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A GENERAL THEOREM ON THE FUZZY STABILITY OF A CLASS OF FUNCTIONAL EQUATIONS INCLUDING QUADRATIC-ADDITIVE FUNCTIONAL EQUATIONS

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Abstract. We will investigate a fuzzy version of Hyers-Ulam stability for a class of functional equations of the form, $\sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big) = 0$, which includes the quadratic-additive functional equations.

1. Introduction

The stability problem of functional equations was first formulated by Ulam [10] in 1940. In the following year, Hyers [3] gave a partial answer to the Ulam's problem when the related functional equation is the Cauchy additive functional equation, namely $f(x+y) = f(x) + f(y)$. Since then, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [2, 9]).

In 2008, Mirmostafaee and Moslehian [7, 8] proved a fuzzy version of Hyers-Ulam stability for the Cauchy additive functional equation and the quadratic functional equation, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.

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For (real) vector spaces V and W, a mapping $f : V \to W$ is called an additive mapping (or a quadratic mapping) provided f satisfies the Cauchy additive functional equation (or the quadratic functional equation) for all $x, y \in V$. Further, a mapping $f : V \to W$ will be called a quadratic-additive mapping if and only if f is represented by the sum of an additive mapping and a quadratic mapping. Similarly, a functional equation will be called a quadratic-additive type functional equation if and only if all of its solutions are quadratic-additive mappings. For example, the mapping $f(x) = ax^2 + bx$ is a quadratic-additive mapping.

In the study of fuzzy stability problems for quadratic-additive type functional equations, we routinely follow out a well known procedure (for example, the direct method) even though we are under different conditions. We can find a lot of references concerning fuzzy version of the stability of functional equations (see [4, 5, 6]).

For a given mapping $f : V \to W$, let us define $Df : V^n \to W$ by

$$Df(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big)$$

for all $x_1, x_2, \ldots, x_n \in V$, where m is a positive integer and c_i and a_{ij} are real constants. We remark that $D(f + g) = Df + Dg$ and $D(cf) = c(Df)$ for all mappings $f, g : V \to W$ and constants $c \in \mathbb{R}$.

In this paper, we prove a general fuzzy stability theorem that can be easily applied to the (generalized) Hyers-Ulam fuzzy stability of a large class of functional equations of the form

$$Df(x_1, x_2, \ldots, x_n) = 0, \tag{1.1}$$

which includes quadratic-additive type functional equations. Indeed, this fuzzy stability theorem may allow us to skip some tedious proofs repeatedly appearing in the fuzzy stability problems for various functional equations including the quadratic, the additive, and the quadratic-additive type functional equations.

2. Preliminaries

We introduce the definition of fuzzy normed spaces to prepare a reasonable fuzzy version of stability of the quadratic-additive type functional equation (1.1).

Definition 2.1. ([1]) Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N1)$ $N(x, c) = 0$ for all $c \le 0$;
$(N2)$ $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

$(N3)$ $N(cx,t) = N\big(x, \frac{t}{|c|}\big)$ for all $c \neq 0$;
$(N4)$ $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
$(N5)$ $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

In this case, the pair (X,N) is called a fuzzy normed space.

Example 2.2. ([7]) Let $(X, \|\cdot\|)$ be a real normed space. We can easily verify that for each $k > 0$,

$$N_k(x,t) = \begin{cases} \dfrac{t}{t + k\|x\|} & (\text{for } t > 0), \\ 0 & (\text{for } t \leq 0) \end{cases}$$

defines a fuzzy norm on X.

Let (X,N) be a fuzzy normed space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an n_0 such that for all integers $n \geq n_0$ and all integers $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. Conversely, if each Cauchy sequence converges, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

3. Fuzzy stability of (1.1)

Throughout this section, let V be a real vector space and (Y,N) be a fuzzy Banach space. For a given mapping $f : V \to Y$, we use the following notations

$$\begin{aligned} Df(x_1, x_2, \ldots, x_n) &:= \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big), \\ f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\ f_o(x) &:= \frac{f(x) - f(-x)}{2} \end{aligned}$$

for all $x_1, x_2, \ldots, x_n, x, y \in V$, where a_{ij} and c_i are fixed real numbers.

Theorem 3.1. *Let V be a real vector space, (Y,N) be a fuzzy Banach space, (Z,N') be a fuzzy normed space, and let k and α be real constants with $|k| > 1$*

and $0 < \alpha < |k|$. *Let* $M : V \to Z$ *and* $\varphi : V^n \to Z$ *be mappings satisfying the inequalities*

$$\begin{aligned} N'\big(M(kx),\, t\big) &\geq N'\big(\alpha M(x),\, t\big), \\ N'\big(\varphi(kx_1, \ldots, kx_n),\, t\big) &\geq N'\big(\alpha\varphi(x_1, \ldots, x_n),\, t\big) \end{aligned} \tag{3.1}$$

for all $x, x_1, x_2, \ldots, x_n \in V$ *and* $t > 0$. *If a mapping* $f : V \to Y$ *with* $f(0) = 0$ *satisfies*

$$N\left(f(kx) - \frac{k^2+k}{2}f(x) - \frac{k^2-k}{2}f(-x),\, t\right) \geq N'\big(M(x), t\big) \tag{3.2}$$

