. Sci. Innovation Tech., **1(1)** (2014), 1-12.
[26] M.E. Özdemir, E. Set and M. Alomari, *Integral inequalities via several kinds of convexity*, Creat. Math. Inform., **20**(**1**) (2011), 62-73.
[27] B.G. Pachpatte, *On some inequalities for convex functions*, RGMIA Res. Rep. Coll., **6** (2003).
[28] C. Peng, C. Zhou and T.S. Du, *Riemann-Liouville fractional Simpson's inequalities through generalized (m,h_1,h_2)-preinvexity*, Ital. J. Pure Appl. Math., **38** (2017), 345-367.
[29] R. Pini, *Invexity and generalized convexity*, Optimization, **22** (1991), 513-525.
[30] F. Qi and B.Y. Xi, *Some integral inequalities of Simpson type for $GA-\epsilon$-convex functions*, Georgian Math. J., **20**(**5**) (2013), 775-788.
[31] E. Set and A. Gözpinar, *A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals*, Submitted.
[32] E. Set, A.O. Akdemir and I. Mumcu, *Ostrowski type inequalities for functions whoose derivatives are convex via conformable fractional integrals*, Submitted.

[33] E. Set, A.O. Akdemir and I. Mumcu, *Chebyshev type inequalities for conformable fractional integrals*, Submitted.

[34] E. Set and I. Mumcu, *Hermite-Hadamard-Fejér type inequalies for conformable fractional integrals*, Submitted.

[35] E. Set, M.Z. Sarikaya and A. Gözpinar, *Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities*, Creat. Math. Inform., Accepted paper.

[36] H.N. Shi, *Two Schur-convex functions related to Hadamard-type integral inequalities*, Publ. Math. Debrecen, **78(2)** (2011), 393-403.

[37] D.D. Stancu, G. Coman and P. Blaga, *Analiză numerică şi teoria aproximării*, Cluj-Napoca: Presa Universitară Clujeană., **2** (2002).

[38] M. Tunç, E. Göv and Ü. Şanal, *On tgs-convex function and their inequalities*, Facta Univ. Ser. Math. Inform., **30(5)** (2015), 679-691.

[39] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl., **326(1)** (2007), 303-311.

[40] X.M. Yang, X.Q. Yang and K.L. Teo, *Generalized invexity and generalized invariant monotonicity*, J. Optim. Theory Appl., **117** (2003), 607-625.

[41] E.A. Youness, *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl., **102** (1999), 439-450.

[42] X.M. Zhang, Y.M. Chu and X.H. Zhang, *The Hermite-Hadamard type inequality of GA-convex functions and its applications*, J. Inequal. Appl., (2010), Article ID 507560, 11 pages.

[43] Y. Zhang, T.S. Du, H. Wang, Y.J. Shen and A. Kashuri, *Extensions of different type parameterized inequalities for generalized (m,h)-preinvex mappings via k-fractional integrals*, J. Inequal. Appl., **2018** : **49** (2018) 1-30.