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EXISTENCE AND NONEXISTENCE RESULTS OF NONLINEAR SECOND-ORDER m-POINT BVP

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Abstract. In this work, we study a class of singular multi-point nonlinear boundary value problems with parameter $\lambda > 0$, the existence and nonexistence results of positive solutions are obtained when the nonlinear term satisfy different requirements of superlinearity and sublinearity and the parameter lies in some intervals.

1. Introduction

This paper considers the existence and nonexistence of positive solutions for the following second-order m-point boundary value problem (BVP):

$$\begin{cases} (p(t)x'(t))' - q(t)x(t) + \lambda h(t)f(t, x(t)) = 0, & t \in (0, 1), \\ ax(0) - bp(0)x'(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \\ cx(1) + dp(1)x'(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.1}$$

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where $\lambda > 0$ is a parameter, $a, c \in [0, +\infty), b, d \in (0, +\infty), \xi_i \in (0, 1), \alpha_i, \beta_i \in [0, +\infty)$ for $(i \in \{1, 2, \cdots, m-2\})$ are given constants, $p \in C^1([0, 1], (0, +\infty)), q \in C([0, 1], (0, +\infty))$ and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, $h(t)$ is allowed to be singular at $t = 0, t = 1$.

If $\lambda = 1, h = p \equiv 1, q \equiv 0, \alpha_i, \beta_i = 0 \ \ (i = 1, 2, \cdots, m-2)$, m-point BVP (1.1) reduces to the two-point BVP

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \qquad t \in (0, 1), \\ ax(0) - bx'(0) = 0, \\ cx(1) + dx'(1) = 0. \end{cases} \tag{1.2}$$

In this case, (1.2) has been intensively studied (see, [4, 6]).

In recent years, singular multi-point boundary value problems have been extensively studied and many optimal results have been obtained (see, [6, 11, 12, 13, 14]) and references therein. In addition, many paper investigated the existence of solutions for the nonsingular multi-point boundary value problems (see,[2, 3, 4, 10]).

Recently, Ma [8] and Ma and Thompson [9] obtained many good results about the existence of positive solutions for the more general m-point boundary value problem (1.1), but they only considered the case the nonlinearity being nonsingular. In this work, we consider the existence and nonexistence of positive solutions for BVP (1.1), here we allow h has singularity at $t = 0, 1$.

This work is organized as follows. In section 2, we present some lemmas that are used to prove our main result. Then in section 3, the existence and nonexistence of positive solutions for BVP (1.1) will be established by using the Krasnoselskii fixed point theory, which we state here for the convenience of the reader.

Lemma 1.1. ([1, 5]) *Suppose that E is a Banach space, K is a cone in E. Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

(i) $\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2.$

(ii) $\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2.$

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries and some lemmas

Let $E = C[0, 1]$ be a real Banach space equipped with the norm $\|x\| = \max_{t \in [0,1]} |x(t)|$ for $x \in C[0, 1]$. We let $P = \{x \in E : x(t) \geq 0, t \in [0, 1]\}$. Clearly P is a cone of E.

In the rest of the paper, we adopt the following assumptions:

$(\mathbf{H}_1)$ $p \in C^1([0,1],(0,+\infty)),\ q \in C([0,1],(0,+\infty))$.

$(\mathbf{H}_2)$ $a, c \in [0,+\infty)$, $b, d \in (0,+\infty)$ with $ac+ad+bc > 0$, $\alpha_i, \beta_i \in [0,+\infty)$ for $i \in \{1,\cdots,m-2\}$.

$(\mathbf{H}_3)$ $f \in C([0,1]\times[0,+\infty),[0,+\infty))$, $h \in C((0,1),[0,+\infty))$ and

$$0 < \int_0^1 G(s,s)h(s)ds < +\infty,$$

where $G(t,s)$ will be given by (2.3).

The following lemmas play important roles to prove our main results, which can be found in papers [8] and [9].