and

$$N\big(Df(x_1, x_2, \ldots, x_n),\, t\big) \geq N'\big(\varphi(x_1, x_2, \ldots, x_n),\, t\big) \tag{3.3}$$

for all $x, x_1, x_2, \ldots, x_n \in V$ *and* $t > 0$, *then there exists a unique mapping* $F : V \to Y$ *such that* $DF(x_1, x_2, \ldots, x_n) = 0$ *for all* $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$,

$$F_e(kx) = k^2 F_e(x) \text{ and } F_o(kx) = kF_o(x) \tag{3.4}$$

for all $x \in V$, *and moreover such that*

$$\begin{aligned} &N\big(F(x) - f(x), t\big) \\ &\geq \sup_{t'<t} \min\big\{N'\big(M(x), (|k| - \alpha)t'\big),\, N'\big(M(-x), (|k| - \alpha)t'\big)\big\} \end{aligned} \tag{3.5}$$

for each $x \in V$ *and* $t > 0$.

Proof. For a given mapping $\varphi : V^n \to Z$ satisfying the second condition in (3.1), we define a mapping $J_i f : V \to Y$ by

$$J_i f(x) := \frac{f(k^i x) + f(-k^i x)}{2k^{2i}} + \frac{f(k^i x) - f(-k^i x)}{2k^i}$$

for all $x \in V$ and all $i \in \mathbb{N}_0$, where we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then $J_0 f(x) = f(x)$ and moreover

$$\begin{aligned} &J_i f(x) - J_{i+1} f(x) \\ &= -\frac{k^{i+1}+1}{2k^{2i+2}}\left(f(k^{i+1}x) - \frac{k^2+k}{2}f(k^i x) - \frac{k^2-k}{2}f(-k^i x)\right) \\ &\quad + \frac{k^{i+1}-1}{2k^{2i+2}}\left(f(-k^{i+1}x) - \frac{k^2+k}{2}f(-k^i x) - \frac{k^2-k}{2}f(k^i x)\right) \end{aligned}$$

for all $x \in V$.

Since $\frac{t}{|k|^{i+1}} = \frac{|k^{i+1}+1|}{2k^{2i+2}}t + \frac{|k^{i+1}-1|}{2k^{2i+2}}t$, together with $(N3)$, $(N4)$, (3.1) and (3.2), the last equality implies that

$$\begin{aligned}
&N\left(J_i f(x) - J_{i+1}f(x),\ \frac{t}{|k|^{i+1}}\right) \\
&\geq \min\left\{N\left[-\frac{k^{i+1}+1}{2k^{2i+2}}\left(f(k^{i+1}x) - \frac{k^2+k}{2}f(k^ix) - \frac{k^2-k}{2}f(-k^ix)\right),\right.\right. \\
&\qquad\left.\frac{|k^{i+1}+1|}{2k^{2i+2}}t\right], \\
&\qquad N\left[\frac{k^{i+1}-1}{2k^{2i+2}}\left(f(-k^{i+1}x) - \frac{k^2+k}{2}f(-k^ix) - \frac{k^2-k}{2}f(k^ix)\right),\right. \\
&\qquad\left.\left.\frac{|k^{i+1}-1|}{2k^{2i+2}}t\right]\right\} \\
&= \min\left\{N\left(f(k^{i+1}x) - \frac{k^2+k}{2}f(k^ix) - \frac{k^2-k}{2}f(-k^ix),\ t\right),\right. \\
&\qquad\left. N\left(f(-k^{i+1}x) - \frac{k^2+k}{2}f(-k^ix) - \frac{k^2-k}{2}f(k^ix),\ t\right)\right\} \\
&\geq \min\left\{N'\big(M(k^ix), t\big),\ N'\big(M(-k^ix), t\big)\right\} \\
&\geq \min\left\{N'\big(\alpha^i M(x), t\big),\ N'\big(\alpha^i M(-x), t\big)\right\} \\
&\geq \min\left\{N'\left(M(x), \frac{t}{\alpha^i}\right),\ N'\left(M(-x), \frac{t}{\alpha^i}\right)\right\} \qquad (3.6)
\end{aligned}$$

for all $x \in V$ and $i \in \mathbb{N}_0$.

Further, in view of $(N3)$ and $(N4)$, (3.6) implies that if $i + j > i \geq 0$, then we have

$$\begin{aligned}
&N\left(J_i f(x) - J_{i+j}f(x),\ \sum_{l=i}^{i+j-1}\frac{\alpha^l t}{|k|^{l+1}}\right) \\
&= N\left(\sum_{l=i}^{i+j-1}\big(J_l f(x) - J_{l+1}f(x)\big),\ \sum_{l=i}^{i+j-1}\frac{\alpha^l t}{|k|^{l+1}}\right) \\
&\geq \min \bigcup_{l=i}^{i+j-1}\left\{N\left(J_l f(x) - J_{l+1}f(x),\ \frac{\alpha^l t}{|k|^{l+1}}\right)\right\} \\
&\geq \min\left\{N'\big(M(x), t\big),\ N'\big(M(-x), t\big)\right\} \qquad (3.7)
\end{aligned}$$

for all $x \in V$ and $t > 0$. By considering $(N5)$, we get

$$\lim_{t\to\infty} \min\Big\{N'\big(M(x),t\big),\, N'\big(M(-x),t\big)\Big\} = 1.$$

