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SOME TOPOLOGICAL RESULTS AND A FIXED POINT THEOREM IN A-METRIC SPACES

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Abstract. In this paper, we prove some topological properties and a common fixed point type theorem for two self mappings on new generalized metric spaces, called $A-$metric spaces.

1. Introduction

The metric space forms an important environment for studying fixed point of single and multi-valued operators and the fixed point theory is important on applied sciences. Many authors have studied this important theory. In 1963, Gahler [3, 4] introduced the notion of a $2-$metric space. He claimed that $2-$metric space is a generalization of an ordinary metric space. On the other hand, Ha et al. [5] and Sharma [13] found some mathematical flaws in theses claims. It was demonstrated that the $2-$metric is not sequentially continuous in each of its arguments, whereas an ordinary metric satisfies this property. To overcome these problems, Dhage [2] introduced the concept of $D-$metric

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space as a generalization of a metric space and claimed that $D-$metric space defines a Haussdorff topology and $D-$metric is sequentially continuous with respect to all it's three variables. He proved some topological property and some fixed point results.

In 2003, Mustafa and Sims [6] introduced a new structure of generalized metric spaces which are called $G-$metric spaces and suggested an important generalization of a metric space. They studied some topological properties of $G-$metric space and afterwards some authors have obtained generalized fixed point theorems in the setup of $G-$metric space, see for examples [7, 14]. Next, Sedghi et al. [11] introduced a D^*-metric space and observed that some condition can be replaced with two axioms. So not every D^*-metric space needs to be a $G-$metric space. To overcome these difficulties, they introduced a new generalized metric space called $S-$metric space [10, 12], they proved that every a $S-$metric space is a generalization of a D^*-metric space and the $G-$metric space. A generalization of the $S-$metric space is called the $A-$metric space (see [1]).

It is our purpose in this paper to study topological properties of an $A-$metric space. We present here the concept of an A-metric space and some of its properties.

2. PRELIMINARIES

For $n \geq 2$, let X^n denotes the cartesian product $X \times X \times X .... \times X$.

Definition 2.1. Let X be a nonempty set. A function $A : X^n \longrightarrow [0, +\infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions holds :

(A1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0$,
(A2) $A(x_1, x_2, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = ... = x_{n-1} = x_n$,
(A3) For any $a \in X$,

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \leq\ & A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) \\ & + A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a) \\ & + A(x_3, x_3, x_3, ..., (x_3)_{n-1}, a) \\ & \ \vdots \\ & + A(x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) \\ & + A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a). \end{aligned}$$

The pair (X, A) is called an A-metric space.

Note that the A-metric space is an n-dimensional S-metric space (see [1]). Therefore the ordinary metric d and S-metric are special cases of an A-metric with $n = 2$ and $n = 3$, respectively.

Example 2.2. Let $X = \mathbb{R}$. Define a function $A : X^n \to [0, +\infty)$ by

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) &= |x_1 - x_2| + |x_1 - x_3| + ... + |x_1 - x_n| \\ &\quad + |x_2 - x_3| + |x_2 - x_4| + ... + |x_2 - x_n| \\ &\quad \vdots \\ &\quad + |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n| + |x_{n-1} - x_n| \\ &= \sum_{i=1}^{n} \sum_{i<j} \mid x_i - x_j \mid . \end{aligned}$$

Then $(\mathbb{R}, A)$ is an A-metric space.

Example 2.3. For a standard ordinary metric d on X, we define a function A_1 on X^n by

$$A_1(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i,j=1,(i<j)}^{n} d(x_i, x_j)$$

for all $x_i \in X, i = 1, 2, ..., n$. Then A_1 is an A-metric on X and is called the standard A-metric on X. Obviously the first two conditions are satisfied. To prove the third condition, let $x_i, a \in X, i = 1, 2, ..., n$, from the triangle inequality, it follows

$$\begin{aligned} A_1(x_1, x_2, ..., x_n) &= d(x_1, x_2) + \cdots + d(x_1, x_n) + d(x_2, x_3) + \cdots + d(x_2, x_n) \\ &\quad + \cdots + d(x_{n-2}, x_{n-1}) + d(x_{n-2}, x_n) + d(x_{n-1}, x_n) \\ &\leq d(x_1, a) + d(a, x_2) + \cdots + d(x_1, a) + d(a, x_n) \\ &\quad + \cdots + d(x_{n-1}, a) + d(a, x_n) \\ &\leq (n-1)d(x_1, a) + (n-1)d(x_2, a) + \cdots + (n-1)d(x_n, a) \\ &\leq A_1(x_1, ..., x_1, a) + A_1(x_2, ..., x_2, a) + \cdots + A_1(x_n, ..., x_n, a). \end{aligned}$$

Hence (X, A_1) is an A-metric space.

Lemma 2.4. ([1]) *Let (X, A) be an A-metric space. Then $A(x, x, x, ..., x, y) = A(y, y, y, ..., y, x)$ for all $x, y \in X$.*

Lemma 2.5. ([1]) *Let (X, A) be an A-metric space. Then, for all $x, y \in X$ we have*

$$A(x, x, x, \cdots, x, z) \leq (n-1)A(x, x, x, \cdots, x, y) + A(z, z, z, \cdots, z, y)$$

and

$$A(x,x,x,\cdots,x,z) \le (n-1)A(x,x,x,\cdots,x,y) + A(y,y,y,\cdots,y,z).$$

Lemma 2.6. ([1]) *Let (X,A) be an A-metric space. Then (X^2, D_A) is an A-metric space on $X \times X$ with the metric D_A given by*

$$D_A((x_1,y_1),(x_2,y_2),...,(x_n,y_n)) = A(x_1,x_2,x_3,...,x_n) + A(y_1,y_2,y_3,...,y_n)$$

for all $x_i, x_j \in X, i,j = 1,...,n$.

Theorem 2.7. *Let X_1, X_2 be two A-metric spaces with A-metrics ρ_1 and ρ_2 respectively. Then (X,ρ) is also an A-metric space, where $X = X_1 \times X_2$ and*

$$\rho((x_1,y_1),(x_2,y_2),...,(x_n,y_n)) = \max\{\rho_1(x_1,x_2,...,x_n), \rho_2(y_1,y_2,...,y_n)\}.$$

Proof. Obviously the conditions of nonnegativity and symmetry are satisfied. To prove the third condition, let $(x_1,y_1),(x_2,y_2),...,(x_n,y_n),(a_1,a_2) \in X = X_1 \times X_2$. Then we have

$$\begin{aligned}
&\rho((x_1,y_1),...,(x_{n-1},y_{n-1}),(x_n,y_n))\\
&= \max\{\rho_1(x_1,...,x_n), \rho_2(y_1,...,y_n)\}\\
&\le \max\{\rho_1(x_1,..,a_1) + ... + \rho_1(x_n,..,a_1), \rho_2(y_1,..,a_2) + ... + \rho_2(y_n,..,a_2)\}\\
&\le \max\{\rho_1(x_1,...,a_1), \rho_2(y_1,...,a_2)\} + ... + max\{\rho_1(x_n,...,a_1), \rho_2(y_n,...,a_2)\}\\
&\le \rho((x_1,y_1),...,(a_1,a_2)) + \rho((x_2,y_2),...,(a_1,a_2)\\
&\quad + ... + \rho((x_n,y_n),...,(a_1,a_2)).
\end{aligned}$$

Hence (X,ρ) is an A-metric space. □

The following useful properties of an A-metric are easily derived from the axioms.

