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COMMON FIXED POINT RESULTS UNDER C-DISTANCE IN TVS-CONE METRIC SPACES

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Abstract. In 2011, Wang and Guo introduced the c-distance in a cone metric spaces and proved common fixed point results. The purpose of this paper is to extend and generalize some common fixed point results in literature for c-distance in tvs-cone metric spaces (with the underlying cone which is not normal) by replacing the constants in contractive conditions with functions.

1. Introduction

In 2007, Huang and Zhang [10] first introduced the concept of cone metric spaces which is more general than the concept of metric space. They also established the Banach contraction mapping principle in this space. Afterwards, several authors have studied fixed point theorems in cone metric spaces (see [1, 2, 4, 8, 9, 11, 16]).

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Du in [8], introduced the concept of tvs-cone metric spaces which is to be improved and extended form of cone metric spaces in the sense of Huang and Zhang [10]. Later on, many authors (See [3, 5, 7, 13, 14, 17]) have generalized and proved fixed point results in tvs-cone metric spaces. However, it should be noted that an old result shows that if the underlying cone of an ordered tvs is solid and normal, then such tvs must be an ordered normed space. Thus, proper generalizations when passing from norm-valued cone metric spaces to tvs-valued cone metric spaces can be obtained only in the case of non-normal cones (for more detail see [13]).

Recently, Wang and Guo [20] introduced the concept of c-distance in a cone metric spaces (also see[6]) and proved common fixed point results in ordered cone metric spaces, which is cone metric version of w-distance of Kada et al. [12]. Then in this direction several authors have proved common fixed point theorems in cone metric spaces as well as in tvs-cone metric spaces (see [7, 16, 18, 19]).

In this paper, we prove some common fixed point results for mapping in tvs-cone metric spaces (with the underlying cone which is not normal) under contractive conditions (in which constants are replaced by functions) expressed in the terms of c-distance. We obtained the result which extend and generalize the main results of Dordevic *et al.* [7] and Fadail *et al.* [9].

2. Preliminaries

We recall some definitions and results from [5, 6, 7, 8, 13, 15], which will be needed in the sequel.

Let E be a tvs with the zero vector θ. A nonempty and closed subset P of E is called a cone if $P + P \subseteq P$ and $\lambda P \subseteq P$ for $\lambda \geq 0$. A cone P is said to be proper if $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P; x \prec y$ will stand for $x \prec y$ and $x \neq y$, while $x \ll y$ stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is said to be solid if it has a nonempty interior. The pair (E, P) is an ordered topological vector space.

Definition 2.1. ([5, 8, 13]) Let X be a nonempty set and (E, P) an ordered tvs. A vector-valued function $d : X \times X \to E$ is said to be a tvs-cone metric, if the following conditions hold:

(C_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(C_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(C_3) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Then the pair (X,d) is called a tvs-cone metric space.

Definition 2.2. ([5, 8, 13]) Let (X,d) be a tvs-cone metric space, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ tvs-cone converges to x whenever for every $c \in E$ *with* $\theta \ll c$, there is a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geqslant n_0$. We denote this by $\lim_{n\to\infty} x_n = x$;

(ii) $\{x_n\}$ is a tvs-cone Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant n_0$;

(iii) (X,d) is tvs-cone complete if every tvs-cone Cauchy sequence in X is tvs-cone convergent.

Let (X,d) be a tvs-cone metric space. The following properties are often used, particularly in the case when the underlying cone is non-normal.

(P_1) If $u, v, w \in E, u \preceq v$ and $v \ll w$ then $u \ll w$.
(P_2) If $u \in E$ and $\theta \preceq u \ll c$ for each $c \in int\, P$ then $u = \theta$.
(P_3) If $u_n, v_n, u, v \in E, \theta \preceq u_n \preceq v_n$ for each $n \in \mathbb{N}$, and $u_n \to u, v_n \to v (n \to \infty)$, then $\theta \preceq u \preceq v$.
(P_4) If $x_n, x \in X, u_n \in E, d(x_n, x) \preceq u_n$ and $u_n \to \theta (n \to \infty)$, then $x_n \to x (n \to \infty)$.
(P_5) If $u \preceq \lambda u$ and $0 \leq \lambda < 1$, then $u = \theta$.
(P_6) If $c \gg \theta$ and $u_n \in E, u_n \to \theta (n \to \infty)$, then there exists n_0 such that $u_n \ll c$ for all $n \geq n_0$.