Hence, for a given real number $\varepsilon > 0$, there exists a $t_0 > 0$ such that

$$\min\Big\{N'\big(M(x),t_0\big),\, N'\big(M(-x),t_0\big)\Big\} \geq 1-\varepsilon.$$

Obviously, the series $\sum_{l=0}^{\infty} \frac{\alpha^l \tilde{t}}{|k|^{l+1}}$ converges for all $\tilde{t} > t_0$. It guarantees that, for an arbitrary $\delta > 0$, there exists an $i_0 \geq 0$ such that

$$\sum_{l=i}^{i+j-1} \frac{\alpha^l \tilde{t}}{|k|^{l+1}} < \delta \tag{3.8}$$

for all integers $i \geq i_0$ and $j \in \mathbb{N}$.

By $(N5)$, (3.7) and (3.8), we have

$$\begin{aligned} N\big(J_i f(x) - J_{i+j} f(x), \delta\big) &\geq N\left(J_i f(x) - J_{i+j} f(x),\, \sum_{l=i}^{i+j-1} \frac{\alpha^l \tilde{t}}{|k|^{l+1}}\right) \\ &\geq \min\Big\{N'\big(M(x),\tilde{t}\big),\, N'\big(M(-x),\tilde{t}\big)\Big\} \\ &\geq \min\Big\{N'\big(M(x),t_0\big),\, N'\big(M(-x),t_0\big)\Big\} \\ &\geq 1-\varepsilon \end{aligned}$$

for all $x \in V$ and all integers $i \geq i_0$ and $j \in \mathbb{N}$. Hence, $\{J_i f(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N), and we can define a mapping $F : V \to Y$ by

$$F(x) := N - \lim_{i\to\infty} J_i f(x)$$

for all $x \in V$. Moreover, putting $i = 0$ in (3.7) and using $(N5)$, we have

$$\begin{aligned} &N\big(f(x) - J_j f(x), t\big) \\ &\geq \min\Big\{N'\big(M(x),(|k|-\alpha)t\big),\, N'\big(M(-x),(|k|-\alpha)t\big)\Big\} \end{aligned} \tag{3.9}$$

for all $x \in V$ and $j \in \mathbb{N}$.

We will now prove that F satisfies the functional equation $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V$. Using $(N4)$, we obtain

$$\begin{aligned}
&N\big(DF(x_1, x_2, \dots, x_n), t\big) \\
&\geq \min\left\{N\left(D(F - J_j f)(x_1, x_2, \dots, x_n), \frac{t}{2}\right),\right. \\
&\qquad\qquad \left.N\left(DJ_j f(x_1, x_2, \dots, x_n), \frac{t}{2}\right)\right\} \\
&\geq \min\bigcup_{i=1}^{m}\left\{N\left(c_i(F - J_j f)\big(a_{i1}x_1 + \cdots + a_{in}x_n\big), \frac{t}{2m}\right),\right. \\
&\qquad\qquad\qquad \left.N\left(DJ_j f(x_1, x_2, \dots, x_n), \frac{t}{2}\right)\right\}
\end{aligned} \tag{3.10}$$

for all $x_1, x_2, \dots, x_n \in V$, $t > 0$ and $j \in \mathbb{N}$. The first terms on the right hand side of (3.10) tend to 1 as $j \to \infty$ by the definition of F and $(N2)$.

By a somewhat tedious calculation, we get

$$\begin{aligned}
DJ_j f(x_1, x_2, \dots, x_n) \;=\;& \frac{Df(k^j x_1, \dots, k^j x_n)}{2k^{2j}} + \frac{Df(-k^j x_1, \dots, -k^j x_n)}{2k^{2j}} \\
&+ \frac{Df(k^j x_1, \dots, k^j x_n)}{2k^j} - \frac{Df(-k^j x_1, \dots, -k^j x_n)}{2k^j}
\end{aligned}$$

for any $j \in \mathbb{N}$. Hence, for the last term on the right hand side of (3.10), we obtain

$$\begin{aligned}
&N\left(DJ_j f(x_1, x_2, \dots, x_n), \frac{t}{2}\right) \\
&\geq \min\left\{N\left(\frac{Df(k^j x_1, \dots, k^j x_n)}{2k^{2j}}, \frac{t}{8}\right),\ N\left(\frac{Df(-k^j x_1, \dots, -k^j x_n)}{2k^{2j}}, \frac{t}{8}\right),\right. \\
&\qquad \left.N\left(\frac{Df(k^j x_1, \dots, k^j x_n)}{2k^j}, \frac{t}{8}\right),\ N\left(\frac{Df(-k^j x_1, \dots, -k^j x_n)}{2k^j}, \frac{t}{8}\right)\right\}
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V$, $t > 0$ and $j \in \mathbb{N}$.