Lemma 2.1. *Let* (H_1) *and* (H_2) *hold. Let* ψ *and* ϕ *be the solutions of the linear problems*

$$\begin{cases} (p(t)\psi'(t))'(t) - q(t)\psi(t) = 0, & t \in (0,1), \\ \psi(0) = b, \quad p(0)\psi'(0) = a, \end{cases} \tag{2.1}$$

and

$$\begin{cases} (p(t)\phi'(t))'(t) - q(t)\phi(t) = 0, & t \in (0,1), \\ \phi(1) = d, \quad p(1)\phi'(1) = -c, \end{cases} \tag{2.2}$$

respectively. Then

(i) ψ *is strictly increasing on [0,1], and* $\psi(t) > 0$ *on* $[0,1]$.
(ii) ϕ *is strictly decreasing on [0,1], and* $\phi(t) > 0$ *on* $[0,1]$.

As in [9], set

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) \\ \rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i) \end{vmatrix}, \quad \rho = p(t)\begin{vmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{vmatrix}.$$

Then, by Liouville's formula, we have

$$\rho = p(0)\begin{vmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{vmatrix} = \text{constant}.$$

Define

$$G(t,s) = \frac{1}{\rho}\begin{cases} \phi(t)\psi(s), & 0 \le s \le t \le 1, \\ \phi(s)\psi(t), & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

It is easy to see that

$$0 \le G(t,s) \le G(s,s), \qquad 0 \le s,t \le 1. \tag{2.4}$$

Lemma 2.2. *Let (H_1) and (H_2) hold. Assume that $\Delta \neq 0$. Then for $y \in C[0,1]$, the problem*

$$\begin{cases} (p(t)x'(t))'(t) - q(t)x(t) + y(t) = 0, \quad t \in (0,1), \\ ax(0) - bp(0)x'(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad cx(1) + dp(1)x'(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{2.5}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)y(s)ds + A(y)\psi(t) + B(y)\phi(t), \tag{2.6}$$

where

$$A(y) = \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i,s)y(s)ds & \rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) \\ \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\xi_i,s)y(s)ds & -\sum_{i=1}^{m-2} \beta_i \phi(\xi_i) \end{vmatrix} \tag{2.7}$$

and

$$B(y) = \frac{1}{\Delta} \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) & \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i,s)y(s)ds \\ \rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) & \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\xi_i,s)y(s)ds \end{vmatrix}. \tag{2.8}$$

Lemma 2.3. *Let (H_1) and (H_2) hold. Assume*

$(\boldsymbol{H}_4)$ $\Delta < 0$, $\rho - \sum_{i=1}^{m-2} \alpha_i \phi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) > 0$.

Then for $y \in C[0,1]$ with $y \geq 0$, the unique solution x of the problem (2.5) satisfies

$$x(t) \geq 0, \quad \textit{for } t \in [0,1].$$

Remark 2.4. *By (2.3) and Lemma 2.1, for any $t \in [0,1]$, we have*

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{\phi(t)}{\phi(s)}, & 0 \leq s \leq t \leq 1, \\ \frac{\psi(t)}{\psi(s)}, & 0 \leq t \leq s \leq 1, \end{cases} \geq \begin{cases} \frac{d}{\phi(0)}, & 0 \leq s \leq t \leq 1, \\ \frac{b}{\psi(1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Let $\gamma = \min\left\{\frac{d}{\phi(0)}, \frac{b}{\psi(1)}\right\}$. Then $G(t,s) \geq \gamma G(s,s), 0 \leq t, s \leq 1$.

Remark 2.5. *Since $\gamma = \min\left\{\frac{d}{\phi(0)}, \frac{b}{\psi(1)}\right\}$, according to the monotonicity of $\psi(t)$, we have $\gamma \leq \frac{b}{\psi(1)} = \frac{\psi(0)}{\psi(1)} \leq \frac{\psi(t)}{\psi(1)}$, so $\psi(t) \geq \gamma\psi(1), t \in [0,1]$. Similarly, by the monotonicity of $\phi(t)$, we have $\gamma \leq \frac{d}{\phi(0)} = \frac{\phi(1)}{\phi(0)} \leq \frac{\phi(t)}{\phi(0)}$, so $\phi(t) \geq \gamma\phi(0), t \in [0,1]$.*

With Lemma 2.2, BVP (1.1) has a solution $x = x(t)$ if and only if x is a solution of the following nonlinear integral equation

$$x(t) = \lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t), \tag{2.9}$$

where $F \triangleq \lambda h(t)f(t,x(t))$, $A(\cdot), B(\cdot)$ are defined by (2.7) and (2.8), respectively.

