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INTEGRAL INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. For $\alpha \in \mathbb{C}$, let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n. If $P(z) \neq 0$ in $|z| < k, k \geq 1$, then it is known for $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + k}{\|z + k\|_p} \right) \|P\|_p .$$

In this paper, we present a refinement of the above inequality valid for $0 \leq p < \infty$ and obtain a bound that depends on some of the coefficients of the polynomial as well. Analogous result for the class of polynomials having no zero in $|z| > k, k \leq 1$ is also obtained.

1. Introduction and preliminaries

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. For $P \in \mathcal{P}_n$, define

$$\|P\|_0 := \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log \left| P(e^{i\theta}) \right| d\theta \right\},$$

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$$\|P\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \; p>0 \;\text{ and }\; \|P\|_\infty := \max_{|z|=1}|P(z)|\,.$$

If $P \in \mathcal{P}_n$, then

$$\left\|P'\right\|_\infty \leq n\left\|P\right\|_\infty \tag{1.1}$$

and

$$\|P'\|_p \leq n\|P\|_p. \tag{1.2}$$

Inequality (1.1) is due to Bernstein (see [13] or [19]) whereas inequality (1.2) is due to Zygmund [20]. Arestov [1] showed that the inequality (1.2) remains valid for $0 < p < 1$ as well. Equality in (1.1) and (1.2) holds for $P(z) = \alpha z^n, \alpha \neq 0$. If we let $p \to \infty$ in (1.2), we get inequality (1.1).

For the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, both the inequalities (1.1) and (1.2) can be sharpened. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then the inequalities (1.1) and (1.2) can be, respectively, replaced by

$$\left\|P'\right\|_\infty \leq \frac{n}{2}\left\|P\right\|_\infty \tag{1.3}$$

and

$$\left\|P'\right\|_p \leq \frac{n}{\left\|1+z\right\|_p}\left\|P\right\|_p, \;\; p \geq 1. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [10] whereas the inequality (1.4) was found out by Bruijn [7]. Rahman and Schmeisser [15] proved the inequality (1.4) remains true for $0 < p < 1$ as well. Both the estimates are sharp and equality in (1.3) and (1.4) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

Malik [11] generalized inequality (1.3) and proved that if $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$, then

$$\left\|P'\right\|_\infty \leq \frac{n}{1+k}\left\|P\right\|_\infty. \tag{1.5}$$

Whereas under the same hypothesis, Govil and Rahman [8] extended inequality (1.5) to L_p-norm by showing that

$$\left\|P'\right\|_p \leq \frac{n}{\left\|k+z\right\|_p}\left\|P\right\|_p, \; p \geq 1. \tag{1.6}$$

As a refinement of inequality (1.6), it was shown by Rather [16] that if $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ for $|z| < k, k \geq 1$, then

$$\left\|P'\right\|_p \leq \frac{n}{\left\|\delta_{k,1}+z\right\|_p}\left\|P\right\|_p, \; p > 0, \tag{1.7}$$

where $\delta_{k,1}$ is defined by

$$\delta_{k,1} = \frac{n\,|a_0|\,k^2 + |a_1|\,k^2}{n\,|a_0| + |a_1|\,k^2}. \tag{1.8}$$

Let $D_\alpha P(z)$ denote the polar differentiation of a polynomial $P(z)$ of degree n with respect to a complex number α. Then

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$$

(see [12]). Note that $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\underset{\alpha\to\infty}{Lim}\frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R, R > 0$.

A. Aziz [2] extended inequalities (1.1) and (1.3) to the polar derivative of a polynomial and proved that if $P \in \mathcal{P}_n$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\|D_\alpha P\|_\infty \leq n\,|\alpha|\,\|P\|_\infty \tag{1.9}$$

and if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\|D_\alpha P\|_\infty \leq \frac{n}{2}(|\alpha| + 1)\,\|P\|_\infty\,. \tag{1.10}$$

Both the inequalities (1.9) and (1.10) are best possible. If we divide the two sides (1.9) and (1.10) by $|\alpha|$ and make $|\alpha| \to \infty$, we get inequalities (1.1) and (1.3) respectively.

A. Aziz [2] also considered the class of polynomials $P \in P_n$ having no zero in $|z| < k$ and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\|D_\alpha P\|_\infty \leq n\left(\frac{|\alpha| + k}{1+k}\right)\|P\|_\infty\,. \tag{1.11}$$

The result is best possible and equality in (1.11) holds for $P(z) = (z+k)^n$, where α is any real number with $\alpha \geq 1$.