Proposition 2.8. *Let (X,A) be an A-metric space. Then for any $x_1,x_2,...,x_n,a \in X$, we have*

(1) $A(x_1,x_2,...,x_n) \le \sum_{j=2}^{n} A(x_1,x_1,...,x_1,x_j)$,
(2) $A(x_1,x_2,...,x_2) \le (n-1)A(x_1,x_1,...,x_1,x_2)$,
(3) $A(x_1,x_2,...,x_n) \le \sum_{j=1}^{n} A(a,a,...,a,x_j)$.

Proof. Let $x_1,x_2,...,x_n,a \in X$. Then
(1)

$$A(x_1,x_2,...,x_n) \le \sum_{j=1}^{n} A(x_j,x_j,...,x_j,a)$$

by taking $a = x_1$, we obtain

$$A(x_1, x_2, ..., x_n) \leq \sum_{j=2}^{n} A(x_j, x_j, ..., x_j, x_1)$$

and by using lemma 2.5, we have

$$A(x_1, x_2, ..., x_n) \leq \sum_{j=2}^{n} A(x_1, x_1, ..., x_1, x_j)$$

also for almost $i = 1, 2, ..., n$, we obtain

$$A(x_1, x_2, ..., x_n) \leq \sum_{j=1, j \neq i}^{n} A(x_i, x_i, ..., x_i, x_j).$$

(2) Using the previous property and by taking $x_j = x_2, \forall j = 3, ..., n$, we obtain

$$A(x_1, x_2, ..., x_2) \leq (n-1) A(x_1, x_1, ..., x_1, x_2).$$

(3) It's obvious. By using the condition (A3) and Lemma 2.5, we obtain the result. □

Next the following lemma is needed to show the continuity of the A-metric function in one variable and in all its variables.

Lemma 2.9. *In an A-metric space X,*

(i)

$$|A(x_1, x_2, ..., x_{n-1}, a) - A(x_1, x_2, ..., x_{n-1}, b)| \leq \sum_{j=1}^{n-1} [A(a, a, ..., a, x_j) + A(b, b, ..., b, x_j)]$$

for all $x_1, ..., x_{n-1}, a, b \in X$,

(ii)

$$|A(x_1, x_2, ..., x_{n-1}, a) - A(y_1, y_2, ..., y_{n-1}, a| \leq \sum_{j=1}^{n-1} [A(a, a, ..., a, x_j) + A(a, a, ..., a, y_j)],$$

for all $x_1, ..., x_{n-1}, y_1, ..., y_{n-1}, a \in X$ and

(iii)

$$|A(x_1, x_2, ..., x_{n-1}, x_n) - A(y_1, y_2, ..., y_{n-1}, y_n)| \leq \sum_{j=1}^{n} [A(x_j, x_j, ..., x_j, y_1) + A(y_j, y_j, ..., y_j, x_1)],$$

for all $x_1, ..., x_{n-1}, x_n, y_1, ..., y_{n-1}, y_n \in X$.

Proof. To prove this Lemma we use the Proposition 2.8.
(i) Let $x_1, ..., x_{n-1}, a, b \in X$. Then by Proposition 2.8 (3) we have

$$\begin{aligned} & |A(x_1, x_2, ..., x_{n-1}, a) - A(x_1, x_2, ..., x_{n-1}, b)| \\ & \leq A(x_1, x_2, ..., x_{n-1}, a) + A(x_1, x_2, ..., x_{n-1}, b) \\ & = \sum_{j=1}^{n-1} [A(a, a, ..., a, x_j) + A(b, b, ..., b, x_j)] . \end{aligned}$$

(ii) Let $x_1, ..., x_{n-1}, y_1, y_2, ..., y_{n-1}, a \in X$. Then by using Proposition 2.8 (3) we obtain

$$\begin{aligned} & |A(x_1, x_2, ..., x_{n-1}, a) - A(y_1, y_2, ..., y_{n-1}, a)| \\ & \leq A(x_1, x_2, ..., x_{n-1}, a) + A(y_1, y_2, ..., y_{n-1}, a) \\ & \leq \sum_{j=1}^{n-1} [A(a, a, ..., a, x_j) + A(a, a, ..., a, y_j)] . \end{aligned}$$

(iii) Let $x_1, ..., x_{n-1}, x_n, y_1, y_2, ..., y_{n-1}, y_n, a, b \in X$. Then by condition (A3) we have

$$\begin{aligned} & |A(x_1, x_2, ..., x_{n-1}, x_n) - A(y_1, y_2, ..., y_{n-1}, y_n)| \\ & \leq A(x_1, x_2, ..., x_{n-1}, x_n) + A(y_1, y_2, ..., y_{n-1}, y_n) \\ & \leq \sum_{j=1}^{n} [A(x_j, x_j, ..., x_j, a) + A(y_j, y_j, ..., y_j, b)] \end{aligned}$$

Take $a = y_1, b = x_1$, then we obtain the result. □

3. The A-metric topology

Definition 3.1. Given a point x_0 in an A-metric space (X, A) and a positive real number r, the set

$$B(x_0, r) = \{y \in X : A(y, y, ..., y, x_0) < r\}$$

is called an open ball centered at x_0 with radius r.
The set

$$\overline{B(x_0, r)} = \{y \in X : A(y, y, ..., y, x_0) \leq r\}$$

is called a closed ball centered at x_0 with radius r.

Let X be an A-metric space with A-metric A. Then the diameter $\delta(X)$ of X is defined by

$$\delta(X) = \sup \{A(x, x, x, ..., x, y) : x, y \in X\} .$$

Definition 3.2. The A-metric space (X, A) is said to be bounded if there exists a constant $r > 0$ such that $A(x, x, ..., x, y) \leq r$ for all $x, y \in X$. Otherwise, X is unbounded.

Theorem 3.3. *Let (X_1, ρ_1) and (X_2, ρ_2) be two bounded A-metric spaces with bounds M_1 and M_2, respectively. Then the A-metric space (X, ρ) is bounded with bound $M = \max\{M_1, M_2\}$, where $X = X_1 \times X_2$ and ρ is defined as in Theorem 2.7.*

Proof. Since (X_1, ρ_1) and (X_2, ρ_2) are bounded, we have

$$\rho_1(x_1, x_1, ..., x_1, x_2) \leq M_1 \quad \text{for all } x_1, x_2 \in X_1,$$

$$\rho_2(y_1, y_1, ..., y_1, y_2) \leq M_2 \quad \text{for all } y_1, y_2 \in X_2.$$

By definition of ρ, we obtain

$$\begin{aligned}\rho((x_1, y_1), ..., (x_1, y_1), (x_2, y_2)) &= \max\{\rho_1(x_1, ..., x_1, x_2), \rho_2(y_1, ..., y_1, y_2)\} \\ &\leq \max\{M_1, M_2\} = M\end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in X = X_1 \times X_2$. This completes the proof. □

Definition 3.4. Let (X, A) be an A-metric space. A subset Ω of X is said to be an open set if for each $x \in \Omega$ there exists an $r > 0$ such that $B(x, r) \subset \Omega$.

Remark 3.5. The open sets so described are those of a topology on X called A-metric topology.