By $(N3)$, (3.1) and (3.3), we have

$$
\begin{aligned}
&N\left(\frac{Df(\pm k^j x_1, \ldots, \pm k^j x_n)}{2k^{2j}}, \frac{t}{8}\right) \\
&\geq \min\left\{N'\left(\varphi(k^j x_1, \ldots, k^j x_n), \frac{|k|^{2j}}{4}t\right),\right. \\
&\qquad \left. N'\left(\varphi(-k^j x_1, \ldots, -k^j x_n), \frac{|k|^{2j}}{4}t\right)\right\} \\
&\geq \min\left\{N'\left(\varphi(x_1, \ldots, x_n), \frac{|k|^{2j}}{4\alpha^j}t\right),\, N'\left(\varphi(-x_1, \ldots, -x_n), \frac{|k|^{2j}}{4\alpha^j}t\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&N\left(\frac{Df(\pm k^j x_1, \ldots, \pm k^j x_n)}{2k^{j}}, \frac{t}{8}\right) \\
&\geq \min\left\{N'\left(\varphi(x_1, \ldots, x_n), \frac{|k|^{j}}{4\alpha^j}t\right),\, N'\left(\varphi(-x_1, \ldots, -x_n), \frac{|k|^{j}}{4\alpha^j}t\right)\right\}
\end{aligned}
$$

for all $x_1, x_2, \ldots, x_n \in V$ and $j \in \mathbb{N}$. Since $0 < \alpha < |k|$, by $(N5)$, we can deduce that the last term of (3.10) also tends to 1 as $j \to \infty$. It follows from (3.10) that

$$N\big(DF(x_1, x_2, \ldots, x_n), t\big) = 1$$

for all $x_1, x_2, \ldots, x_n \in V$ and $t > 0$. By making use of $(N2)$, this implies that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V$.

We will now estimate the difference between f and F in a fuzzy sense. For any given $x \in V$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since $F(x)$ is the limit of $\{J_i f(x)\}$, there is a positive integer i such that

$$N\big(F(x) - J_i f(x), t - t'\big) \geq 1 - \varepsilon.$$

By $(N4)$, $(N5)$ and (3.9), we have

$$
\begin{aligned}
&N\big(F(x) - f(x), t\big) \\
&\geq \min\Big\{N\big(F(x) - J_i f(x), t - t'\big),\, N\big(J_i f(x) - f(x), t'\big)\Big\} \\
&\geq \min\Big\{1 - \varepsilon,\, N'\big(M(x), (|k| - \alpha)t'\big),\, N'\big(M(-x), (|k| - \alpha)t'\big)\Big\}
\end{aligned}
$$

for any $x \in V$ and $t > 0$. Since $0 < \varepsilon < 1$ is arbitrary, inequality (3.5) is true.

Finally, we will prove the uniqueness of F. Let $F' : V \to Y$ be another mapping satisfying $DF'(x_1, x_2, \ldots, x_n) = 0$ and equalities in (3.4) as well as inequality (3.5). Then by (3.4), we get

$$\begin{aligned} J_iF'(x) &= \frac{F'(k^ix)+F'(-k^ix)}{2k^{2i}} + \frac{F'(k^ix)-F'(-k^ix)}{2k^i} \\ &= \frac{1}{k^{2i}}F'_e(k^ix) + \frac{1}{k^i}F'_o(k^ix) \\ &= F'_e(x) + F'_o(x) \\ &= F'(x) \end{aligned} \tag{3.11}$$

for all $x \in V$ and $i \in \mathbb{N}$. Together with ($N3$), ($N4$), (3.1), (3.5), and (3.11), it holds that

$$\begin{aligned} &N\big(F'(x) - J_if(x), t\big) \\ &= N\big(J_iF'(x) - J_if(x), t\big) \\ &\geq \min\left\{ N\left(\frac{(F'-f)(k^ix)}{2k^{2i}}, \frac{t}{4}\right),\ N\left(\frac{(F'-f)(-k^ix)}{2k^{2i}}, \frac{t}{4}\right),\right. \\ &\qquad\qquad \left. N\left(\frac{(F'-f)(k^ix)}{2k^i}, \frac{t}{4}\right),\ N\left(\frac{(F'-f)(-k^ix)}{2k^i}, \frac{t}{4}\right)\right\} \\ &\geq \sup_{t'<t}\min\left\{ N'\left(M(x), \frac{(|k|-\alpha)|k|^it'}{2\alpha^i}\right),\ N'\left(M(-x), \frac{(|k|-\alpha)|k|^it'}{2\alpha^i}\right)\right\} \end{aligned}$$

for all $x \in V$ and $i \in \mathbb{N}$. Observe that, for $0 < \alpha < |k|$, the last term of the above inequality tends to 1 as $i \to \infty$ by ($N5$). This implies that $\lim_{i\to\infty} N\big(F'(x) - J_if(x), t\big) = 1$ and so, we get

$$F'(x) = N - \lim_{i\to\infty} J_if(x)$$

for all $x \in V$ by ($N2$). This completes the proof. □

In the following theorem, we assume that k, β and γ are nonzero real constants with $1 < |k| < \beta \leq \gamma < |k|^2$.