For polynomials $P \in \mathcal{P}_n$ having all their zeros in disk, Aziz and Rather [4] proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$\|D_\alpha P\|_\infty \geq n\left(\frac{|\alpha| - k}{1+k}\right)\|P\|_\infty\,. \tag{1.12}$$

The result is sharp and equality in (1.12) holds for $P(z) = (z-k)^n$ with real $\alpha \geq k$.

As an extension of inequality (1.10) to the L_p-norm, Aziz and Rather [5] proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p . \tag{1.13}$$

Aziz *et al.* [6] also extended inequality (1.11) to the L_p- norm and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then for, $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p \geq 1$,

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p . \tag{1.14}$$

Rather [17,18] showed that inequalities (1.13) and (1.14) remain valid for $0 < p < 1$ as well.

The bound in inequality (1.14) depends upon the zero of smallest modulus. It is interesting to obtain a bound which depends upon some or all the coefficients of the polynomial $P \in \mathcal{P}_n$ in addition to the zero of smallest modulus as well.

We need the following lemmas.

Lemma 1.1. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \neq 0$, $1 \leq \mu \leq n$ *in* $|z| < k$ *where* $k \geq 1$ *and* $Q(z) = z^n \overline{P(1/\overline{z})}$, *then for* $|z| = 1$,

$$\delta_{k,\mu} \left| P'(z) \right| \leq \left| Q'(z) \right|$$

where

$$\delta_{k,\mu} = \left(\frac{n |a_0| k^{\mu+1} + \mu |a_\mu| k^{2\mu}}{n |a_0| + \mu |a_\mu| k^{\mu+1}} \right) \ (\geq k^\mu) \tag{1.15}$$

and

$$\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^\mu \leq 1, \qquad 1 \leq \mu \leq n.$$

Lemma 1.1 follows easily on using argument similar to that used in [14, Lemma 1].

Lemma 1.2. *If* a, b *are any two positive real numbers such that* $a \geq bt$ *where* $t \geq 1$, *then for any* $x \geq 1, p > 0$ *and* $0 \leq \beta < 2\pi$,

$$(a + bx)^p \int_0^{2\pi} \left| t + e^{i\beta} \right|^p d\beta \leq (t + x)^p \int_0^{2\pi} \left| a + be^{i\beta} \right|^p d\beta.$$

Proof. By hypothesis $t \geq 1$ and $x \geq 1$, it can be easily seen that

$$Re \left(\frac{1}{t + e^{i\beta}} \right) \geq \frac{1}{t+1} \geq \frac{1}{t+x}.$$

Now using the fact that $a > 0, b > 0$ and $a \geq bt$, we get

$$\begin{aligned}\left|\frac{a+be^{i\beta}}{t+e^{i\beta}}\right| &\geq Re\left(\frac{a+be^{i\beta}}{t+e^{i\beta}}\right) = Re\left(b+\frac{a-bt}{t+e^{i\beta}}\right) \\ &\geq b+(a-bt)\left(\frac{1}{t+x}\right) \\ &= \frac{a+bx}{t+x}.\end{aligned}$$

This implies that for each $p > 0$,

$$(a+bx)^p \left|t+e^{i\beta}\right|^p \leq (t+x)^p \left|a+be^{i\beta}\right|^p.$$

which on integration leads to the desired result. □

Next two lemmas are due to Aziz and Rather [3].

Lemma 1.3. *If* $P \in \mathcal{P}_n$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$, *then for every* $p > 0$ *and real* β *with* $0 \leq \beta < 2\pi$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta \leq 2\pi n^p \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta.$$

Lemma 1.4. *If* $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$ $(1 \leq \mu \leq n)$ *has all zeros in* $|z| \leq k$ *where* $k \leq 1$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$, *then for* $|z| = 1$,

$$t_{k,\mu}\left|P'(z)\right| \geq \left|Q'(z)\right|$$

where $t_{k,\mu}$ *is given by*

$$t_{k,\mu} = \left(\frac{n\left|a_n\right|k^{2\mu}+\mu\left|a_{n-\mu}\right|k^{\mu-1}}{n\left|a_n\right|k^{\mu-1}+\mu\left|a_{n-\mu}\right|}\right). \tag{1.16}$$

We also need the following lemma due to Aziz [2].

