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TRIPLED FIXED POINT RESULTS IN PARTIALLY ORDERED S-METRIC SPACES

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Abstract. We establish a tripled fixed point result for a mixed monotone mapping satisfying nonlinear contractions in ordered S-metric spaces. Also, some examples are given to support our result.

1. Introduction

In 1922. Banach [4] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups. Considering different mappings ets.

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Many mathematical problems require one to find a distance between two or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continues to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [6, 9, 10, 11, 12, 16, 21]. Sedghi *et al.* [19] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

In present era, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. Fixed point problems have also been considered in partially ordered probabilistic metric spaces [8] , partially ordered G-metric spaces [3, 17], partially ordered cone metric spaces [7, 14, 22], partially ordered fuzzy metric spaces and partially ordered non-Archimedean fuzzy metric spaces [1, 2].

Mixed monotone operators were introduce by Guo and Lakshmikantham in [13]. Their study has not only important theoretical meaning but alse wide applications in engineering, nuclear physics, biological chemistry technology, etc. Particularly, a coupled fixed point result in partially ordered metric spaces was established by Bhaskar and Lakshmikantham [5]. After the publication of this work, several coupled fixed point and coincidence point results have appeared in the recent literature.

In this paper, we establish a tripled fixed point result for a mapping having a mixed monotone mapping property in S-metric spaces. Also, some examples are given to support our result.

2. Preliminaries

In the sequel, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}^*$ denote the set of real numbers, the set of nonnegative real numbers, and the set of nonnegative integers, respectively.

Definition 2.1. ([19]) Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,

(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Some examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X. Then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X. Then $S(x, y, z) = ||x - z|| + ||y - z||$ is an S-metric on X.
(3) Let X be a nonempty set and d be an ordinary metric on X. Then $S(x, y, z) = d(x, y) + d(y, z)$ is an S-metric on X.

Lemma 2.2. ([19]) In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Definition 2.3. ([19]) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S\ [x, r]$ with center x and radius r as follows, respectively:

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(y, y, x) \leq r\}. \end{aligned}$$

Example 2.4. ([19]) Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_s(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} \\ &= (0, 2). \end{aligned}$$

Definition 2.5. ([19]) Let (X, S) be an S-metric space and $A \subset X$.

(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subset A$, then the subset A is called an open subset of X.
(2) A subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in X converges to x if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$\forall n \geq n_0 \Longrightarrow S(x_n, x_n, x) < \varepsilon,$$
and we denote by $\lim\limits_{n\to\infty} x_n = x$.
(4) A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.
(5) An S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X(induced by the S-metric S).

Definition 2.6. ([20]) Let (X, S) and $(X\ ', S\ ')$ be two S-metric spaces and let $f : (X, S) \to (X\ ', S\ ')$ be a function. Then f is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X, $S(x_n, x_n, a) \to 0$ implies $S\ '(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous on X or continuous function if it is continuous at every $a \in X$.

Lemma 2.7. ([20]) Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Definition 2.8. Let (X, S) be an S-metric space. A mapping $F : X \times X \times X \longrightarrow X$ is said to be continuous if for any three S-convergent sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converging to x, y, and z, respectively, $\{F(x_n, y_n, z_n)\}$ is S-convergent to $F(x, y, z)$.

Definition 2.9. ([5]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \longrightarrow X$. The mapping F is said to have the mixed monotone property if, for any $x, y, z \in X$,

$$x_1 \preceq x_2 \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z), \quad for \quad x_1, x_2 \in X,$$

$$y_1 \preceq y_2 \Longrightarrow F(x, y_2, z) \preceq F(x, y_1, z), \quad for \quad y_1, y_2 \in X$$

and

$$z_1 \preceq z_2 \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2), \quad for \quad y_1, y_2 \in X.$$

Definition 2.10. ([5]) An element $(x, y, z) \in X \times X \times X$ is said to be a tripled fixed point of the mapping $F : X \times X \times X \longrightarrow X$ if

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z.$$

3. Main results

Let Φ be the set of all non-decreasing functions $\phi : [0, +\infty) \to [0, +\infty)$ such that $\lim\limits_{n\to+\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then we have the followings [15].

(i) $\phi(t) < t$ for all $t \in (0, +\infty)$,
(ii) $\phi(0) = 0$.