Define an operator $T : P \to P$ by

$$(Tx)(t) = \lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t). \tag{2.10}$$

It is easy to prove that the existence of solution to BVP (1.1) is equivalent to the existence of solutions to Eq.(2.9). That is the fixed point of operator T.

Let

$$K = \{x \in P : x(t) \geq \gamma\|x\|, t \in [0,1]\}. \tag{2.11}$$

It is obvious that K is a subcone of P. Let $K_r = \{x \in K : \|x\| < r\}$ for $r > 0$.

Lemma 2.6. *$T(K) \subset K$, and $T : K \to K$ is completely continuous.*

Proof. For any $x \in K$, (H_3) and (H_4) imply that $(Tx)(t) \geq 0$. From (2.4), (2.10) and the monotonicity of $\psi(t)$ and $\phi(t)$, we have

$$(Tx)(t) \leq \lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0),$$

which implies

$$\|Tx\| \leq \lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0). \tag{2.12}$$

By Remarks 2.4 and 2.5, we have

$$\begin{aligned}(Tx)(t) &= \lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t)\\ &\geq \gamma\lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\gamma\psi(1) + B(F)\gamma\phi(0) \qquad (2.13)\\ &\geq \gamma\left[\lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0)\right].\end{aligned}$$

Then, (2.12) and (2.13) yield that

$$(Tx)(t) \geq \gamma\|Tx\|.$$

Thus, $Tx \in K$. Therefore, $T(K) \subset K$. The complete continuity of $T : K \to K$ is obvious. $\square$

For convenience, we introduce the following symbols :

$$A = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i & \rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) \\ \sum_{i=1}^{m-2}\beta_i & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i) \end{vmatrix}, \tag{2.14}$$

$$B = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \sum_{i=1}^{m-2}\alpha_i \\ \rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & \sum_{i=1}^{m-2}\beta_i \end{vmatrix}, \tag{2.15}$$

$$L = \int_0^1 G(s,s)h(s)ds. \tag{2.16}$$

3. Main results

In this section, we present our main results as follows:

Theorem 3.1. *Suppose that $(H_1)-(H_4)$ hold. Besides, we assume that $f^0 < \infty, f_\infty > 0$ and*

$$\gamma^2 f_\infty \max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} > (1 + A\psi(1) + B\phi(0))f^0.$$

Then BVP (1.1) has at least one positive solution for any

$$\frac{1}{\gamma^2 L f_\infty \max\{1, \gamma A\psi(1), \gamma B\phi(0)\}} < \lambda < \frac{1}{(1 + A\psi(1) + B\phi(0))Lf^0}, \tag{3.1}$$

where γ is defined in Remark 2.4 and A, B, L are defined by (2.14), (2.15) and (2.16), respectively.

Proof. Let λ satisfies (3.1), we choose $\varepsilon_1 > 0$ such that $f_\infty - \varepsilon_1 > 0$ and

$$\frac{1}{\gamma^2 L(f_\infty - \varepsilon_1)\max\{1, \gamma A\psi(1), \gamma B\phi(0)\}} \le \lambda \le \frac{1}{(1 + A\psi(1) + B\phi(0))L(f^0 + \varepsilon_1)}. \tag{3.2}$$

Since $f^0 < \infty$, there exists $r_1 > 0$ such that

$$f(t,x) \le (f^0 + \varepsilon_1)x, \ \text{for } 0 \le t \le 1, \ 0 < x \le r_1. \tag{3.3}$$