Lemma 1.5. *If* $P \in \mathcal{P}_n$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$, *then for every non-zero complex number* γ *and* $0 \leq \theta < 2\pi$,

$$\left|D_\gamma Q(e^{i\theta})\right| = |\gamma|\left|D_{\frac{1}{\overline{\gamma}}}P(e^{i\theta})\right|.$$

2. Main results

Theorem 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + \delta_{k,1}}{\|\delta_{k,1} + z\|_p} \right) \|P\|_p \,, \tag{2.1}$$

where $\delta_{k,1}$ is given by (1.8). *In the limiting case, when $p \to \infty$, the result is sharp and equality in* (2.1) *holds for $P(z) = (z+k)^n$ with real $\alpha \geq 1$.*

Remark 2.2. Since $\delta_{k,1} \geq k \geq 1$, setting $a = \delta_{k,1}, b = 1, t = k$ and $x = |\alpha|$ in Lemma 1.2, we get

$$(\delta_{k,1} + |\alpha|)^p \int_0^{2\pi} \left| k + e^{i\beta} \right|^p d\beta \leq (k + |\alpha|)^p \int_0^{2\pi} \left| \delta_{k,1} + e^{i\beta} \right|^p d\beta.$$

Equivalently,

$$\frac{(\delta_{k,1} + |\alpha|)^p}{\int_0^{2\pi} |\delta_{k,1} + e^{i\beta}|^p \, d\beta} \leq \frac{(k + |\alpha|)^p}{\int_0^{2\pi} |k + e^{i\beta}|^p \, d\beta},$$

that is,

$$\frac{(\delta_{k,1} + |\alpha|)}{\|\delta_{k,1} + z\|_P} \leq \frac{(k + |\alpha|)}{\|k + z\|_p},$$

which shows that Theorem 2.1 sharpens the inequality (1.14).

Instead of proving Theorem 2.1, we prove a more general result for the class of lacunary type polynomials

$$\mathcal{P}_{n,\mu} := \left\{ P \in \mathcal{P}_n : P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \ 1 \leq \mu \leq n \right\},$$

which also extends an L_p- inequality due Rather [15] to the polar derivatives of a polynomial valid for $0 \leq p < \infty$. More precisely, we prove:

Theorem 2.3. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + \delta_{k,\mu}}{\|\delta_{k,\mu} + z\|_p} \right) \|P\|_p \,. \tag{2.2}$$

where $\delta_{k,\mu}$ is given by (1.15). *In the limiting case, when $p \to \infty$, the result is sharp and equality in* (2.2) *holds for $P(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of μ and real $\alpha \geq 1$.*

Proof. By hypothesis, $P \in \mathcal{P}_n$ does not vanish in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, therefore, by Lemma 1.1, we have

$$\delta_{k,\mu}\left|P'(z)\right| \leq \left|Q'(z)\right|,$$

where $\delta_{k,\mu}$ is given by (1.15). Further, since $\delta_{k,\mu} \geq k^{\mu} \geq 1, 1 \leq \mu \leq n$, by Lemma 1.2 with $a = \left|Q'(e^{i\theta})\right|$, $b = \left|P'(e^{i\theta})\right|$, $t = \delta_{k,\mu}$ and $x = |\alpha|$, we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,

$$\begin{aligned}
&\Big(|Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|^p\Big)\int_0^{2\pi}|\delta_{k,\mu} + e^{i\beta}|^p d\beta \\
&\leq (|\alpha| + \delta_{k,\mu})^p\int_0^{2\pi}|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})|^p d\beta \\
&= (|\alpha| + \delta_{k,\mu})^p\int_0^{2\pi}|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})|^p d\beta. \qquad (2.3)
\end{aligned}$$

Now for every $p > 0$, we have

$$\begin{aligned}
&\int_0^{2\pi}\left|\delta_{k,\mu} + e^{i\beta}\right|^p d\beta\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^p d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi}\left|\delta_{k,\mu} + e^{i\beta}\right|^p\left|D_\alpha P(e^{i\theta})\right|^p d\beta d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi}\left|\delta_{k,\mu} + e^{i\beta}\right|^p\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) + \alpha P'(e^{i\theta})\right|^p d\beta d\theta \\
&\leq \int_0^{2\pi}\int_0^{2\pi}\left|\delta_{k,\mu} + e^{i\beta}\right|^p\left\{\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right| + |\alpha|\left|P'(e^{i\theta})\right|\right\}^p d\beta d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi}\left|\delta_{k,\mu} + e^{i\beta}\right|^p\left\{\left|Q'(e^{i\theta})\right| + |\alpha|\left|P'(e^{i\theta})\right|\right\}^p d\beta d\theta. \qquad (2.4)
\end{aligned}$$