Theorem 3.2. *Let V be a real vector space, (Y, N) be a fuzzy Banach space, (Z, N') be a fuzzy normed space, and let k, β, and γ be nonzero real numbers such that $1 < |k| < \beta \leq \gamma < |k|^2$. Let $M : V \to Z$ and $\varphi : V^n \to Z$ be mappings satisfying the conditions*

$$\begin{aligned} N'\big(M(kx), \gamma t\big) &\geq N'\big(M(x), t\big) \geq N'\big(M(kx), \beta t\big), \\ N'\big(\varphi(kx_1, \ldots, kx_n), \gamma t\big) &\geq N'\big(\varphi(x_1, \ldots, x_n), t\big) \geq N'\big(\varphi(kx_1, \ldots, kx_n), \beta t\big) \end{aligned} \tag{3.12}$$

for all $x, x_1, x_2, \ldots, x_n \in V$ and $t > 0$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and if inequalities (3.2) *and* (3.3) *hold for all $x, x_1, x_2, \ldots, x_n \in$*

V and $t > 0$, then there exists a unique mapping $F : V \to Y$ such that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, equalities in (3.4) hold for all $x \in V$, and such that

$$\begin{aligned} &N\big(F(x) - f(x), t\big) \qquad (3.13)\\ &\geq \sup_{t'<t} \min\left\{N'\left(M(x),\ \frac{t'}{\frac{1}{k^2-\gamma} + \frac{1}{\beta-|k|}}\right),\ N'\left(M(-x),\ \frac{t'}{\frac{1}{k^2-\gamma} + \frac{1}{\beta-|k|}}\right)\right\} \end{aligned}$$

for each $x \in V$ and $t > 0$.

Proof. Let φ satisfy the second condition in (3.12) and let $J_i f : V \to Y$ be a mapping defined by

$$J_i f(x) := \frac{f(k^i x) + f(-k^i x)}{2k^{2i}} + \frac{k^i}{2}\left(f\left(\frac{x}{k^i}\right) - f\left(\frac{-x}{k^i}\right)\right)$$

for all $x \in V$ and $i \in \mathbb{N}_0$. Then, we have $J_0 f(x) = f(x)$ and by $(N3)$, (3.2) and (3.12), we get

$$\begin{aligned} &N\left(J_i f(x) - J_{i+1} f(x), \frac{\gamma^i t}{k^{2i+2}} + \frac{|k|^i t}{\beta^{i+1}}\right)\\ &\geq \min\left\{N\left[\frac{-1}{2k^{2i+2}}\left(f(k^{i+1}x) - \frac{k^2+k}{2} f(k^i x) - \frac{k^2-k}{2} f(-k^i x)\right), \frac{\gamma^i t}{2k^{2i+2}}\right],\right.\\ &\qquad N\left[\frac{-1}{2k^{2i+2}}\left(f(-k^{i+1}x) - \frac{k^2+k}{2} f(-k^i x) - \frac{k^2-k}{2} f(k^i x)\right), \frac{\gamma^i t}{2k^{2i+2}}\right],\\ &\qquad N\left[\frac{k^i}{2}\left(f\left(\frac{x}{k^i}\right) - \frac{k^2+k}{2} f\left(\frac{x}{k^{i+1}}\right) - \frac{k^2-k}{2} f\left(\frac{-x}{k^{i+1}}\right)\right), \frac{|k|^i t}{2\beta^{i+1}}\right],\\ &\qquad \left. N\left[-\frac{k^i}{2}\left(f\left(\frac{-x}{k^i}\right) - \frac{k^2+k}{2} f\left(\frac{-x}{k^{i+1}}\right) - \frac{k^2-k}{2} f\left(\frac{x}{k^{i+1}}\right)\right), \frac{|k|^i t}{2\beta^{i+1}}\right]\right\}\\ &\geq \min\left\{N'\big(M(k^i x), \gamma^i t\big),\ N'\big(M(-k^i x), \gamma^i t\big),\right.\\ &\qquad \left. N'\left(M\left(\frac{x}{k^{i+1}}\right), \frac{t}{\beta^{i+1}}\right),\ N'\left(M\left(\frac{-x}{k^{i+1}}\right), \frac{t}{\beta^{i+1}}\right)\right\}\\ &\geq \min\left\{N'\big(M(x), t\big),\ N'\big(M(-x), t\big)\right\} \qquad (3.14) \end{aligned}$$

for all $x \in V$, $t > 0$ and $i \in \mathbb{N}_0$. Together with $(N4)$, inequality (3.14) implies that if $i + j > i \geq 0$ then

$$\begin{aligned} &N\left(J_i f(x) - J_{i+j} f(x), \sum_{l=i}^{i+j-1} \left(\frac{\gamma^l}{k^{2l+2}} + \frac{|k|^l}{\beta^{l+1}}\right) t\right) \\ &= N\left(\sum_{l=i}^{i+j-1} \big(J_l f(x) - J_{l+1} f(x)\big), \sum_{l=i}^{i+j-1} \left(\frac{\gamma^l t}{k^{2l+2}} + \frac{|k|^l t}{\beta^{l+1}}\right)\right) \\ &\geq \min \bigcup_{l=i}^{i+j-1} \left\{ N\left(J_l f(x) - J_{l+1} f(x), \frac{\gamma^l t}{k^{2l+2}} + \frac{|k|^l t}{\beta^{l+1}}\right) \right\} \\ &\geq \min \Big\{ N'\big(M(x), t\big), N'\big(M(-x), t\big) \Big\} \end{aligned} \tag{3.15}$$