The aim of this paper is to prove the following theorem.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and (X, S) be an S-metric space. Let $F : X \times X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $\phi \in \Phi$ such that for all $x, y, z, a, b, c, u, v, w \in X$ with $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$, we have*

$$\begin{aligned} &S(F(x,y,z), F(a,b,c), F(u,v,w)) \\ &\leq \phi[\max(S(x,a,u), S(y,b,v), S(z,c,w))]. \end{aligned} \tag{3.1}$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$, and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y, \quad F(z,y,x) = z. \tag{3.2}$$

Proof. Suppose $x_0, y_0, z_0 \in X$ are such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$. Define $x_1 = F(x_0, y_0, z_0), y_1 = F(y_0, x_0, y_0)$, and $z_1 = F(z_0, y_0, x_0)$. Then $x_0 \preceq x_1$, $y_0 \succeq y_1$ and $z_0 \preceq z_1$. Again, define $x_2 = F(x_1, y_1, z_1), y_2 = F(y_1, x_1, y_1)$, and $z_2 = F(z_1, y_1, x_1)$. Since F has the mixed monotone property, we have $x_0 \preceq x_1 \preceq x_2$, $y_2 \preceq y_1 \preceq y_0$, and $z_0 \preceq z_1 \preceq z_2$. Continuing in this process, we can construct three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X such that

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}) \preceq x_{n+1} = F(x_n, y_n, z_n), \\ y_{n+1} &= F(y_n, x_n, y_n) \preceq y_n = F(y_{n-1}, x_{n-1}, y_{n-1}), \\ z_n &= F(z_{n-1}, y_{n-1}, x_{n-1}) \preceq z_{n+1} = F(z_n, y_n, x_n). \end{aligned} \tag{3.3}$$

If, for some integer n, we have

$$(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n),$$

then

$$x_n = F(x_n, y_n, z_n), \quad y_n = F(y_n, x_n, y_n) \quad \textit{and} \quad z_n = F(z_n, y_n, x_n),$$

that is, (x_n, y_n, z_n) is a tripled fixed point of F. Thus we will assume that $(x_{n+1}, y_{n+1}, z_{n+1}) \neq (x_n, y_n, z_n)$ for all $n \in \mathbb{N}$, that is, we assume that either $x_{n+1} \neq x_n$, $y_{n+1} \neq y_n$ or $z_{n+1} \neq z_n$. For any $n \in \mathbb{N}^*$, we have from (3.1) that

$$\begin{aligned} &S(x_{n+1}, x_{n+1}, x_n) \\ &= S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\ &\leq \phi[\max(S(x_n, x_n, x_{n-1}), S(y_n, y_n, y_{n-1}), S(z_n, z_n, z_{n-1}))], \end{aligned} \tag{3.4}$$

$$\begin{aligned} &S(y_{n+1}, y_{n+1}, y_n) \\ &= S(F(y_n, x_n, y_n), F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1})) \\ &\leq \phi[\max(S(y_n, y_n, y_{n-1}), S(x_n, x_n, x_{n-1}), S(z_n, z_n, z_{n-1}))]. \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&S(z_{n+1}, z_{n+1}, z_n) \\
&= S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\
&\leq \phi[\max(S(z_n, z_n, z_{n-1}), S(y_n, y_n, y_{n-1}), S(x_n, x_n, x_{n-1}))]. \qquad (3.6)
\end{aligned}$$

It follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned}
&\max(S(x_{n+1}, x_{n+1}, x_n), S(y_{n+1}, y_{n+1}, y_n), S(z_{n+1}, z_{n+1}, z_n)) \\
&\leq \phi[\max(S(x_n, x_n, x_{n-1}), S(y_n, y_n, y_{n-1}), S(z_n, z_n, z_{n-1}))]. \qquad (3.7)
\end{aligned}$$

By repeating (3.7) n-times and using the fact that ϕ is non-decreasing, we get

$$\begin{aligned}
&\max(S(x_{n+1}, x_{n+1}, x_n), S(y_{n+1}, y_{n+1}, y_n), S(z_{n+1}, z_{n+1}, z_n)) \\
&\leq \phi[\max(S(x_n, x_n, x_{n-1}), S(y_n, y_n, y_{n-1}), S(z_n, z_n, z_{n-1}))] \\
&\leq \phi^2[\max(S(x_{n-1}, x_{n-1}, x_{n-2}), S(y_{n-1}, y_{n-1}, y_{n-2}), S(z_{n-1}, z_{n-1}, z_{n-2}))] \\
&\quad \vdots \\
&\leq \phi^n[\max(S(x_1, x_1, x_0), S(y_1, y_1, y_0), S(z_1, z_1, z_0))]. \qquad (3.8)
\end{aligned}$$

Now, we will show that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ is an S-Cauchy sequences in X. Let $\varepsilon > 0$. Since

$$\lim_{n \longrightarrow \infty} \phi^n[\max(S(x_1, x_1, x_0), S(y_1, y_1, y_0), S(z_1, z_1, z_0))] = 0, \qquad (3.9)$$

and $\epsilon > \phi(\epsilon)$, there exists $n_0 \in \mathbb{N}$ such that

$$\phi^n[\max(S(x_1, x_1, x_0), S(y_1, y_1, y_0), S(z_1, z_1, z_0))] < \varepsilon, \quad \forall n \geq n_0. \qquad (3.10)$$