For any $x \in \partial K_{r_1}$, by (3.2) and (3.3) we obtain

$$\begin{aligned}\|Tx\| \leq & \lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0) \\ \leq & \lambda(f^0+\varepsilon_1)\int_0^1 G(s,s)h(s)ds\|x\| \\ & + \lambda(f^0+\varepsilon_1)A\psi(1)\int_0^1 G(s,s)h(s)ds\|x\| \\ & + \lambda(f^0+\varepsilon_1)B\phi(0)\int_0^1 G(s,s)h(s)ds\|x\| \\ = & \lambda(f^0+\varepsilon_1)L(1+A\psi(1)+B\phi(0))\|x\| \\ \leq & \|x\|. \end{aligned} \tag{3.4}$$

On the other hand, since $\max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} > 0$. Without loss of generality, we assume that $\max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} = 1$. By $f_\infty > 0$, there exists r_2 satisfying $\gamma r_2 > r_1 > 0$ and

$$f(t,x) \geq (f_\infty - \varepsilon_1)x, \text{ for } x \geq \gamma r_2, \ 0 \leq t \leq 1. \tag{3.5}$$

For any $x \in \partial K_{r_2}$, by (3.2) and (3.5) we have

$$\begin{aligned}\|Tx\| \geq & \lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t) \\ \geq & \lambda\gamma \int_0^1 G(s,s)h(s)f(s,x(s))ds \\ \geq & \lambda\gamma^2(f_\infty - \varepsilon_1)\int_0^1 G(s,s)h(s)ds\|x\| \\ \geq & \|x\|. \end{aligned} \tag{3.6}$$

It follows from (3.4), (3.6) and Lemma 1.1 that the operator T has a fixed point in $\overline{K}_{r_2} \backslash K_{r_1}$, which is a positive solution of BVP (1.1). ☐

Corollary 3.2. *Suppose that* $(H_1) - (H_4)$ *hold and* $f^0 = 0, f_\infty = \infty$. *Then BVP* (1.1) *has at least one positive solution for* $\lambda > 0$.

Theorem 3.3. *Suppose that* $(H_1) - (H_4)$ *hold. In addition, we assume that* $f^\infty < +\infty, f_0 > 0$ *and*

$$\gamma^2 f_0 \max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} > (1 + A\psi(1) + B\phi(0))f^\infty.$$

Then BVP (1.1) *has at least one positive solution for any*

$$\frac{1}{\gamma^2 L f_0 \max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\}} < \lambda < \frac{1}{(1 + A\psi(1) + B\phi(0))Lf^\infty}, \tag{3.7}$$

where γ *is defined in Remark 2.4 and* A, B, L *are defined by (2.14), (2.15) and (2.16), respectively.*

Proof. Let λ satisfies (3.7), we choose $\varepsilon_2 > 0$ such that $f_0 - \varepsilon_2 > 0$ and

$$\frac{1}{\gamma^2 L(f_0 - \varepsilon_2)\max\{1, \gamma A\psi(1), \gamma B\phi(0)\}} \le \lambda \le \frac{1}{(1 + A\psi(1) + B\phi(0))L(f^\infty + \varepsilon_2)}. \tag{3.8}$$

Since $f^\infty < \infty$, there exists $r_1 > 0$ such that

$$f(t,x) \le (f^\infty + \varepsilon_2)x, \text{ for } 0 \le t \le 1, \ 0 < x \le \gamma r_1. \tag{3.9}$$

For any $x \in \partial K_{r_1}$, by (3.8) and (3.9) we obtain

$$\begin{aligned} \|Tx\| \le& \lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0) \\ \le& \lambda(f^\infty + \varepsilon_2)\int_0^1 G(s,s)h(s)ds\|x\| \\ &+ \lambda(f^\infty + \varepsilon_2)A\psi(1)\int_0^1 G(s,s)h(s)ds\|x\| \\ &+ \lambda(f^\infty + \varepsilon_2)B\phi(0)\int_0^1 G(s,s)h(s)ds\|x\| \\ =& \lambda(f^\infty + \varepsilon_2)L(1 + A\psi(1) + B\phi(0))\|x\| \\ \le& \|x\|. \end{aligned} \tag{3.10}$$