for all $x \in V$, $i \in \mathbb{N}_0$ and $t > 0$. By a similar argument presented after (3.6), we can define the limit $F(x) := N - \lim_{i \to \infty} J_i f(x)$ of the Cauchy sequence $\{J_i f(x)\}$ in the fuzzy Banach space Y. Moreover, putting $i = 0$ in (3.15), we have

$$\begin{aligned} &N\big(f(x) - J_j f(x), t\big) \\ &\geq \min\left\{ N'\left(M(x), \frac{t}{\sum_{l=0}^{j-1} \left(\frac{\gamma^l}{k^{2l+2}} + \frac{|k|^l}{\beta^{l+1}}\right)}\right), \right. \\ &\qquad \left. N'\left(M(-x), \frac{t}{\sum_{l=0}^{j-1} \left(\frac{\gamma^l}{k^{2l+2}} + \frac{|k|^l}{\beta^{l+1}}\right)}\right) \right\} \\ &\geq \min\left\{ N'\left(M(x), \frac{(k^2-\gamma)(\beta-|k|)}{k^2-\gamma+\beta-|k|} t\right), \right. \\ &\qquad \left. N'\left(M(-x), \frac{(k^2-\gamma)(\beta-|k|)}{k^2-\gamma+\beta-|k|} t\right) \right\} \end{aligned} \tag{3.16}$$

for each $x \in V$ and $j \in \mathbb{N}$.

In order to prove that F satisfies $DF(x_1, x_2, \ldots, x_n) = 0$, it suffices to show that the last term of (3.10) in Theorem 3.1 tends to 1 as $j \to \infty$. By $(N3)$,

$(N4)$, (3.3) and (3.12), we get

$$\begin{aligned}
&N\left(DJ_jf(x_1,x_2,\ldots,x_n),\ \frac{t}{2}\right)\\
&\geq \min\left\{N\left(\frac{Df(k^jx_1,\ldots,k^jx_n)}{2k^{2j}},\ \frac{t}{8}\right),\right.\\
&\qquad N\left(\frac{Df(-k^jx_1,\ldots,-k^jx_n)}{2k^{2j}},\ \frac{t}{8}\right),\\
&\qquad N\left(\frac{k^j}{2}Df\left(\frac{x_1}{k^j},\ldots,\frac{x_n}{k^j}\right),\ \frac{t}{8}\right),\\
&\qquad \left.N\left(\frac{-k^j}{2}Df\left(\frac{-x_1}{k^j},\ldots,\frac{-x_n}{k^j}\right),\ \frac{t}{8}\right)\right\}\\
&\geq \min\left\{N'\left(\varphi(x_1,\ldots,x_n),\frac{|k|^{2j}}{4\gamma^j}t\right),\right.\\
&\qquad N'\left(\varphi(-x_1,\ldots,-x_n),\frac{|k|^{2j}}{4\gamma^j}t\right),\\
&\qquad N'\left(\varphi(x_1,\ldots,x_n),\frac{\beta^j}{4|k|^j}t\right),\\
&\qquad \left.N'\left(\varphi(-x_1,\ldots,-x_n),\frac{\beta^j}{4|k|^j}t\right)\right\}
\end{aligned}$$

for all $x_1,x_2,\ldots,x_n\in V$, $j\in\mathbb{N}$ and $t>0$.

Observe that each term on the right hand side of the above inequality tend to 1 as $j\to\infty$, since $|k|<\beta\leq\gamma<|k|^2$. Hence, together with the similar argument after (3.11), we can say that

$$DF(x_1,x_2,\ldots,x_n)=0$$

for all $x_1,x_2,\ldots,x_n\in V$. Recall that the inequality (3.5) in Theorem 3.1 follows from (3.9). By the same argument, the validity of inequality (3.13) follows from (3.16).

Let us prove the uniqueness of F. We assume that F' is another mapping satisfying $DF'(x_1,x_2,\ldots,x_n)=0$, (3.4) and (3.13). Then by (3.4), we get (3.11) for all $x\in V$ and $i\in\mathbb{N}$. Together with the definition of J_i, $(N3)$, $(N4)$

and (3.13), we further have

$$\begin{aligned}
&N\big(F'(x) - J_i f(x), t\big) \\
&= N\big(J_i F'(x) - J_i f(x), t\big) \\
&\geq \min\left\{ N\left(\frac{(F'-f)(k^i x)}{2k^{2i}}, \frac{t}{4}\right),\ N\left(\frac{(F'-f)(-k^i x)}{2k^{2i}}, \frac{t}{4}\right),\right. \\
&\qquad\quad \left. N\left(\frac{k^i}{2}(F'-f)\left(\frac{x}{k^i}\right), \frac{t}{4}\right),\ N\left(\frac{k^i}{2}(F'-f)\left(\frac{-x}{k^i}\right), \frac{t}{4}\right) \right\} \\
&\geq \min\left\{ \sup_{t'<t} \min\left\{ N'\left(M(x),\ \left(\frac{k^2}{\gamma}\right)^i \frac{t'}{\frac{2}{k^2-\gamma} + \frac{2}{\beta-|k|}}\right),\right.\right. \\
&\qquad\qquad\qquad \left. N'\left(M(-x),\ \left(\frac{k^2}{\gamma}\right)^i \frac{t'}{\frac{2}{k^2-\gamma} + \frac{2}{\beta-|k|}}\right)\right\}, \\
&\qquad\quad \sup_{t'<t} \min\left\{ N'\left(M(x),\ \left(\frac{\beta}{|k|}\right)^i \frac{t'}{\frac{2}{k^2-\gamma} + \frac{2}{\beta-|k|}}\right),\right. \\
&\qquad\qquad\qquad \left.\left. N'\left(M(-x),\ \left(\frac{\beta}{|k|}\right)^i \frac{t'}{\frac{2}{k^2-\gamma} + \frac{2}{\beta-|k|}}\right)\right\}\right\}
\end{aligned}$$