Using (2.3) in (2.4) and the property of definite integrals, we obtain for each $p > 0$ and $|\alpha| \geq 1$,

$$\begin{aligned}
&\int_0^{2\pi}\left|\delta_{k,\mu} + e^{i\beta}\right|^p d\beta\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^p d\theta \\
&\leq (|\alpha| + \delta_{k,\mu})^p\int_0^{2\pi}\int_0^{2\pi}\left|\left|P'(e^{i\theta})\right| + e^{i\beta}\left|Q'(e^{i\theta})\right|\right|^p d\theta d\beta \\
&= (|\alpha| + \delta_{k,\mu})^p\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^p d\theta d\beta.
\end{aligned}$$

This gives with the help of Lemma 1.3 for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $p > 0$,

$$\int_0^{2\pi} \left|\delta_{k,\mu} + e^{i\beta}\right|^p d\beta \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^p d\theta \leq 2\pi n^p \left(|\alpha| + \delta_{k,\mu}\right)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta,$$

which immediately leads to (2.2) and this completes the proof of Theorem 2.3 for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$. □

The following result, which extends a result due to Qazi [14] to the polar derivatives of a polynomial, immediately follows from Theorem 2.3 by letting $p \to \infty$ in (2.2).

Corollary 2.4. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,*

$$\|D_\alpha P\|_\infty \leq n \left(\frac{|\alpha| + \delta_{k,\mu}}{1 + \delta_{k,\mu}}\right) \|P\|_\infty \tag{2.5}$$

where $\delta_{k,\mu}$ is given by (1.15)*. The result is best possible and equality in* (2.5) *holds for $P(z) = (z^\mu + k^\mu)^{n/\mu}$ where n is a multiple of μ and $\alpha \geq 1$.*

For $\mu = 1$, Corollary 2.4 sharpens the inequality (1.11).

Using Lemma 1.2 and the fact that $\delta_{k,\mu} \geq k^\mu \geq 1$, $1 \leq \mu \leq n$, the follwing result which is a generalization of inequality (1.14) also follows from Theorem 2.3.

Corollary 2.5. *If $P \in \mathcal{P}_{n,\mu}$ does not vanish in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $0 \leq p < \infty$,*

$$\|D_\alpha P\|_p \leq n \left(\frac{|\alpha| + k^\mu}{\|k^\mu + z\|_p}\right) \|P\|_p\,, \tag{2.6}$$

where $\delta_{k,\mu}$ is given by (1.15)*. In the limiting case, when $p \to \infty$, the result is sharp and equality in* (2.6) *holds for $P(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of μ and real $\alpha \geq 1$.*

For the class of polynomials

$$\mathcal{P}^*_{n,\mu} := \left\{P \in \mathcal{P}_n : P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ 1 \leq \mu \leq n\right\},$$

we also establish the following result:

Theorem 2.6. *If $P \in \mathcal{P}^*_{n,\mu}$ and $P(z)$ has all its zeros in $0 < |z| \le k$ where $k \le 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$ and $0 \le p < \infty$,*

$$\|D_\alpha P\|_p \le n\left(\frac{|\alpha| + t_{k,\mu}}{\|t_{k,\mu} + z\|_p}\right)\|P\|_p\,, \tag{2.7}$$

where $t_{k,\mu}$ is defined by (1.16).

Proof. Let $Q(z) = z^n\overline{P(1/\overline{z})}$. Since all the zeros of polynomial $P(z) = a_0 + a_1 z + \cdots + a_{n-\mu}z^{n-\mu} + a_n z^n$ of degree n lie in $0 < |z| \le k$, therefore, $Q(z) = \overline{a_n} + \overline{a_{n-\mu}}z^\mu + \cdots + \overline{a_1}z^{n-1} + \overline{a_0}z^n$ is a polynomial of degree n which does not vanish in $|z| < (1/k)$ where $(1/k) \ge 1$. Applying Theorem 2.3 to the polynomial $Q(z)$ and using the fact that $\left|Q(e^{i\theta})\right| = \left|P(e^{i\theta})\right|$ for $0 \le \theta < 2\pi$ and $\|z + 1/t_{k,\mu}\|_p = \frac{1}{t_{k,\mu}}\|z + t_{k,\mu}\|_p$, we get for $\gamma \in \mathbb{C}$ with $|\gamma| \ge 1$ and $p > 0$,

$$\left\{\int_0^{2\pi}\left|D_\gamma Q(e^{i\theta})\right|^p d\theta\right\}^{1/p} \le n\left(\frac{t_{k,\mu}|\gamma| + 1}{\|z + t_{k,\mu}\|_p}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}.$$