On the other hand, since $\max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} > 0$. Without loss of generality, we assume that $\max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} = \gamma A\psi(1)$. Since $f_0 > 0$, there exists $0 < r_2 < \gamma r_1$ such that

$$f(t,x) \ge (f_0 - \varepsilon_2)x, \text{ for } \ t \in [0,1], x \in [0, r_2]. \tag{3.11}$$

For any $x \in \partial K_{r_2}$, by (3.8) and (3.11) we have

$$\begin{aligned} \|Tx\| \ge& \lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t) \\ \ge& A(\lambda h(s)f(s,x(s)))\gamma\psi(1) \\ \ge& \lambda\gamma^3(f_0 - \varepsilon_2)\int_0^1 G(s,s)h(s)ds\|x\|A\psi(1) \\ \ge& \|x\|. \end{aligned} \tag{3.12}$$

It follows from (3.10), (3.12) and Lemma 1.1 that the operator T has a fixed point in $\overline{K}_{r_1}\backslash K_{r_2}$, which is a positive solution of BVP (1.1). $\square$

Corollary 3.4. *Suppose that* $(H_1)-(H_4)$ *hold and* $f_0=\infty, f^\infty=0$. *Then BVP* (1.1) *has at least one positive solution for* $\lambda>0$.

Theorem 3.5. *Suppose that* $(H_1)-(H_4)$ *hold. Besides, we assume that* $f^0=0$ *or* $f^\infty=0$. *Then there exists* $\lambda_0>0$ *such that BVP* (1.1) *has at least one positive solution for* $\lambda>\lambda_0$.

Proof. Choose $r_1>0$ and

$$\lambda_0=r_1\left(L\gamma(1+A\gamma\psi(1)+B\gamma\phi(0))\min_{(t,x)\in D}f(t,x)\right)^{-1},$$

where $D=\{(t,x):t\in[0,1],x\in[\gamma r_1,r_1]\}$. For $\lambda>\lambda_0, x\in\partial K_{r_1}$ we have

$$\begin{aligned}\|Tx\|\geq&\lambda\int_0^1 G(t,s)h(s)f(s,x(s))ds+A(F)\psi(t)+B(F)\phi(t)\\ \geq&\lambda\gamma\int_0^1 G(s,s)h(s)f(s,u(s))ds+\lambda\gamma\int_0^1 G(s,s)h(s)f(s,x(s))dsA\gamma\psi(1)\\ &+\lambda\gamma\int_0^1 G(s,s)h(s)f(s,x(s))dsB\gamma\phi(0)\\ >&\lambda_0L\gamma(1+A\gamma\psi(1)+B\gamma\phi(0))\min_{(t,x)\in D}f(t,x)\\ =&r_1=\|x\|.\end{aligned}\tag{3.13}$$

If $f^0=0$. Taking $r_2\in(0,\gamma r_1)$ such that $f(t,x)<\varepsilon x,$ for $t\in[0,1],x\in[0,r_2]$, where ε satisfying $\varepsilon\lambda L(1+A\psi(1)+B\phi(0))<1$. For $x\in\partial K_{r_2}$ we have

$$\begin{aligned}\|Tx\|\leq&\lambda\int_0^1 G(s,s)h(s)f(s,x(s))ds+A(F)\psi(1)+B(F)\phi(0)\\ \leq&\lambda L\varepsilon r_2+\lambda LA\psi(1)\varepsilon r_2+\lambda LB\phi(0)\varepsilon r_2\\ =&\lambda L(1+A\psi(1)+B\phi(0))\varepsilon r_2\\ <&r_2=\|x\|.\end{aligned}\tag{3.14}$$

(3.13), (3.14) and Lemma 1.1 yield that the operator T has a fixed point in $\overline{K}_{r_1}\backslash K_{r_2}$, which is a positive solution of BVP (1.1).