for all $x \in V$, $i \in \mathbb{N}$ and $t > 0$. Since

$$\lim_{i\to\infty} \left(\frac{k^2}{\gamma}\right)^i = \lim_{i\to\infty} \left(\frac{\beta}{|k|}\right)^i = \infty,$$

both terms on the right hand side of the above inequality tend to 1 as $i \to \infty$ by $(N5)$. This implies that $\lim_{i\to\infty} N\big(F'(x) - J_i f(x), t\big) = 1$ and so $F'(x) = N - \lim_{i\to\infty} J_i f(x)$ for all $x \in V$ by $(N2)$. □

In the following theorem, let k and δ be nonzero real constants with $1 < |k| < |k|^2 < \delta$. Under appropriate conditions, we will prove the generalized Hyers-Ulam fuzzy stability of the functional equation (1.1).

Theorem 3.3. *Let V be a real vector space, (Y, N) a fuzzy Banach space, (Z, N') a fuzzy normed space, and let k and δ be nonzero real numbers such that $|k| > 1$ and $|k|^2 < \delta$. Let $M : V \to Z$ and $\varphi : V^n \to Z$ be mappings satisfying the inequalities*

$$\begin{aligned}
N'\big(\delta M(x), t\big) &\geq N'\big(M(kx), t\big), \\
N'\big(\delta\varphi(x_1, x_2, \ldots, x_n), t\big) &\geq N'\big(\varphi(kx_1, kx_2, \ldots, kx_n), t\big)
\end{aligned} \tag{3.17}$$

for all $x, x_1, x_2, \ldots, x_n \in V$ and $t > 0$. If a mapping $f : V \to Y$ with $f(0) = 0$ satisfies inequalities (3.2) *and* (3.3) *for all $x, x_1, x_2, \ldots, x_n \in V$ and $t > 0$, then there is a unique mapping $F : V \to Y$ such that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V \backslash \{0\}$, such that equalities in* (3.4) *hold for all $x \in V$, and such that*

$$N\big(F(x) - f(x), t\big) \geq \sup_{t'<t} \min \Big\{ N'\big(M(x), (\delta - k^2)t'\big), \ N'\big(M(-x), (\delta - k^2)t'\big) \Big\} \tag{3.18}$$

for each $x \in V$ and $t > 0$.

Proof. Assume that φ satisfies the second condition in (3.17) and define $J_i f : V \to Y$ by

$$J_i f(x) := \frac{k^{2i}}{2}\left(f\left(\frac{x}{k^i}\right) + f\left(\frac{-x}{k^i}\right)\right) + \frac{k^i}{2}\left(f\left(\frac{x}{k^i}\right) - f\left(\frac{-x}{k^i}\right)\right)$$

for all $x \in V$ and $i \in \mathbb{N}_0$. Then, by $(N3)$, (3.2) and (3.17), we have $J_0 f(x) = f(x)$ and

$$\begin{aligned} &N\left(J_i f(x) - J_{i+1} f(x), \frac{k^{2i} t}{\delta^{i+1}}\right) \\ &\geq \min \Bigg\{ N\Bigg[\frac{k^{2i} + k^i}{2} \left(f\left(\frac{x}{k^i}\right) - \frac{k^2 + k}{2} f\left(\frac{x}{k^{i+1}}\right) - \frac{k^2 - k}{2} f\left(\frac{-x}{k^{i+1}}\right)\right), \\ &\qquad \frac{(k^{2i} + k^i)t}{2\delta^{i+1}}\Bigg], \\ &\qquad N\Bigg[\frac{k^{2i} - k^i}{2} \left(f\left(\frac{-x}{k^i}\right) - \frac{k^2 + k}{2} f\left(\frac{-x}{k^{i+1}}\right) - \frac{k^2 - k}{2} f\left(\frac{x}{k^{i+1}}\right)\right), \\ &\qquad \frac{(k^{2i} - k^i)t}{2\delta^{i+1}}\Bigg] \Bigg\} \\ &\geq \min \left\{ N'\left(M\left(\frac{x}{k^{i+1}}\right), \frac{t}{\delta^{i+1}}\right), \ N'\left(M\left(\frac{-x}{k^{i+1}}\right), \frac{t}{\delta^{i+1}}\right)\right\} \\ &\geq \min \Big\{ N'\big(M(x), t\big), \ N'\big(M(-x), t\big)\Big\} \end{aligned} \tag{3.19}$$

for all $x \in V$, $i \in \mathbb{N}_0$ and $t > 0$.