In [20], Wang and Guo introduced the notion of c-distance on a cone metric space, which is a generalization of w-distance of Kada *et al.* [12]. Then Cho *et al.* [6] converted it in to the setting of ordered cone metric spaces.

Definition 2.3. ([6]) Let (X,d) be a tvs-cone metric space. A function $q : X \times X \to E$ is called a c-distance in X if:

(q_1) $\theta \preceq q(x,y)$ for all $x, y \in X$;
(q_2) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x, y, z \in X$;
(q_3) if a sequence $\{y_n\}$ in X converges to a point $y \in X$, and for some $x \in X$ and $u = u_x \in P, q(x, y_n) \preceq u$ holds for each $n \in \mathbb{N}$, then $q(x,y) \preceq u$;
(q_4) for each $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \preceq e$, such that $q(z,x) \ll e$ and $q(z,y) \ll e$ implies $d(x,y) \ll c$.

Example 2.4. ([7]) Let (X, d) be a tvs-cone metric space such that the metric d(.,.) is a continuous function in second variable. Then $q(x, y) = d(x, y)$ is a c-distance. Indeed, only property (q_3) is non-trivial and it follows from $q(x, y_n) = d(x, y_n) \leq u$, passing to the limit when $n \to \infty$ and using continuity of d.

A sequence $\{u_n\}$ in P a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$. It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in E and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

The following lemma is a tvs-cone metric version of lemmas from [6, 12].

Lemma 2.5. ([7]) *Let (X, d) be a tvs-cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are c-sequences in P. Then the following hold:*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$.*
(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*
(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n > n_0$, then $\{x_n\}$ is a Cauchy sequence in X.*
(4) *If $q(y, x_n) \preceq u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 2.6. ([20])

(1) $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$;
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Now we are ready to state and prove our main results.

3. Main Results

Theorem 3.1. *Let (X, d) be a complete tvs-cone metric space and q is a c-distance on X. Let $f, g : X \to X$ be two continuous self-maps and suppose that there exist mappings $k, l : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \leq k(x), l(fx) \leq l(x), k(gx) \leq k(x), l(gx) \leq l(x)$ *for all* $x \in X$;
(b) $(k + 2l)(x) < 1$ *for all* $x \in X$;
(c) (i) $q(fx, gy) \preceq k(x)q(x, y) + l(x)[q(fx, y) + q(x, gy)]$
    (ii) $q(gy, fx) \preceq k(y)q(y, x) + l(y)[q(y, fx) + q(gy, x)]$ *for all* $x, y \in X$.

Then f and g have a common fixed point in X. If $fu = gu = u$, then $q(u, u) = \theta$.

Proof. Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \geq 0$. Denote

$$u_n = q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n})$$

and

$$v_n = q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1}).$$

Putting $x = x_{2n+2}, y = x_{2n+1}$ in (c)-(i), we get

$$\begin{aligned} q(fx_{2n+2}, gx_{2n+1}) &= q(x_{2n+3}, x_{2n+2}) \\ &\preceq k(x_{2n+2})q(x_{2n+2}, x_{2n+1}) + l(x_{2n+2})[q(fx_{2n+2}, x_{2n+1}) \\ &\quad + q(x_{2n+2}, gx_{2n+1})] \\ &= k(gx_{2n+1})q(x_{2n+2}, x_{2n+1}) + l(gx_{2n+1})[q(x_{2n+3}, x_{2n+1}) \\ &\quad + q(x_{2n+2}, x_{2n+2})] \\ &\preceq k(x_{2n+1})q(x_{2n+2}, x_{2n+1}) + l(x_{2n+1})[q(x_{2n+3}, x_{2n+2}) \\ &\quad + q(x_{2n+2}, x_{2n+1})]. \end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned} q(x_{2n+3}, x_{2n+2}) \preceq\ & k(x_0)q(x_{2n+2}, x_{2n+1}) + l(x_0)[q(x_{2n+3}, x_{2n+2}) \\ & +q(x_{2n+2}, x_{2n+1})]. \end{aligned} \tag{3.1}$$