If $f^\infty=0$. Taking $r_3\in(\frac{r_1}{\gamma},\infty)$ such that $f(t,x)<\varepsilon x,$ for $t\in[0,1],x\in[\gamma r_3,\infty)$, where ε satisfying $\varepsilon\lambda L(1+A\psi(1)+B\phi(0))<1$. For $x\in\partial K_{r_3}$, we have $\gamma r_3 \le x(t) \le r_3, t \in [0,1]$. So

$$\begin{aligned}\|Tx\| \le&\lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0)\\ \le&\lambda L\varepsilon r_3 + \lambda LA\psi(1)\varepsilon r_3 + \lambda LB\phi(0)\varepsilon r_3\\ =&\lambda L(1 + A\psi(1) + B\phi(0))\varepsilon r_3\\ <&r_3 = \|x\|.\end{aligned} \tag{3.15}$$

By (3.13), (3.15) and Lemma 1.1 that the operator T has a fixed point in $\overline{K}_{r_1} \backslash K_{r_2}$, which is a positive solution of BVP (1.1). □

Theorem 3.6. *Suppose that* $(H_1)-(H_4)$ *hold. Besides, we assume that* $f_0 = 0$ *or* $f_\infty = 0$. *Then there exists* $\lambda_0 > 0$ *such that BVP* (1.1) *has at least one positive solution for* $0 < \lambda < \lambda_0$.

Proof. Choose $r_1 > 0$ and

$$\lambda_0 = r_1 \left(L(1 + A\psi(1) + B\phi(0)) \max_{(t,x)\in D} f(t,x) \right)^{-1},$$

where $D = \{(t,x) : t \in [0,1], x \in [0,r_1]\}$. For $0 < \lambda < \lambda_0, x \in \partial K_{r_1}$,

$$\begin{aligned}\|Tx\| \le&\lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0)\\ \le&\lambda L \max_{(t,x)\in D} f(t,x) + \lambda LA\psi(1) \max_{(t,x)\in D} f(t,x) + \lambda LB\phi(0) \max_{(t,x)\in D} f(t,x)\\ <&\lambda_0 L(1 + A\psi(1) + B\phi(0)) \max_{(t,x)\in D} f(t,x)\\ =&r_1 = \|x\|.\end{aligned} \tag{3.16}$$

On the other hand, since $\max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} > 0$. Without loss of generality, we assume that $\max\Big\{1, \gamma A\psi(1), \gamma B\phi(0)\Big\} = \gamma B\phi(0)$.

Case (i) If $f_0 = \infty$, there exists $r_2 \in (0, r_1)$ such that $f(t,x) \ge \xi x$, for $t \in [0,1], x \in [0,r_2]$, where ξ satisfying $\lambda\gamma^3\xi LB\phi(0) > 1$. For any $x \in \partial K_{r_2}$,

$$\begin{aligned}\|Tx\| \ge&\lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t)\\ \ge&B(\lambda h(s)f(s,x(s)))\gamma\phi(0)\\ \ge&\lambda\gamma^3\xi \int_0^1 G(s,s)h(s)dsB\phi(0)\|x\|\\ >&\|x\|.\end{aligned} \tag{3.17}$$

By (3.16), (3.17) and Lemma 1.1 that the operator T has a fixed point in $\overline{K}_{r_1} \backslash K_{r_2}$, which is a positive solution of BVP (1.1).

Case (ii) If $f_\infty = 0$, there exists $M > 0$ such that $f(t,x) \geq \xi x$, for $0 \leq t \leq 1, x \geq M$, where ξ satisfying $\lambda\xi\gamma L > 1$. Let $r_3 = max\{\frac{r_1}{\gamma}, \frac{M}{\gamma}\}$. For $x \in \partial K_{r_3}$, $\min_{t\in[0,1]} x(t) \geq \gamma\|x\| \geq M$. So

$$\begin{aligned}
\|Tx\| \geq &\lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t) \\
\geq &B(\lambda h(s)f(s,x(s)))\gamma\phi(0) \\
\geq &\lambda\gamma^3\xi \int_0^1 G(s,s)h(s)dsB\phi(0)\|x\| \\
> &\|x\|.
\end{aligned} \tag{3.18}$$

By (3.16), (3.18) and Lemma 1.1 that the operator T has a fixed point in $\overline{K}_{r_3} \backslash K_{r_1}$, which is a positive solution of BVP (1.1). □

Theorem 3.7. *Suppose that* $(H_1)-(H_4)$ *hold. Besides, we assume that* $f^0 = f^\infty = 0$. *Then there exists* $\lambda_0 > 0$ *such that BVP* (1.1) *has at least two positive solutions for* $\lambda > \lambda_0$.