Similarly as the previous theorems, we define the mapping $F : V \to Y$ by $F(x) := N - \lim_{i\to\infty} J_i f(x)$ and obtain the inequality

$$N\big(f(x) - J_j f(x), t\big) \geq \min\left\{ N'\left(M(x), \frac{t}{\sum_{l=0}^{j-1} \frac{k^{2l}}{\delta^{l+1}}} \right),\ N'\left(M(-x), \frac{t}{\sum_{l=0}^{j-1} \frac{k^{2l}}{\delta^{l+1}}} \right) \right\} \tag{3.20}$$

for all $x \in V$, $j \in \mathbb{N}$ and $t > 0$. Notice that by (3.3) and (3.17), we have

$$\begin{aligned} &N\left(DJ_i f(x_1, x_2, \ldots, x_n),\ \frac{t}{2} \right) \\ &\geq \min\left\{ N\left(\frac{k^{2i} + k^i}{2} Df\left(\frac{x_1}{k^i}, \ldots, \frac{x_n}{k^i} \right),\ \frac{t}{4} \right),\right. \\ &\qquad \left. N\left(\frac{k^{2i} - k^i}{2} Df\left(\frac{-x_1}{k^i}, \ldots, \frac{-x_n}{k^i} \right),\ \frac{t}{4} \right) \right\} \\ &\geq \min\left\{ N'\left(\varphi(x_1, \ldots, x_n), \frac{\delta^i t}{4(k^{2i} + k^i)} \right),\right. \\ &\qquad \left. N'\left(\varphi(-x_1, \ldots, -x_n), \frac{\delta^i t}{4(k^{2i} - k^i)} \right) \right\} \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in V$ and $t > 0$. Since $\delta > k^2$, each term on the right hand side tends to 1 as $i \to \infty$, which implies that the last term tends to 1 as $i \to \infty$. Therefore, we can say that $DF(x_1, \ldots, x_n) = 0$. Moreover, using the similar argument after (3.9) in Theorem 3.1, we obtain inequality (3.18) from (3.20). It now remains to prove the uniqueness of F. Assume that $F' : V \to Y$ is another mapping satisfying $DF'(x_1, x_2, \ldots, x_n) = 0$, (3.4) and (3.18). Then by (3.4), (3.11) holds for all $x \in V$ and $i \in \mathbb{N}$. By (3.18), we get

$$\begin{aligned} &N\big(F'(x) - J_i f(x), t\big) \\ &= N\big(J_i F'(x) - J_i f(x), t\big) \\ &\geq \min\left\{ N\left(\frac{k^{2i}}{2} (F' - f)\left(\frac{x}{k^i} \right),\ \frac{t}{4} \right),\ N\left(\frac{k^{2i}}{2} (F' - f)\left(\frac{-x}{k^i} \right),\ \frac{t}{4} \right),\right. \\ &\qquad \left. N\left(\frac{k^{i}}{2} (F' - f)\left(\frac{x}{k^i} \right),\ \frac{t}{4} \right),\ N\left(\frac{k^{i}}{2} (F' - f)\left(\frac{-x}{k^i} \right),\ \frac{t}{4} \right) \right\} \\ &\geq \sup_{t' < t} \min\left\{ N'\left(M(x),\ \frac{\delta^i}{2k^{2i}} (\delta - k^2) t' \right),\ N'\left(M(-x),\ \frac{\delta^i}{2k^{2i}} (\delta - k^2) t' \right) \right\} \end{aligned}$$

for all $x \in V$, $i \in \mathbb{N}$ and $t > 0$. Observe that, for $\delta > k^2$, the last term tends to 1 as $i \to \infty$ by $(N5)$. This implies that $\lim_{i\to\infty} N(F'(x) - J_i f(x), t) = 1$ and so $F'(x) = N - \lim_{i\to\infty} J_i f(x)$ for all $x \in V$ by $(N2)$. □

4. Conclusions

We prove a general fuzzy stability theorem that can be easily applied to the (generalized) Hyers-Ulam fuzzy stability of a large class of functional equations of the form (1.1) which includes quadratic-additive type functional equations. This fuzzy stability theorem allows us to skip some tedious proofs repeatedly appearing in the fuzzy stability problems for various functional equations including the quadratic, the additive, and the quadratic-additive type functional equations.

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References

[1] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces,* J. Fuzzy Math., **11(3)** (2003), 687–705.
[2] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,* J. Math. Anal. Appl., **184** (1994), 431–436.
[3] D.H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.
[4] S.-S. Jin and Y.-H. Lee, *Fuzzy stability of a general quadratic functional equation,* Adv. Fuzzy Syst., 2011, Art. ID 791695, 9 pp.
[5] S.-S. Jin and Y.-H. Lee, *Fuzzy stability of a mixed type functional equation,* J. Inequal. Appl., **2011**, 2011:70.
[6] Y.-H. Lee and S.-M. Jung, *Fuzzy stability of an n-dimensional quadratic and additive functional equation,* Adv. Fuzzy Syst., 2012, Art. ID 150815, 9 pp.
[7] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy almost quadratic functions,* Results Math., **52** (2008), 161–177.
[8] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem,* Fuzzy Sets and Systems, **159** (2008), 720–729.
[9] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., **72** (1978), 297–300.
[10] S.M. Ulam, *A Collection of Mathematical Problems,* Interscience, New York, 1960.