This gives by using Lemma 1.5 for $|\gamma| \ge 1$, and $p > 0$,

$$\left\{\int_0^{2\pi}|\gamma|\left|D_{1/\overline{\gamma}} P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \le n\left(\frac{t_{k,\mu}|\gamma| + 1}{\|z + t_{k,\mu}\|_p}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{2.8}$$

Replacing $1/\overline{\gamma}$ by α, we obtain from (2.8),

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \le n\left(\frac{|\alpha| + t_{k,\mu}}{\|z + t_{k,\mu}\|_p}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p},$$

for $|\alpha| \le 1$ and $p > 0$. This proves Theorem 2.6 for $p > 0$. The extension to $p = 0$ obtains by letting $p \to 0+$. □

The following result is an immediate consequence of Theorem 2.6.

Corollary 2.7. *If $P \in \mathcal{P}^*_{n,\mu}$ and $P(z)$ has all its zeros in $0 < |z| \le k$ where $k \le 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$,*

$$\|D_\alpha P\|_\infty \le n\left(\frac{|\alpha| + t_{k,\mu}}{1 + t_{k,\mu}}\right)\|P\|_\infty\,,$$

where $t_{k,\mu}$ is given by (1.16). *The result is sharp.*

Finally, we present following integral inequality, which yields a refinement of the inequality (1.12) as a special case.

Theorem 2.8. *If $P \in \mathcal{P}^*_{n,\mu}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| > t_{k,\mu}$ and $0 \leq p < \infty$,*

$$n(|\alpha| - t_{k,\mu}) \left\| \frac{P}{D_\alpha P} \right\|_p \leq \|1 + t_{k,\mu} z\|_p \tag{2.9}$$

where $t_{k,\mu}$ is given by (1.16).

Proof. Let $Q(z) = z^n \overline{P(1/\overline{z})}$. Then it can be easly verified for $|z| = 1$,

$$\left|Q'(z)\right| = \left|nP(z) - zP'(z)\right| \quad \text{and} \quad \left|P'(z)\right| = \left|nQ(z) - zQ'(z)\right|. \tag{2.10}$$

Since all the zeros of polynomial $P \in \mathcal{P}^*_{n,\mu}$ lie in $|z| \leq 1$, by Lemma 1.4, we have

$$t_{k,\mu} \left|P'(z)\right| \geq \left|Q'(z)\right| \quad \text{for} \quad |z| = 1 \tag{2.11}$$

where $t_{k,\mu}$ is given by (1.16). This gives with the help of (2.10),

$$\left|Q'(z)\right| \leq t_{k,\mu} \left|nQ(z) - zQ'(z)\right| \quad \text{for} \quad |z| = 1. \tag{2.12}$$

Also, since all the zeros of $P(z)$ lie in $|z| \leq k \leq 1$, by Gauss-Lucas theorem all the zeros of polynomial $P'(z)$ also lie $|z| \leq k \leq 1$. This shows that all the zeros of polynomial $z^{n-1}\overline{P'(1/\overline{z})} = nQ(z) - zQ'(z)$ lie in $|z| \geq (1/k) \geq 1$, Therefore, the function

$$f(z) = \frac{zQ'(z)}{t_{k,\mu}(nQ(z) - zQ'(z))}$$

is analytic in $|z| \leq 1$ and by (2.12), we have $|f(z)| \leq 1$ for $|z| = 1$. Further, $f(0) = 0$. Thus the function $1 + t_{k,\mu} f(z)$ is subordinate to the function $1 + t_{k,\mu} z$. Hence by property of subordination [9], we have for each $p > 0$,

$$\int_0^{2\pi} \left|1 + t_{k,\mu} f(e^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|1 + t_{k,\mu} e^{i\theta}\right|^p d\theta. \tag{2.13}$$

Now

$$1 + t_{k,\mu} f(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)},$$

which by using (2.10) gives,

$$n\,|Q(z)| = |1 + t_{k,\mu} f(z)| \left|nQ(z) - zQ'(z)\right| = |1 + t_{k,\mu} f(z)| \left|P'(z)\right|, \; |z| = 1. \tag{2.14}$$