Proof. Choose two numbers $0 < r_3 < \gamma r_4$. Let

$$\lambda_0 = r_4 \left(L\gamma(1 + A\gamma\psi(1) + B\gamma\phi(0)) \min_{(t,x)\in D} f(t,x) \right)^{-1},$$

where $D = \{(t,x) : t \in [0,1], x \in [\gamma r_3, r_3] \cup [\gamma r_4, r_4]\}$. Similar to (3.13), we have

$$\|Tx\| \geq \|x\|, \quad \text{for } \lambda > \lambda_0, \quad x \in \partial K_{r_3}, \tag{3.19}$$

and

$$\|Tx\| \geq \|x\|, \quad \text{for } \lambda > \lambda_0, \quad x \in \partial K_{r_4}. \tag{3.20}$$

From the proof of Theorem 3.5 we know, if $f^0 = f^\infty = 0$, choose $r_1 \in (0, \gamma r_3)$, $r_2 \in (\frac{r_4}{\gamma}, \infty)$, respectively. Then we have

$$\|Tx\| \leq \|x\|, \quad x \in \partial K_{r_1}, \tag{3.21}$$

and

$$\|Tx\| \leq \|x\|, \quad x \in \partial K_{r_2}. \tag{3.22}$$

By (3.19), (3.21) and (3.20), (3.22), T has at least one fixed point in $\overline{K}_{r_3} \backslash K_{r_1}$ and $\overline{K}_{r_2} \backslash K_{r_4}$, respectively. Therefore, BVP (1.1) has at least two positive solutions x_1, x_2 satisfying $r_1 \leq \|x_1\| \leq r_3 < r_4 \leq \|x_2\| \leq r_2$. □

Theorem 3.8. *Suppose that* $(H_1)-(H_4)$ *hold. Besides, we assume that* $f_0 = f_\infty = 0$. *Then there exists* $\lambda_0 > 0$ *such that BVP* (1.1) *has at least one positive solution for* $0 < \lambda < \lambda_0$.

Proof. Choose two numbers $0 < r_3 < r_4$. Let

$$\lambda_0 = r_3 \left(L(1 + A\psi(1) + B\phi(0)) \max_{(t,x)\in D} f(t,x) \right)^{-1},$$

where $D = \{(t,x) : t \in [0,1], x \in [0,r_4]\}$. Similar to (3.16), we have

$$\|Tx\| \leq \|x\|, \quad \text{for } 0 < \lambda < \lambda_0, \quad x \in \partial K_{r_3}, \tag{3.23}$$

and

$$\|Tx\| \leq \|x\|, \quad \text{for } 0 < \lambda < \lambda_0, \quad x \in \partial K_{r_4}. \tag{3.24}$$

From the proof of Theorem 3.6 we know, if $f_0 = f_\infty = \infty$, choose $r_1 \in (0, r_3)$, $r_2 \in (\frac{r_4}{\gamma}, \infty)$, respectively. Then we have

$$\|Tx\| \geq \|x\|, \quad x \in \partial K_{r_1}, \tag{3.25}$$

and

$$\|Tx\| \geq \|x\|, \quad x \in \partial K_{r_2}. \tag{3.26}$$

By (3.23), (3.25) and (3.24), (3.26), T has at least one fixed point in $\overline{K}_{r_3} \backslash K_{r_1}$ and $\overline{K}_{r_2} \backslash K_{r_4}$, respectively. Therefore BVP (1.1) has at least two positive solutions x_1, x_2 satisfying $r_1 \leq \|x_1\| \leq r_3 < r_4 \leq \|x_2\| \leq r_2$. □

Theorem 3.9. *Suppose that* $(H_1) - (H_4)$ *hold. Besides, we assume that* $f^0 < \infty$ *and* $f^\infty < \infty$ *Then there exists* $\lambda_0 > 0$ *such that BVP* (1.1) *has no positive solution for* $0 < \lambda < \lambda_0$.