Theorem 3.6. *The open sets of an A-metric space X are exactly the union of open balls.*

Proof. First, each open ball is an open set in X ([1]). Then, any union of open balls is open and, if Ω is an open set, for all $x \in \Omega$, there exists an $r_x > 0$ such that $B(x, r_x) \subset \Omega$ from where $\Omega \subset \bigcup_{x\in\Omega} B(x, r_x) \subset \Omega$ and we obtain equality $\Omega = \bigcup_{x\in\Omega} B(x, r_x)$. □

Theorem 3.7. *The A-metric function $A(x_1, x_2, ..., x_n)$ is continuous in all its variables.*

Proof. Let $\epsilon > 0$ be given and let $x_1^0, ..., x_{n-1}^0, x_n^0 \in X$. Then for $x_1, x_2, ..., x_n \in X$ such that

$$x_j \in \bigcap_{i=1}^{n} B(x_i^0, \frac{\epsilon}{2n})$$

for $j = 1, 2, ..., n$ and using lemma 2.9 (iii), we obtain

$$\begin{aligned} & |A(x_1, x_2, ..., x_{n-1}, x_n) - A(x_1^0, x_2^0, ..., x_{n-1}^0, x_n^0)| \\ & \leq \sum_{j=1}^{n} \left[A(x_j, ..., x_j, x_1^0) + A(x_j^0, x_j^0, ..., x_j^0, x_1) \right] \\ & < \sum_{j=1}^{n} (\frac{\epsilon}{2n} + \frac{\epsilon}{2n}) = \epsilon. \end{aligned}$$

This completes the proof. □

We denote also another important problem that is the A-metrizability of the topological space which is satisfied under a condition given in the following theorem.

Theorem 3.8. *If the topological space X is metrizable then it is A-metrizable.*

Proof. Suppose that X is a metrizable space and denote the ordinary metric on X by d, where d induces the topology of X. Using an A-metric A_1 on X defined as in example 2.3. This A-metric generate the same topology on that of X. We deduce that X is A-metrizable. □

Theorem 3.9. (Kolmogorov space) *An A-metric space X is a T_0-space.*

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \neq y_0$. Suppose that $A(y_0, y_0, ..., y_0, x_0) = r > 0$, then $y_0 \notin B(x_0, r)$, where $B(x_0, r)$ is an open ball in X defined by

$$B(x_0, r) = \{y \in X : A(y, y, ..., y, x_0) < r\}.$$

Hence X is a T_0-space. □

Theorem 3.10. (Frechet space) *An A-metric space X is T_1-space.*

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \neq y_0$. Suppose that

$$A(y_0, y_0, ..., y_0, x_0) = A(x_0, x_0, ..., x_0, y_0) = r_1 > 0.$$

Then $y_0 \notin B(x_0, r_1)$, where $B(x_0, r_1) = \{y \in X : A(y, y, ..., y, x_0) < r_1\}$. Similarly, $x_0 \notin B(y_0, r_1)$, where $B(y_0, r_1) = \{x \in X : A(x, x, ..., x, y_0) < r_1\}$. Since $B(x_0, r_1)$ and $B(y_0, r_1)$ are two open balls in X containing x_0 and y_0, respectively, we deduce that X is T_1-space. □

Theorem 3.11. (Haussdorff space) *An A-metric space X is T_2-space.*

Proof. Let $x_0, y_0 \in X$ such that $x_0 \neq y_0$. Consider two sets B_1^* and B_2^* as follows :

$$B_1^* = \{x \in X : A(x,x,...,x,x_0) < A(x,x,...,x,y_0)\}$$

and

$$B_2^* = \{x \in X : A(x,x,...,x,y_0) < A(x,x,...,x,x_0)\}$$

It is clear that B_1^* and B_2^* contains x_0 and y_0, respectively. To prove that $B_1^* \cap B_2^* = \varnothing$, suppose there exists $z \in B_1^* \cap B_2^*$, then

$$A(z,z,...,z,x_0) < A(z,z,...,z,y_0)$$

and

$$A(z,z,...,z,y_0) < A(z,z,...,z,x_0)$$

which is absurd, because there are two contradictory statements. Then $B_1^* \cap B_2^* = \varnothing$. It remains to prove that B_1^* and B_2^* are open sets. For this, let $x \in B_1^*$. Then we have

$$A(x,x,...,x,x_0) < A(x,x,...,x,y_0)$$

and set $s = \dfrac{A(x,x,...,x,y_0) - A(x,x,...,x,x_0)}{2(n-1)} > 0$. It is clear that $B(x,s) \subset B_1^*$, because for $z \in B(x,s)$, we have

$$A(z,z,...,z,x) < \frac{A(x,x,...,x,y_0) - A(x,x,...,x,x_0)}{2(n-1)} \tag{3.1}$$

therefore $2(n-1)A(z,z,...,z,x) < A(x,x,...,x,y_0) - A(x,x,...,x,x_0)$, which implies that

$$(n-1)A(z,...,z,x) + A(x,...,x,x_0) < A(x,...,x,y_0) - (n-1)A(z,...,z,x) \tag{3.2}$$

Now from (3.2), Lemma 2.4 and condition (A3), we obtain

$$\begin{aligned} A(z,...,z,x_0) &\leq (n-1)A(z,...,z,x) + A(x_0,...,x_0,x) \\ &< A(x,...,x,y_0) - (n-1)A(z,...,z,x) \\ &\leq (n-1)A(x,...,x,z) + A(z,...,z,y_0) - (n-1)A(z,...,z,x) \\ &= A(z,...,z,y_0). \end{aligned}$$

So

$$A(z,...,z,x_0) < A(z,...,z,y_0),$$

which is the desired result. This proves that B_1^* is an open set contains x_0. Similarly, we can show that B_2^* is also an open set contains y_0. Hence, any A-metric space is T_2-space. □

3.1. **Completeness of A-metric spaces.**

Definition 3.12. Let (X, A) be an A-metric space. A sequence $\{x_k\}$ in X is said to converge to a point $x \in X$, if $A(x_k, x_k, ..., x_k, x) \longrightarrow 0$ as $k \longrightarrow +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$ we have $A(x_k, x_k, ..., x_k, x) \leq \epsilon$ and we write $\lim_{k \to +\infty} x_k = x$.

Lemma 3.13. ([1]) *Let (X, A) be an A-metric space. If the sequence $\{x_k\}$ in X converges to a point x, then x is unique.*

Definition 3.14. Let (X, A) be an A-metric space. A sequence $\{x_k\}$ in X is called a Cauchy sequence if $A(x_k, x_k, ..., x_k, x_m) \longrightarrow 0$ as $k, m \longrightarrow +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $k, m \geq n_0$ we have $A(x_k, x_k, ..., x_k, x_m) \leq \epsilon$.

Lemma 3.15. ([1]) *Every convergent sequence in A-metric space is a Cauchy sequence. The converse does not hold in general.*

Definition 3.16. The A-metric space (X, A) is said to be complete if every Cauchy sequence in X is convergent.

Lemma 3.17. ([1]) *Let (X, A) be an A-metric space. Then the function $A(x, x, ..., x, y)$ is continuous if there exist $\{x_k\}$ and $\{y_k\}$such that $\lim_{k \to \infty} x_k = x$ and $\lim_{k \to \infty} y_k = y$ then $\lim_{k \to \infty} A(x_k, x_k, ..., x_k, y_k) = A(x, x, ..., x, y)$.*

The following lemma shows that every metric space is an A-metric space.