Since $|Q(z)| = |P(z)|$ for $|z| = 1$, we get from (2.14),

$$n\,|P(z)| = |1 + t_{k,\mu} f(z)| \left|P'(z)\right| \quad \mathit{for} \quad |z| = 1. \tag{2.15}$$

Further, for $\alpha \in \mathbb{C}$ with $|\alpha| > t_{k,\mu}$ and for $|z| = 1$,

$$\begin{aligned}|D_\alpha P(z)| &= \left|nP(z) - zP'(z) + \alpha P'(z)\right| \\ &\geq |\alpha| \left|P'(z)\right| - \left|nP(z) - zP'(z)\right|.\end{aligned}$$

Combining this with (2.10) and (2.11), we obtain for $|z| = 1$,

$$\begin{aligned}|D_\alpha P(z)| &\geq |\alpha| \left|P'(z)\right| - \left|Q'(z)\right| \\ &\geq |\alpha| \left|P'(z)\right| - t_{k,\mu} \left|P'(z)\right| \\ &= (|\alpha| - t_{k,\mu}) \left|P'(z)\right|. \end{aligned} \tag{2.16}$$

Using (2.16) in (2.15), it follows that,

$$n\left(|\alpha| - t_{k,\mu}\right)|P(z)| \leq |1 + t_{k,\mu} f(z)|\,|D_\alpha P(z)|\,, \ |z| = 1. \tag{2.17}$$

From (2.13) and (2.17), we deduce for each $p > 0$,

$$n^p\left(|\alpha| - t_{k,\mu}\right)^p \int_0^{2\pi} \left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^p d\theta \leq \int_0^{2\pi} \left|1 + t_{k,\mu} e^{i\theta}\right|^p d\theta,$$

which is equivalent to the desired result. This completes the proof of Theorem 2.8. To establish this result for $p = 0$, we simply let $p \to 0+$. □

Since $|D_\alpha P(z)| \leq \|D_\alpha P\|_\infty$ for $|z| = 1$, we immediately get the following result from Theorem 2.8.

Corollary 2.9. *If $P \in \mathcal{P}^*_{n,\mu}$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq t_{k,\mu}$ and $0 \leq p < \infty$,*

$$n(|\alpha| - t_{k,\mu})\,\|P\|_p \leq \|1 + t_{k,\mu} z\|_p\,\|D_\alpha P\|_\infty \tag{2.18}$$

where $t_{k,\mu}$ is given by (1.16).

Letting $p \to \infty$ in (2.18) and taking $\mu = 1$, we obtain the following refinement of the inequality (1.12).

Corollary 2.10. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq t_{k,1}$,*

$$\|D_\alpha P\|_\infty \geq n\left(\frac{|\alpha| - t_{k,1}}{1 + t_{k,1}}\right)\|P\|_\infty \tag{2.19}$$

where

$$t_{k,1} = \left(\frac{n\,|a_n|\,k^2 + |a_{n-1}|}{n\,|a_n| + |a_{n-1}|}\right). \tag{2.20}$$

The result is best possible and equality in (2.19) *holds for $P(z) = (z-k)^n$ with real $\alpha \geq t_{k,1}$.*

Remark 2.11. From (2.13) and (2.17), we deduce for each $p > 0$,

$$n^p\left(|\alpha| - t_{k,\mu}\right)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|1 + t_{k,\mu}e^{i\theta}\right|^p \left|D_\alpha P(e^{i\theta})\right|^p d\theta,$$

which gives with the help of Holder's inequality for $r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$,

$$\begin{aligned} &n^p\left(|\alpha| - t_{k,\mu}\right)^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \\ &\leq \left\{\int_0^{2\pi} \left|1 + t_{k,\mu}e^{i\theta}\right|^{pr} d\theta\right\}^{1/r} \times \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{ps} d\theta\right\}^{1/s}. \end{aligned}$$

Thus we have proved the following generalization of Corollary 2.9 which also leads to an extension of the inequality (1.12) to L_p mean of $|P(z)|$ on $|z| = 1$.

Theorem 2.12. *If $P \in \mathcal{P}^*_{n,\mu}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq t_{k,\mu}$, $0 \leq p < \infty$, and $r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$,*

$$\begin{aligned} &n\left(|\alpha| - t_{k,\mu}\right) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \\ &\leq \left\{\int_0^{2\pi} \left|1 + t_{k,\mu}e^{i\theta}\right|^{pr} d\theta\right\}^{1/pr} \times \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{ps} d\theta\right\}^{1/ps}. \end{aligned} \tag{2.21}$$

where $t_{k,\mu}$ is given by (1.16).

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