Proof. Since $f^0 < \infty, f^\infty < \infty$, there exist positive numbers ρ_1, ρ_2, r_1, r_2 such that $r_1 < r_2$ and $f(t,x) \leq \rho_1 x, t \in [0,1], x \in [0, r_1]$ and $f(t,x) \leq \rho_2 x, t \in [0,1], x \in [r_2, \infty)$. Let $\rho_3 = \max\left\{\rho_1, \rho_2, \max_{(t,x)\in D} \frac{f(t,x)}{x}\right\}$, where $D = \{(t,x) : 0 \leq t \leq 1, r_1 \leq x \leq r_2\}$. Then $f(t,x) \leq \rho_3 x,\ t \in [0,1], x \in [0,\infty)$. Let $\lambda_0 = (L\rho_3(1 + A\psi(1) + B\phi(0)))^{-1} > 0$.

Suppose that $x(t)$ is a positive solution of BVP (1.1), i.e., $(Tx)(t) = x(t), t \in [0,1]$. For $0 < \lambda < \lambda_0$, it follows that

$$\begin{aligned}
\|x\| = \|Tx\| \leq& \lambda \int_0^1 G(s,s)h(s)f(s,x(s))ds + A(F)\psi(1) + B(F)\phi(0) \\
\leq& \lambda L\rho_3\|x\| + \lambda LA\psi(1)\rho_3\|x\| + \lambda LB\phi(0)\rho_3\|x\| \\
<& \lambda_0 L\rho_3(1 + A\psi(1) + B\phi(0))\|x\| \\
=& \|x\|,
\end{aligned}$$

which is a contradiction. □

Theorem 3.10. *Suppose that* $(H_1) - (H_4)$ *hold. Besides, we assume that* $f_0 > 0$ *and* $f_\infty > 0$. *Then there exists* $\lambda_0 > 0$ *such that BVP* (1.1) *has no positive solution for* $\lambda > \lambda_0$.

Proof. Since $f_0 > 0, f_\infty > 0$, there exist positive numbers $\sigma_1, \sigma_2, r_1, r_2$ such that $r_1 < r_2$ and $f(t,x) \geq \sigma_1 x, t \in [0,1], x \in [0, r_1]$ and $f(t,x) \geq \sigma_2 x, t \in [0,1], x \in [r_2, \infty)$. Let $\sigma_3 = \min\left\{\sigma_1, \sigma_2, \min_{(t,x)\in D} \frac{f(t,x)}{x}\right\}$, where $D = \{(t,x) : 0 \leq t \leq 1, r_1 \leq x \leq r_2\}$. Then $f(t,x) \geq \sigma_3 x, t \in [0,1], x \in [0,\infty)$. Let $\lambda_0 = \Big(L\gamma^2(1 + A\gamma\psi(1) + B\gamma\phi(0))\sigma_3\Big)^{-1}$.

Suppose that $x(t)$ is a positive solution of BVP (1.1), i.e., $(Tx)(t) = x(t), t \in [0,1]$. For $\lambda > \lambda_0$, it follows that

$$\begin{aligned}
\|x\| = \|Tx\| \geq & \lambda \int_0^1 G(t,s)h(s)f(s,x(s))ds + A(F)\psi(t) + B(F)\phi(t) \\
\geq & \lambda\gamma \int_0^1 G(s,s)h(s)f(s,x(s))ds \\
& + \lambda\gamma \int_0^1 G(s,s)h(s)f(s,x(s))dsA\gamma\psi(1) \\
& + \lambda\gamma \int_0^1 G(s,s)h(s)f(s,x(s))dsB\gamma\phi(0) \\
> & \lambda_0 L\gamma^2(1 + A\gamma\psi(1) + B\gamma\phi(0))\sigma_3\|x\| \\
= & \|x\|,
\end{aligned}$$

which is a contradiction. □

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