Lemma 3.18. *Let (X, d) be a metric space. Then we have*

(1) *$A_d(x_1, x_2, ..., x_n) = \sum_{i=1}^{n-1} d(x_i, x_n)$ for all $x_1, ..., x_n \in X$ is an A-metric on X.*

(2) *$x_n \longrightarrow x$ in (X, d) if and only if $x_n \longrightarrow x$ in (X, A_d).*

(3) *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, A_d).*

(4) *(X, d) is complete if and only if (X, A_d) is complete.*

Proof. (1) Obviously, the first and the second conditions are satisfied. For the third condition we have:

$$\begin{aligned}
A_d(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n-1} d(x_i, x_n) \\
&\leq \sum_{i=1}^{n-1} [d(x_i, a) + d(a, x_n)] \\
&= \sum_{i=1}^{n-1} d(x_i, a) + \sum_{i=1}^{n-1} d(a, x_n) \\
&\leq \sum_{i=1}^{n-1} [d(x_i, a) + ... + d(x_i, a)] + \sum_{i=1}^{n-1} d(a, x_n) \\
&= \sum_{i=1}^{n-1} A_d(x_i, x_i, ..., x_i, a) + A_d(x_n, x_n, ..., x_n, a) \\
&= \sum_{i=1}^{n} A_d(x_i, x_i, ..., x_i, a).
\end{aligned}$$

(2) We have

$$\begin{aligned}
x_n \longrightarrow x \text{ in } (X, d) &\iff d(x_n, x) \longrightarrow 0 \\
&\iff d(x_n, x) + ... + d(x_n, x) \longrightarrow 0 \text{ in } (X, d) \\
&\iff A_d(x_n, .., x_n, x) \longrightarrow 0
\end{aligned}$$

where $A_d(x_n, x_n, ..., x_n, x) = (n-1)d(x_n, x)$, that is $x_n \longrightarrow x$ in (X, A_d).
(3) We have

$$\begin{aligned}
\{x_n\} \text{ is Cauchy sequence in } (X, d) &\iff d(x_n, x_m) \longrightarrow 0 \text{ as } n, m \longrightarrow +\infty \\
&\iff A_d(x_n, .., x_m) = (n-1)d(x_n, x_m) \\
&\qquad \rightarrow 0
\end{aligned}$$

as $n, m \longrightarrow +\infty$, that is $\{x_n\}$ is Cauchy in (X, A_d).
(4) It is a consequence of (2) and (3). □

The following example proves that the inverse implication of the precedent lemma does not hold.

Example 3.19. Let $X = \mathbb{R}$ and

$$A(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sum_{i<j}^{n} |x_i - x_j|$$

for all $x_1, x_2, ..., x_n \in X$. A is an A-metric (see[1], p7). Suppose that there exists a metric d with $A(x_1, x_2, ..., x_n) = \sum_{i=1}^{n-1} d(x_i, x_n)$ for all $x_1, ..., x_n \in X$. Then $A(x_i, x_i, ..., x_i, x_n) = d(x_i, x_n) + d(x_i, x_n) + ... + d(x_i, x_n)$ and so

$$d(x_i, x_n) = \frac{1}{n-1} A(x_i, x_i, ..., x_i, x_n).$$

We have also

$$\begin{aligned}\sum_{i=1}^{n-1} d(x_i, x_n) &= \frac{1}{n-1} \sum_{i=1}^{n-1} A(x_i, x_i, ..., x_i, x_n) \\ &= \frac{1}{n-1} A(x_1, ..., x_1, x_n) + ... + \frac{1}{n-1} A(x_{n-1}, ..., x_{n-1}, x_n) \\ &= \frac{1}{n-1}|x_1 - x_n| + \frac{1}{n-1}|x_2 - x_n + ... + \frac{1}{n-1}|x_{n-1} - x_n|\end{aligned}$$

Clearly, $A(x_1, x_2, ..., x_n) \neq \sum_{i=1}^{n} d(x_i, x_n)$, and this is a contradiction.

Next we show that the A-metric space is normal. Let C be a closed subset of an A-metric space X. We define a function $A(x, x, x, ..., x, C)$ by

$$A(x, x, ..., x, C) = \inf \{A(x, x, ..., x, c) : c \in C\}.$$

Then it is clear that

$$A(x, x, ..., x, C) = 0 \Longleftrightarrow x \in C.$$

We need the following lemma in the sequel.

Lemma 3.20. *$x \longmapsto A(x, x, ..., x, C)$ is a continuous function in an A-metric space X.*

Proof. Let $c \in C$. Then by the condition (A3), Lemma 2.4 and Lemma 2.5 we have

$$A(x, x, ..., x, c) \leq (n-1)A(x, x, ..., x, y) + A(y, y, ..., y, c) \tag{3.3}$$

and

$$A(y, y, ..., y, c) \leq (n-1)A(y, y, ..., y, x) + A(x, x, ..., x, c). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$A(x, x, ..., x, C) - A(y, y, ..., y, C) \leq (n-1)A(x, x, ..., x, y)$$

and

$$A(y, y, ..., y, C) - A(x, x, ..., x, C) \leq (n-1)A(y, y, ..., y, x).$$

And then we obtain

$$|A(x, x, ..., C) - A(y, y, ..., y, C)| \leq (n-1)A(x, x, ..., x, y).$$

Therefore, if $\{x_i\}$ is a sequence such that $x_i \longrightarrow y$ and

$$|A(x_i, x_i, ..., x_i, C) - A(y, y, ..., y, C)| \leq (n-1)A(x_i, x_i, ..., x_i, y),$$

then we obtain $A(x_i, x_i, ..., x_i, C) \longrightarrow A(y, y, ..., y, C)$. This shows that $x \longrightarrow A(x, x, ..., x, C)$ is a continuous function on X. □

Theorem 3.21. *Let C and B be two closed subsets of an A-metric space X such that $C \cap B = \varnothing$. Then there exists a continuous real function $f : X \longrightarrow R$ such that $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in B$.*

Proof. Define a function $f : X \longrightarrow R$ by

$$f(x) = \frac{A(x, x, ..., x, C)}{A(x, x, ..., x, C) + A(x, x, ..., x, B)}.$$

Since the function $x \longmapsto A(x, x, ..., x, C)$ is continuous and denominator is continuous and positive, the function f is continuous on X and satisfied $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in B$. □

Theorem 3.22. *An A-metric space X is normal.*

Proof. Let A and B be two closed and disjoint subsets of X. Using the Theorem 3.21, there exists a continuous real function $f : X \longrightarrow R$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. Define the open sets U and V in X by

$$U = \left\{ x \in X / f(x) < \frac{3}{4} \right\}$$

and

$$V = \left\{ x \in X / f(x) > \frac{3}{4} \right\}$$

It is clear that, $A \subset U$ and $B \subset V$ and $U \cap V = \varnothing$. Hence, X is normal. □

Theorem 3.23. *If a Cauchy sequence in an A-metric space contains a convergent subsequence, then the sequence is convergent.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in an A-metric space X. Then, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $k, m \geq n_0$ we have

$$A(x_k, x_k, ..., x_k, x_m) < \frac{\epsilon}{2(n-1)}.$$

Since the subsequence $\left\{x_{\varphi(n)}\right\}$ of $\{x_n\}$ converging to a point $x \in X$, and also, at the same $\epsilon > 0$ is associated r_0 such that

$$\forall r \geq r_0, A(x_{\varphi(r)}, x_{\varphi(r)}, ..., x_{\varphi(r)}, x) < \frac{\epsilon}{2}.$$

As φ is strictly increasing, there exist $r_1 \geq r_0$ such that $\varphi(r_1) \geq n_0$, then for all $k \geq n_0$,

$$\begin{aligned} A(x_k, x_k, ..., x_k, x) \leq (n-1)&A(x_k, x_k, ..., x_k, x_{\varphi(r_1)}) \\ &+ A(x_{\varphi(r_1)}, x_{\varphi(r_1)}, ..., x_{\varphi(r_1)}, x) \end{aligned}$$

$$\leq \frac{(n-1)\epsilon}{2(n-1)} + \frac{\epsilon}{2} = \epsilon.$$

Finaly, for all $\epsilon > 0$ there exist $n_0 \in N$ such that for all $k \geq n_0$ we have $A(x_k, x_k, ..., x_k, x) < \epsilon$. □

Theorem 3.24. *Let X_1, X_2 be two A-metric spaces with A-metrics ρ_1 and ρ_2, respectively. Define A on $X_1 \times X_2$ by*

$$A((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) = \max\{\rho_1(x_1, x_2, ..., x_n), \rho_2(y_1, y_2, ..., y_n)\}$$

for $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in X_1 \times X_2$. Then (X, A) is complete if and only if (X_1, ρ_1) and (X_2, ρ_2) are complete.