Similarly, putting $y = x_{2n+1}$ and $x = x_{2n+2}$ in (c)-(ii), we obtain

$$\begin{aligned} q(gx_{2n+1}, fx_{2n+2}) &= q(x_{2n+2}, x_{2n+3}) \\ &\preceq k(x_{2n+1})q(x_{2n+1}, x_{2n+2}) + l(x_{2n+1})[q(x_{2n+1}, fx_{2n+2}) \\ &\quad + q(gx_{2n+1}, x_{2n+2})] \\ &= k(fx_{2n})q(x_{2n+1}, x_{2n+2}) + l(fx_{2n})[q(x_{2n+1}, x_{2n+3}) \\ &\quad + q(x_{2n+2}, x_{2n+2})] \\ &\preceq k(x_{2n})q(x_{2n+1}, x_{2n+2}) + l(x_{2n})[q(x_{2n+1}, x_{2n+2}) \\ &\quad + q(x_{2n+2}, x_{2n+3})]. \end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned} q(x_{2n+2}, x_{2n+3}) \preceq\ & k(x_0)q(x_{2n+1}, x_{2n+2}) + l(x_0)[q(x_{2n+1}, x_{2n+2}) \\ & +q(x_{2n+2}, x_{2n+3})]. \end{aligned} \tag{3.2}$$

By adding up (3.1) and (3.2), we get

$$u_{n+1} \preceq (k(x_0) + l(x_0))v_n + l(x_0)u_{n+1},$$

i.e. $u_{n+1} \preceq hv_n$, for all $n \in \mathbb{N}$, where $0 < h = \frac{k(x_0)+l(x_0)}{1-l(x_0)} < 1$, since $(k + 2l)(x) < 1$ for all $x \in X$.

By a similar procedure, starting with $x = x_{2n}$ and $y = x_{2n+1}$, one can get $v_n \preceq hu_n, n \in \mathbb{N}$. Combining the last two inequalities, it follows that

$u_{n+1} \preceq h^2 u_n$ and $v_n \preceq h^2 v_{n-1}$, and we obtain that $\{u_n\}$ and $\{v_n\}$ are c-sequences.

We have that $q(x_{2n}, x_{2n+1}) \preceq u_n, q(x_{2n+1}, x_{2n+2}) \preceq v_n$, and it follows that $q(x_n, x_{n+1}) \preceq u_n + v_n$, where $u_n + v_n$ is a c-sequence. Using Lemma 2.5(3), we get that $\{x_n\}$ is a Cauchy sequence in X. Hence $x_n \to x^* \in X (n \to \infty)$. Since f and g are continuous, it easily follows from the definition of $\{x_n\}$ that $fx^* = gx^* = x^*$. Thus, mappings f and g have a common fixed point.

Suppose that $u \in X$ is another point satisfying $fu = gu = u$. Then, (c)-(i) implies that

$$q(fu, gu) = q(u, u) \preceq k(u)q(u, u) + l(u)[q(u, u) + q(u, u)] = (k + 2l)(u)q(u, u)$$

and since $0 < (k + 2l)(x) < 1$, property (P_5) implies that $q(u, u) = \theta$. □

As corollary, we obtain common fixed point result for self maps f and g satisfying

(i) $q(fx, gy) \preceq k(x)q(x, y) + l(x)[q(x, fx) + q(y, gy)]$,
(ii) $q(gy, fx) \preceq k(y)q(y, x) + l(y)[q(fx, x) + q(gy, y)]$

for all $x, y \in X$ and $(k + 2l)(x) < 1$.