By (3.8), this implies that for all $n \geq n_0$.

$$\max(S(x_{n+1}, x_{n+1}, x_n), S(y_{n+1}, y_{n+1}, y_n), S(z_{n+1}, z_{n+1}, z_n)) < \varepsilon. \qquad (3.11)$$

For $m, n \in \mathbb{N}$, we will prove that

$$\max(S(x_n, x_n, x_m), S(y_n, y_n, y_m), S(z_n, z_n, z_m)) < \varepsilon, \forall m \geq n \geq n_0. \qquad (3.12)$$

By Definition 2.1-(3), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\le 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\vdots \\ &\le 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\le 2\sum_{i=n}^{m-1} S(x_i, x_i, x_{i+1}) \\ &= 2\sum_{i=n}^{m-1} S(x_{i+1}, x_{i+1}, x_i) \\ &\le 2\sum_{i=n}^{m-1} \phi[\max(S(x_i, x_i, x_{i-1}), S(y_i, y_i, y_{i-1}), S(z_i, z_i, z_{i-1}))] \\ &\vdots \\ &\le 2\sum_{i=n}^{m-1} \phi^n[\max(S(x_1, x_1, x_0), S(y_1, y_1, y_0), S(z_1, z_1, z_0))] \\ &\to 0. \end{aligned} \tag{3.13}$$

Similarly, we show that

$$S(y_n, y_n, y_m) < \varepsilon \quad \text{and} \quad S(z_n, z_n, z_m) < \varepsilon. \tag{3.14}$$

Hence, we have

$$\max(S(x_n, x_n, x_m), S(y_n, y_n, y_m), S(z_n, z_n, z_m)) < \varepsilon. \tag{3.15}$$

Thus (3.12) holds for all $m \ge n \ge n_0$. Hence $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are S-Cauchy sequences in X. Since X is a complete S-metric space, there exists $x, y, z \in X$ such that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to x, y and z, respectively. Finally, we show that (x, y, z) is a tripled fixed point of F. Since F is continuous and $(x_n, y_n, z_n) \longrightarrow (x, y, z)$. We have

$$x_{n+1} = F(x_n, y_n, z_n) \longrightarrow F(x, y, z).$$

By the uniqueness of limit, we get that $x = F(x, y, z)$. Similarly, we show that $y = F(y, x, y)$ and $z = F(z, y, x)$. So (x, y, z) is a tripled fixed point of F. □

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and (X, S) be an S-metric space. Let $F : X \times X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Supppose that there exists $k \in [0, 1)$ such that for*

$x, y, z, a, b, c, u, v, w \in X$, with $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$ we have

$$\begin{aligned} &S(F(x,y,z), F(a,b,c), F(u,v,w)) \\ &\leq k[\max(S(x,a,u), S(y,b,v), S(z,c,w))]. \end{aligned} \tag{3.16}$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y, \quad F(z,y,x) = z. \tag{3.17}$$

Proof. It follows from Theorem 3.1 by taking $\phi(t) = kt$. □

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set and (X, S) be an S-metric space. Let $F : X \times X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Supppose that there exists $k \in [0, 1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$, we have*

$$\begin{aligned} &S(F(x,y,z), F(a,b,c), F(u,v,w)) \\ &\leq \frac{k}{3}[S(x,a,u) + S(y,b,v) + S(z,c,w)]. \end{aligned} \tag{3.18}$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y, \quad F(z,y,x) = z. \tag{3.19}$$

Proof. Note that

$$\begin{aligned} &S(x,a,u) + S(y,b,v) + S(z,c,w) \\ &\leq 3[\max(S(x,a,u), S(y,b,v), S(z,c,w))]. \end{aligned} \tag{3.20}$$

Then, the proof follows from Corollary 3.2. □

By adding an additional hypothesis, the continuity of F in Theorem 3.1 can be dropped.

Theorem 3.4. *Let $(X, \preceq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X \times X \times X \longrightarrow X$ be a mapping with the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$\begin{aligned} &S(F(x,y,z), F(a,b,c), F(u,v,w)) \\ &\leq \phi[\max(S(x,a,u), S(y,b,v), S(z,c,w))], \end{aligned} \tag{3.21}$$

for all $x, y, z, a, b, c, u, v, w \in X$, with $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$. Assume also that X has the following properties:

(i) *if a nondecreasing sequence $x_n \longrightarrow x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,*
(ii) *if a nondecreasing sequence $y_n \longrightarrow y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ *and* $z_0 \preceq F(z_0, y_0, x_0)$, *then* F *has a tripled fixed point.*