Proof. From the definition of completeness, we can prove this theorem. □

Definition 3.25. A sequence $\{F_n\}$ of closed sets in an A-metric space X is said to be nested if

$$F_1 \supset F_2 \supset ... \supset F_n \supset ....$$

Theorem 3.26. (Intersection theorem) *Let X be an A-metric space and let $\{F_n\}$ be a nested sequence of nonempty subsets of X such that $\delta(F_n) \longrightarrow 0$ as $n \longrightarrow \infty$. If X is complete, then $\bigcap_{i=1}^{\infty} F_n$ is a singleton.*

Proof. Let X be complete. For each $n \in N$, there exists $x_n \in F_n$ which is nonempty. Then, for all $m \geq n$ we have $x_m \in F_m \subset F_n$. So, for all $m \geq n$ and $k \geq n$ we get $A(x_m, x_m, ..., x_m, x_k) \leq \delta(F_n)$ such that $\delta(F_n) \longrightarrow 0$ as $n \to \infty$ that is to say, for all $\epsilon > 0$ there exist $n_0 \in N$ such that for all $n \geq n_0$ we have $\delta(F_n) \leq \epsilon$, a fortiori, we will have for all $m \geq n_0$ and $k \geq n_0$ we get $A(x_m, x_m, ..., x_m, x_k) \leq \epsilon$. Then $\{x_n\}$ is a Cauchy sequence in a complete space X and then $\{x_n\}$ converges. Let x be the limit of $\{x_n\}$. As for all $m \geq n$ we have $x_m \in F_n$ and then $x \in \overline{F_n} = F_n$ (F_n closed), from where $x \in \cap_{n \in N} F_n$, which is nonempty. Finally, if $y \in \cap_{n \in N} F_n$ we get

$$A(x, x, ..., x, y) \leq \delta(F_n)$$

for all $n \in N$, so if n tends to infinity, we obtain

$$A(x, x, ..., x, y) \leq 0.$$

It follows from $A(x, x, ..., x, y) = 0$ that $x = y$. Therefore the intersection is a singleton. □

3.2. **Compactness in A-metric spaces.**

Definition 3.27. Let (X, A) be an A-metric space, and let $\epsilon > 0$ be given. Then a set $\Omega \subseteq X$ is called an ϵ-net of (X, A) if given any x in X there is at least one point a in Ω such that $x \in B(a, \epsilon)$. If the set Ω is finite then Ω is called a finite ϵ-net of (X, A). Note that if Ω is an ϵ-net then $\Omega = \bigcup_{a \in A} B(a, \epsilon)$.

Definition 3.28. An A-metric space (X, A) is called A-totally bounded if for every $\epsilon > 0$ there exists a finite ϵ-net.

Definition 3.29. An A-metric space (X, A) is said to be a compact A-metric space if it is A-complete and A-totally bounded.

Theorem 3.30. *Every sequentially compact A-metric space X is A-totally bounded.*

Proof. If X is not A-totally bounded, there exists $\epsilon > 0$ such that X has no ϵ-net. Let $x_0 \in X$. Then there must exists a point $x_1 \in X$, distinct from x_0, such that $A(x_1, x_1, ..., x_1, x_0) \geq \epsilon$, for otherwise, $\{x_0\}$ would be an $\epsilon - net$ for X. In the same way, there exists a point $x_2 \in X$, distinct from x_0 and x_1 such that $A(x_2, x_2, ..., x_2, x_1) \geq \epsilon$, for otherwise $\{x_0, x_1\}$ would be an ϵ-net for X. Continuing this process, we obtain a sequence $\{x_0, x_1, ...\}$ with the property $A(x_j, x_j, ..., x_j, x_i) \geq \epsilon$, $i \neq j$. Then $\{x_n\}$ cannot contain any convergent sequence. Hence X is not sequentially compact. $\square$

Below we give a theorem in an A-metric space without proof, since its proof is similar to ordinary metric space case with appropriate modifications.

Theorem 3.31. *For an A-metric space (X, A), the following are equivalent:*

(1) *X is compact,*
(2) *X is countably compact,*
(3) *X has Bolzano-Weierstrass property,*
(4) *X is sequentially compact.*

Before stating our main result, we recall the following definitions which will be useful later.

Definition 3.32. ([8]) A pair of maps f and g is called weakly compatible if they commute at coincidence points.

Example 3.33. Let $(X = [0, 1], d)$ be a metric space with $d(x, y) = |x - y|$. Define $f, g : [0, 1] \to [0, 1]$ by

$$f(x) = x, g(x) = 1 - x \quad \text{if } x \in \left[0, \frac{1}{2}\right],$$

and

$$f(x) = g(x) = \frac{1}{2} \quad \text{if } x \in \left[\frac{1}{2}, 1\right].$$

Then, for any $x \in \left[\frac{1}{2}, 1\right]$, $fg(x) = gf(x)$, therefore f, g are weakly compatible maps on $[0, 1]$.

Example 3.34. Let $X = R$. Define $f, g : R \to R$ by $f(x) = x^2 - 1, x \in R$ and $g(x) = x - 1, x \in R$. 0 a nd 1 are two coincidence points for the maps f, g. We have $fg(1) = gf(1) = -1$, but $fg(0) = 0$ and $gf(0) = -2$. Hence f and g are not weakly compatible maps on R.

Now,we present the concept of weakly A-contractive for mapping $f : X \to X$ as follows:

Definition 3.35. Let (X, A) be an A-metric space. A mapping $f : X \to X$ is said to be weakly A-contractive type if for all $x_i \in X, i = 1, ..., n$, the following inequality holds :

$$\begin{aligned} A(fx_1, fx_2, ..., fx_n) \leq \frac{1}{2n} A(x_1, ..., x_1, fx_2) + \frac{1}{2n} A(x_2, ..., x_2, fx_3) \\ + ... + \frac{1}{2n} A(x_{n-1}, ..., x_{n-1}, fx_n) + \frac{1}{2n} A(x_n, ..., x_n, fx_1) \\ - \phi\left(A(x_1, ..., fx_2), ..., A(x_{n-1}, ..., fx_n), A(x_n, ..., fx_1)\right), \end{aligned}$$

where $\phi : [0, +\infty)^n \to [0, +\infty)$ is a continuous function with $\phi(\alpha_1, \alpha_2, ..., \alpha_n) = 0$ if and only if $\alpha_i = 0$ for all $i = 1, ..., n$.