Proof. Following proof of Theorem 3.1 step by step, we construct three S-Cauchy sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X with

$$\begin{aligned} &x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots, \\ &y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq \cdots, \\ &z_1 \preceq z_2 \preceq \cdots \preceq z_n \preceq \cdots, \end{aligned} \tag{3.22}$$

such that $x_n \longrightarrow x \in X$, $y_n \longrightarrow y \in X$, and $z_n \longrightarrow z \in X$. By the hypotheses on X, we have $x_n \preceq x$, $y_n \succeq y$, and $z_n \preceq z$ for all $n \in \mathbb{N}$. If for some $n \geq 0$, $x_n = x$, $y_n = y$, and $z_n = z$, then

$$\begin{aligned} &x = x_n \preceq x_{n+1} \preceq x = x_n, \\ &y = y_n \succeq y_{n+1} \succeq y = y_n, \\ &z = z_n \preceq z_{n+1} \preceq z = z_n, \end{aligned} \tag{3.23}$$

which implies that $x_n = x_{n+1} = F(x_n, y_n, z_n)$, $y_n = y_{n+1} = F(y_n, x_n, y_n)$, and $z_n = z_{n+1} = F(z_n, y_n, x_n)$, that is, (x_n, y_n, z_n) is a tripled fixed point of F.

Now, assume that, for all $n \geq 0$, $(x_n, y_n, z_n) \neq (x, y, z)$. Thus, for each $n \geq 0$,

$$\max(S(x, x, x_n), S(y, y, y_n), S(z, z, z_n)) > 0. \tag{3.24}$$

From (3.21), we have

$$\begin{aligned} &S(F(x, y, z), F(x, y, z), x_{n+1}) \\ &= S(F(x, y, z), F(x, y, z), F(x_n, y_n, z_n)) \\ &\leq \phi[\max(S(x, x, x_n), S(y, y, y_n), S(z, z, z_n))], \end{aligned} \tag{3.25}$$

$$\begin{aligned} &S(F(y, x, y), F(y, x, y), y_{n+1}) \\ &= S(F(y, x, y), F(y, x, y), F(y_n, x_n, y_n)) \\ &\leq \phi[\max(S(y, y, y_n), S(x, x, x_n))] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &S(F(z, y, x), F(z, y, x), z_{n+1} \\ &= S(F(z, y, x), F(z, y, x), F(z_n, y_n, x_n)) \\ &\leq \phi[\max(S(x, x, x_n), S(y, y, y_n), S(z, z, z_n))]. \end{aligned} \tag{3.27}$$

Letting $n \longrightarrow +\infty$ in (3.25), (3.26), (3.27) and using (3.24) in the fact that $\phi(t) < t$ for all $t > 0$, it follows that $x = F(x, y, z)$, $y = F(y, x, y)$, and $z = F(z, y, x)$. Hence (x, y, z) is a tripled fixed point of F. $\square$

Now we give some examples illustraiting our results.

Example 3.5. Take $X = [0, +\infty)$ endowed with the complete S-metric

$$S(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\},$$

for all $x, y, z \in X$. Set $k = \frac{1}{2}$ and $F : X \times X \times X \longrightarrow X$ defined by $F(x, y, z) = \frac{1}{6}x$. The mapping F has the mixed monotone property. We have

$$\begin{aligned} &S(F(x, y, z), F(a, b, c), F(u, v, w)) \\ &= \frac{1}{6}S(x, a, u) \leq \frac{k}{3}[\max(S(x, a, u), S(y, b, v), S(z, c, w))], \end{aligned} \tag{3.28}$$

for all $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$, that is, (3.16) holds. Take $x_0 = y_0 = z_0 = 0$, then all the hypotheses of Corollary 3.2 are verified, and $(0, 0, 0)$ is the unique tripled fixed point of F.

Example 3.6. As in Example 3.5, take $X = [0, +\infty)$ and

$$S(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\},$$

for all $x, y, z \in X$. Set $k = \frac{1}{2}$ and $F : X \times X \times X \longrightarrow X$ defined by $F(x, y, z) = \frac{1}{36}(6x - 6y + 6z + 5)$. The mapping F has the mixed monotone property. For all $x \succeq a \succeq u$, $y \preceq b \preceq v$, and $z \succeq c \succeq w$, we have

$$\begin{aligned} &S(F(x, y, z), F(a, b, c), F(u, v, w)) \\ &\leq \frac{1}{6}(|x - u| + |y - v| + |z - w|) \\ &= \frac{1}{6}(S(x, a, u), S(y, b, v), S(z, c, w)) \\ &= \frac{k}{3}(S(x, a, u), S(y, b, v), S(z, c, w)), \end{aligned}$$

that is, (3.18) holds. Take $x_0 = y_0 = z_0 = \frac{1}{6}$, then all the hypotheses of Corollary 3.3 hold, and $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ is the unique tripled fixed point of F.

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