The following definition of the altering distance function was introduced in ([9]).

Definition 3.36. The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied :

(1) ψ is continuous and increasing;
(2) $\psi(t) = 0$ if and only if $t = 0$.

4. Main Results

Let (X, A) be an A-metric space and $f, g : X \longrightarrow X$ be two mappings. We say that f is a generalized weakly contraction mapping (g.w.c.m.) with respect to g if for all $x, y \in X$, the following inequality holds:

$$\begin{aligned}&\psi\left(A(fx,...,fx,fy)\right)\\&\leq \psi\left(\frac{1}{2n}\left[(n-2)A(gx,...,fx)+A(gx,...,fy)+A(gy,...,fx)\right]\right)\\&\quad-\phi\left(A(gx,...,fx),...,A(gx,...,fx),A(gx,...,fy),A(gy,...,fx)\right),\end{aligned} \tag{4.1}$$

where

(1) ψ is an altering distance function;
(2) $\phi : [0, +\infty)^n \longrightarrow [0, +\infty)$ is a continuous function with $\phi(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$, for all $i = 1, ..., n$.

Theorem 4.1. *Let (X, A) be an A-metric space and $f, g : X \longrightarrow X$ be two mappings such that f is a g.w.c.m. with respect to g. Assume that*

(i) $f(X) \subset g(X)$,
(ii) $g(X)$ *is a complete subset of* (X, A),
(iii) f *and* g *are weakly compatible maps.*

Then f and g have a unique common fixed point.

Proof. By using the assumption (i), we can construct a sequence $\{x_m\}$ in X such that $gx_{m+1} = fx_m$, for any $m \in N$. If for some m, $gx_{m+1} = gx_m$, we obtain $gx_m = fx_m$ and then f and g have a common fixed point. Assume that $gx_{m+1} \neq gx_m$ for any $m \in N$. For $m \in N$ and by (4.1) and (A_3), we get

$$\begin{aligned}&\psi\left(A(gx_m,...,gx_m,gx_{m+1})\right)\\&=\psi\left(A(fx_{m-1},...,fx_{m-1},fx_m)\right)\\&\leq \psi\left(\frac{1}{2n}\left[(n-2)A(gx_{m-1},...,gx_m)+A(gx_{m-1},...,gx_{m+1})+A(gx_m,...,gx_m)\right]\right)\\&\quad-\phi\left(A(gx_{m-1},...,gx_m),..,A(gx_{m-1},..,gx_{m+1}),A(gx_m,...,gx_m)\right)\\&\leq \psi\left(\frac{1}{2n}\left[(2n-3)A(gx_{m-1},...,gx_{m-1},gx_m)+A(gx_{m+1},...,gx_{m+1},gx_m)\right]\right).\end{aligned}$$

Since ψ is increasing, by (4.2) and Lemma 2.5 we obtain

$$\begin{aligned}&A(gx_m,...,gx_{m+1})\\&\leq \frac{1}{2n}\left[(n-2)A(gx_{m-1},...,gx_m)+A(gx_{m-1},...,gx_{m+1})\right]\\&\leq \frac{1}{2n}\left[(2n-3)A(gx_{m-1},...,gx_m)+A(gx_m,...,gx_{m+1})\right].\end{aligned} \tag{4.2}$$

Then we have

$$\begin{aligned}(1-\frac{1}{2n})A(gx_m,...,gx_m,gx_{m+1}) &\leq \frac{2n-3}{2n}A(gx_{m-1},...,gx_{m-1},gx_m)\\ &\leq \frac{2n-1}{2n}A(gx_{m-1},...,gx_{m-1},gx_m),\end{aligned}$$

it implies that

$$A(gx_m,...,gx_m,gx_{m+1}) \leq A(gx_{m-1},...,gx_{m-1},gx_m)$$

for any $n \geq 1$. Therefore $\{A(gx_m,...,gx_m,gx_{m+1}), n \in N\}$ is a non-increasing sequence. Hence there exists $\varrho \geq 0$ such that

$$\lim_{m\to\infty} A(gx_m,...,gx_m,gx_{m+1}) = \varrho. \tag{4.3}$$

Letting $m \to +\infty$ in (4.2), we obtain

$$\lim_{m\to\infty} A(gx_{m-1},...,gx_{m-1},gx_{m+1}) = n\varrho. \tag{4.4}$$

We also have from (4.2)

$$\begin{aligned}&\psi\left(A(gx_m,...,gx_{m+1})\right)\\ &\leq \psi\left(\frac{1}{2n}\left[(n-2)A(gx_{m-1},...,gx_m)+A(gx_{m-1},...,gx_{m+1})\right]\right)\\ &\quad -\phi\left(A(gx_{m-1},...,gx_m),...,A(gx_{m-1},...,gx_m),A(gx_{m-1},...,gx_{m+1}),0\right).\end{aligned}$$

Letting $m \to +\infty$ and using (4.3), (4.4) and the continuity of ψ and ϕ, we get

$$\psi(\varrho) \leq \psi(\varrho) - \phi(\varrho,...,\varrho,n\varrho,0),$$

hence $\phi(\varrho,\varrho,...,\varrho,n\varrho,0) = 0$. By a property of ϕ, we deduce that $\varrho = 0$, that is

$$\lim_{m\to\infty} A(gx_m,...,gx_m,gx_{m+1}) = 0. \tag{4.5}$$

To prove that $\{gx_m\}$ is a Cauchy sequence, we proceed as follows : Suppose $\{gx_m\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ such that, for all $i \in N$, there exists two subsequences $\{gx_{p(i)}\}$ and $\{gx_{q(i)}\}$ of $\{gx_m\}$ such that $q(i)$ is the smallest index for which $q(i) > p(i) > i$,

$$A(gx_{p(i)},...,gx_{p(i)},gx_{q(i)}) \geq \epsilon. \tag{4.6}$$

Therefore we have

$$A(gx_{p(i)},...,gx_{p(i)},gx_{q(i)-1}) < \epsilon. \tag{4.7}$$

Using (4.6), (4.7) and the condition (A_3), we have

$$
\begin{aligned}
\epsilon &\leq A(gx_{p(i)},...,gx_{p(i)},gx_{q(i)})\\
&\leq (n-1)A(gx_{p(i)},...,gx_{p(i)},gx_{p(i)-1}) + A(gx_{q(i)},...,gx_{q(i)},gx_{p(i)-1})\\
&\leq (n-1)A(gx_{p(i)},...,gx_{p(i)},gx_{p(i)-1}) + (n-1)A(gx_{q(i)},...,gx_{q(i)},gx_{q(i)-1})\\
&\quad + A(gx_{p(i)-1},...,gx_{p(i)-1},gx_{q(i)-1})\\
&\leq (n-1)A(gx_{p(i)},...,gx_{p(i)},gx_{p(i)-1}) + (n-1)A(gx_{q(i)},...,gx_{q(i)},gx_{q(i)-1})\\
&\quad + (n-1)A(gx_{p(i)-1},...,gx_{p(i)-1},gx_{p(i)}) + A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{p(i)})\\
&< (n-1)A(gx_{p(i)},...,gx_{p(i)},gx_{p(i)-1}) + (n-1)A(gx_{q(i)},...,gx_{q(i)},gx_{q(i)-1})\\
&\quad + (n-1)A(gx_{p(i)-1},...,gx_{p(i)-1},gx_{p(i)}) + \epsilon.
\end{aligned}
$$

Letting $i \to +\infty$ in the precedent inequalities and using (4.5), we obtain

$$
\begin{aligned}
&\lim_{i\to+\infty} A(gx_{p(i)},...,gx_{p(i)},gx_{q(i)})\\
&= \lim_{i\to+\infty} A(gx_{q(i)},...,gx_{q(i)},gx_{p(i)-1})\\
&= \lim_{i\to+\infty} A(gx_{p(i)-1},...,gx_{p(i)-1},gx_{q(i)-1})\\
&= \epsilon.
\end{aligned}
\tag{4.8}
$$

Now, by (4.1) we have

$$
\begin{aligned}
&\psi\left(A(gx_{q(i)},...,gx_{q(i)},gx_{p(i)})\right)\\
&= \psi\left(A(fx_{q(i)-1},...,fx_{q(i)-1},fx_{p(i)-1})\right)\\
&\leq \psi\left(\frac{n-2}{2n}A(gx_{q(i)-1},...,fx_{q(i)-1}) + \frac{1}{2n}A(gx_{q(i)-1},...,fx_{p(i)-1})\right.\\
&\quad \left.+\frac{1}{2n}A(gx_{p(i)-1},...,fx_{q(i)-1})\right) - \phi\left(A(gx_{q(i)-1},...,fx_{q(i)-1}),\ ...,\right.\\
&\quad \left.A(gx_{q(i)-1},...,fx_{p(i)-1}), A(gx_{p(i)-1},...,fx_{q(i)-1})\right)\\
&= \psi\Bigg(\frac{1}{2n}\left[(n-2)A(gx_{q(i)-1},..,gx_{q(i)}) + A(gx_{q(i)-1},..,gx_{p(i)})\right.\\
&\quad \left.+A(gx_{p(i)-1},..,gx_{q(i)})\right]\Bigg) - \phi\left(A(gx_{q(i)-1},...,gx_{q(i)}),\ ...,\right.\\
&\quad \left.A(gx_{q(i)-1},...,gx_{p(i)}), A(gx_{p(i)-1},...,gx_{q(i)})\right)\\
&\leq \psi\Bigg(\frac{1}{2n}\left[(n-2)A(gx_{q(i)-1},..,gx_{q(i)}) + A(gx_{q(i)-1},..,gx_{p(i)})\right.\\
&\quad \left.+A(gx_{p(i)-1},..,gx_{q(i)})\right]\Bigg).
\end{aligned}
\tag{4.9}
$$

Since ψ is increasing and by (A_3), we obtain

$$\begin{aligned}
A(gx_{q(i)},...,gx_{q(i)},gx_{p(i)}) &\leq \frac{n-2}{2n}A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{q(i)}) \\
&\quad + \frac{1}{2n}A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{p(i)}) \\
&\quad + \frac{1}{2n}A(gx_{p(i)-1},...,gx_{p(i)-1},gx_{q(i)}) \\
&\leq \frac{n-2}{2n}A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{q(i)}) \\
&\quad + \frac{n-1}{2n}A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{p(i)-1}) \\
&\quad + \frac{1}{2n}A(gx_{p(i)},...,gx_{p(i)},gx_{p(i)-1}) \\
&\quad + \frac{n-1}{2n}A(gx_{p(i)-1},...,gx_{p(i)-1},gx_{q(i)-1}) \\
&\quad + \frac{1}{2n}A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{q(i)}).
\end{aligned}$$

Letting $i \to \infty$ in the precedent inequalities and using (4.5) and (4.8), we get

$$\begin{aligned}
\epsilon &\leq \frac{1}{2n}\left[\lim_{i\to+\infty} A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{p(i)}) + \epsilon\right] \\
&\leq \frac{1}{2n}\left[2(n-1)\epsilon\right].
\end{aligned}$$

It implies that

$$\lim_{i\to+\infty} A(gx_{q(i)-1},...,gx_{q(i)-1},gx_{p(i)}) = 2n\epsilon - \epsilon. \tag{4.10}$$

Letting $i \to +\infty$ in (4.9) and by (4.5), (4.8), (4.10) and the continuity of ψ and ϕ, we get

$$\psi(\epsilon) \leq \psi\left(\frac{1}{2n}\left[2n\epsilon - \epsilon + \epsilon\right]\right) - \phi\left(0,...,0,2n\epsilon-\epsilon,\epsilon\right).$$

Therefore $\epsilon = 0$, this is a contradiction. We deduce that $\{gx_m\}$ is a Cauchy sequence in $g(X)$, which is a complete subset of (X, A). So we obtain the existence of $t, u \in X$ such that $\{gx_m\}$ converges to $t = gu$ and then

$$\lim_{m\to+\infty} A(gx_m,...,gx_m,gu) = 0. \tag{4.11}$$

By using Lemma 3.17 we have

$$\lim_{m\to+\infty} A(gx_m,...,gx_m,fu) = A(gu,...,gu,fu). \tag{4.12}$$

Now, we show that $fu = t$. By (4.1), we get

$$\begin{aligned}
&\psi\left(A(gx_{m+1},...,fu)\right)\\
&= \psi\left(A(fx_m,...,fu)\right)\\
&\leq \psi\left(\frac{n-2}{2n}A(gx_m,...,fx_m) + \frac{1}{2n}A(gx_m,...,fu) + \frac{1}{2n}A(gu,...,fx_m)\right)\\
&\quad - \phi\left(A(gx_m,...,fx_m),...,A(gx_m,...,fu),A(gu,...,fx_m)\right)\\
&= \psi\left(\frac{n-2}{2n}A(gx_m,...,gx_{m+1}) + \frac{1}{2n}A(gx_m,...,fu) + \frac{1}{2n}A(gu,...,fx_{m+1})\right)\\
&\quad - \phi\left(A(gx_m,...,gx_{m+1}),...,A(gx_m,...,fu),A(gu,...,gx_{m+1})\right).
\end{aligned}$$

Letting $m \to +\infty$ and using (4.5), (4.11), (4.12) and the continuity of ψ and ϕ and the fact that ψ is increasing, we get

$$\begin{aligned}
&\psi\left(A(gu,...,gu,fu)\right)\\
&\leq \psi\left(\frac{1}{2n}\left[A(gu,...,gu,fu)\right]\right) - \phi\left(0,...,0,A(gu,...,gu,fu),0\right).
\end{aligned} \tag{4.13}$$

Then, $A(gu,...,gu,fu) = 0$ and hence $fu = gu = t$. Therefore, u is a coincidence point of f and g. And since the pair $\{f,g\}$ is weakly compatible, we have $ft = gt$.

Now, to prove that t is a common fixed point of f and g, we have by (4.1)

$$\begin{aligned}
&\psi\left(A(gt,..,gt,gx_{m+1})\right)\\
&= \psi\left(A(ft,..,ft,fx_m)\right)\\
&\leq \psi\left(\frac{1}{2n}\left[(n-2)A(gt,..,gt,ft) + A(gt,..,gt,fx_m) + A(gx_m,..,gx_m,ft)\right]\right)\\
&\quad - \phi\left(A(gt,..,gt,ft),..,A(gt,...,gt,fx_m),A(gx_m,..,gx_m,ft)\right)\\
&= \psi\left(\frac{1}{2n}\left[(0 + A(gt,..,gt,gx_{m+1}) + A(gx_m,..,gx_m,gt)\right]\right)\\
&\quad - \phi\left(0,...,0,A(gt,..,gt,gx_{m+1}),A(gx_m,..,gx_m,gt)\right).
\end{aligned}$$

Letting $m \to +\infty$ and by Lemma 2.5 and the fact that ψ is increasing, we obtain

$$\begin{aligned}
\psi\left(A(gt,...,gt,gu)\right) &\leq \psi\left(\frac{1}{2n}\left[0 + A(gt,...,gt,gu) + A(gu,...,gu,gt)\right]\right)\\
&\quad - \phi\left(0,...,0,A(gt,...,gt,gu),A(gu,...,gu,gt)\right)\\
&< \psi\left([A(gt,...,gu)]\right) - \phi\left(0,...,0,A(gt,...,gu),A(gu,...,gt)\right).
\end{aligned}$$

Then $\phi\left(0,...,0,A(gt,...,gt,gu),A(gu,...,gu,gt)\right) = 0$ and with the property of ϕ, we obtain $A(gt,...,gt,gu) = 0$. Therefore, $gt = gu = t$. We deduce that $ft = fu = t$ and then $ft = gt = t$, so the result follows.

To prove the uniqueness, suppose that w is another common fixed point of f and g. Using (4.1), we get

$$\begin{aligned}\psi(A(t,...,w)) &= \psi(A(ft,...,fw))\\ &\leq \psi\left(\frac{1}{2n}\left[(n-2)A(ft,...,ft)+A(ft,...,fw)+A(fw,...,ft)\right]\right)\\ &\quad -\phi(A(ft,...,ft),...,A(ft,...,ft),A(ft,...,fw),A(fw,...,ft))\\ &\leq \psi\left(\frac{1}{n}A(ft,...,fw)\right)-\phi(0,...,0,A(ft,...,fw),A(fw,...,ft))\\ &< \psi(A(t,...,w))-\phi(0,...,0,A(t,...,w),A(w,...,t)).\end{aligned}$$

Then by the property of ϕ, we have $\phi(0,...,0,A(t,...,t,w),A(w,...,w,t))=0$. This implies that $A(t,...,t,w)=0$ and then $t=w$. □

Corollary 4.2. *Let (X,A) be an A-metric space and $f,g: X\to X$ be two mappings. Suppose that $g(X)$ is a complete subspace of (X,A), $f(X)\subset g(X)$ and the pair $\{f,g\}$ is weakly compatible. By putting*

$$\psi(t)=t,\quad \phi(t_1,t_2,...,t_n)=\left(\frac{1}{2n}-\beta\right)\sum_{i=1}^{n}t_i,$$

where $\beta\in\left[0,\frac{1}{2n}\right)$ in the inequality (4.1) and by Theorem 4.1, we conclude that f and g have a unique common fixed point.

Corollary 4.3. *Let (X,A) be an A-metric space and $f: X\to X$ be a mapping such that*

$$\begin{aligned}&\psi(A(fx,fx,...,fx,fy))\\ &\leq \psi\left(\frac{1}{2n}\left[(n-2)A(x,...,x,fx)+A(x,...,x,fy)\ +A(y,...,y,fx)\right]\right)\\ &\quad -\phi(A(x,...,x,fx),...,A(x,...,x,fx),A(x,...,x,fy),A(y,...,y,fx)),\end{aligned}$$

where ψ is an altering distance function and $\phi:[0,+\infty)^n\longrightarrow[0,+\infty)$ is a continuous function with $\phi(x_1,x_2,...,x_n)=0$ if and only if $x_i=0$, for all $i=1,...,n$. Then f has a unique fixed point.

Proof. By taking $g=Id_X$, the identity mapping on X in Theorem 4.1, the result follows immediately. □

Example 4.4. Let $X=[0,3]$ and the A-metric on X define by :

$$A(x_1,x_2,...,x_n)=\sum_{i=1}^{n}\sum_{i<j}|x_i-x_j|$$

for $n \geq 2$ and $x_1, x_2, ..., x_n \in X$. By taking $\psi(t) = t$, $\phi(t_1, t_2, ..., t_n) = \frac{\sum_{i=1}^{n} t_i}{k}$, $k \geq 2n$, $fx = 2$ and $g = Id_X$, then we obtain

$$\psi\left(A(fx, fx, ..., fx, fy)\right) = \psi\left(A(2, 2, ..., 2)\right) = 0$$

and

$$\begin{aligned}&\psi\left(\tfrac{1}{2n}\left[(n-2)A(x, ..., x, 2) + A(x, ..., x, 2) + A(y, ..., y, 2)\right]\right)\\ &= \psi\left(\tfrac{1}{2n}\left[(n-1)^2|x-2| + (n-1)|y-2|\right]\right)\\ &= \frac{(n-1)^2|x-2| + (n-1)|y-2|}{2n}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\phi\left(A(x, ..., x, 2), ..., A(x, ..., x, 2), A(y, ..., y, 2)\right)\\ &= \phi\left((n-1)|x-2|, ..., (n-1)|x-2|, (n-1)|y-2|\right)\\ &= \frac{(n-1)^2|x-2| + (n-1)|y-2|}{k},\end{aligned}$$

which means that the condition (4.1) is satisfied. Also we have, $f(X) = \{2\}, g(X) = [0,3]$, $f(X) \subset g(X)$, $g(X)$ is a complete subset of (X, A) and the pair $\{f, g\}$ is weakly compatible. Then f and g have a unique common fixed point $x = 2$.

References

[1] M. Abbas, B. Ali and Y. Suleiman, *Generalized coupled common fixed point results in partially ordered A-metric spaces,* Fixed Point Theory and Appl., **64** (2015), 1–24.

[2] B.C. Dhage, *Generalized metric spaces and topological structure1,* An. Stiint. Univ. 'Al. I. CUZA.' IASI, Mat., **46** (2000), 3–24.

[3] S. Gahler, 2-*Metrische raume and ihre topologische struktur,* Math. Nachr., **26** (1963), 115–148.

[4] S. Gahler, *Zur geometric* 2-*metrische raume,* Fund. **11** (1966), 665–667.

[5] K.S. Ha, Y.J. Cho and A. White, *Strictly convex and 2-convex 2-normed spaces,* Math. Jpn., **33**(3) (1988), 375–384.

[6] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear Convex Anal., **7** (2012), 289–297.

[7] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point results in G-metric spaces,* Int. J. Math. Sci., **2009** Article ID283028, (2009), 10 pages.

[8] G. Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity,* Indian J. Pure Appl. Math., **29**(3)(1998), 227–238.

[9] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Aust. Math. Soc., **30** (1984), 1–9.

[10] M.M. Rezaee and S. Sedghi, *Tripled fixed point results in partially ordered S-metric spaces*, Nonlinear Funct. Anal. and Appl., **23**(2) (2018), 395-405.

[11] S. Sedghi, N. Shobe and H. Zhou, *A common fixed point theorem in $D^\star$-metric spaces,* Fixed Point Theory Appl., **2007** Article ID 27906 (2007), 1–13.

[12] S. Sedghi, N. Shobe and T. Dosenovic,*Fixed point results in S-metric spaces*, Nonlinear Funct. Anal. and Appl., **20**(1) (2015), 55-67.

[13] A.K. Sharma, *A note on fixed ponts in 2-metric spaces,* Indian J. Pure Appl. Math., **11**(2) (1980), 1580–1583.

[14] D. Singh, V. Joshi and J. K. Kim, *Existence of solution to Bessel-type boundary value problem via G-l cyclic F-contractive mapping with graphical verification,* Nonlinear Funct. Anal. and Appl., **23**(2) (2018), 205-224.