Theorem 3.2. *Let (X, d) be a complete tvs-cone metric space and q is a c-distance on X. Let $f, g : X \to X$ be two continuous self-maps and suppose that there exists mapping $k, l, r : X \to [0, 1)$such that the following conditions hold:*

(a) $k(fx) \leq k(x), l(fx) \leq l(x), r(fx) \leq r(x)$ *and*
$k(gx) \leq k(x), l(gx) \leq l(x), r(gx) \leq r(x)$ *for all* $x \in X$;
(b) $(k + 2l + 2r)(x) < 1$ *for all* $x \in X$;
(c) (i) $q(fx, gy) \preceq k(x)q(x, y) + l(x)[q(x, gy) + q(y, fx)] + r(x)[q(x, fx) + q(y, gy)]$
(ii) $q(gy, fx) \preceq k(y)q(y, x) + l(y)[q(gy, x) + q(fx, y)] + r(y)[q(fx, x) + q(gy, y)]$
for all $x, y \in X$.

Then f and g have a common fixed point in X. If $fu = gu = u$, then $q(u, u) = \theta$.

Proof. Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \geq 0$. Denote

$$u_n = q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n})$$

and

$$v_n = q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1}).$$

Putting $x = x_{2n+2}, y = x_{2n+1}$ in (c)-(i), we get

$$\begin{aligned} q(fx_{2n+2}, gx_{2n+1}) &= q(x_{2n+3}, x_{2n+2}) \\ &\preceq k(x_{2n+2})q(x_{2n+2}, x_{2n+1}) \\ &\quad + l(x_{2n+2})[q(x_{2n+2}, gx_{2n+1}) + q(x_{2n+1}, fx_{2n+2})] \\ &\quad + r(x_{2n+2})[q(x_{2n+2}, fx_{2n+2}) + q(x_{2n+1}, gx_{2n+1})] \\ &= k(gx_{2n+1})q(x_{2n+2}, x_{2n+1}) \\ &\quad + l(gx_{2n+1})[q(x_{2n+2}, x_{2n+2}) + q(x_{2n+1}, x_{2n+3})] \\ &\quad + r(gx_{2n+1})[q(x_{2n+2}, x_{2n+3}) + q(x_{2n+1}, x_{2n+2})] \\ &\preceq k(x_{2n+1})q(x_{2n+2}, x_{2n+1}) \\ &\quad + l(x_{2n+1})[q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+3})] \\ &\quad + r(x_{2n+1})[q(x_{2n+2}, x_{2n+3}) + q(x_{2n+1}, x_{2n+2})]. \end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned} q(x_{2n+3}, x_{2n+2}) &\preceq k(x_0)q(x_{2n+2}, x_{2n+1}) \\ &\quad + l(x_0)[q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+3})] \\ &\quad + r(x_0)[q(x_{2n+2}, x_{2n+3}) + q(x_{2n+1}, x_{2n+2})]. \end{aligned} \tag{3.3}$$

Similarly, putting $y = x_{2n+1} \, and \, x = x_{2n+2}$ in (c)-(ii), we get

$$\begin{aligned} q(gx_{2n+1}, fx_{2n+2}) &= q(x_{2n+2}, x_{2n+3}) \\ &\preceq k(x_{2n+1})q(x_{2n+1}, x_{2n+2}) \\ &\quad + l(x_{2n+1})[q(gx_{2n+1}, x_{2n+2}) + q(fx_{2n+2}, x_{2n+1})] \\ &\quad + r(x_{2n+1})[q(fx_{2n+2}, x_{2n+2}) + q(gx_{2n+1}, x_{2n+1})] \\ &= k(fx_{2n})q(x_{2n+1}, x_{2n+2}) \\ &\quad + l(fx_{2n})[q(x_{2n+2}, x_{2n+2}) + q(x_{2n+3}, x_{2n+1})] \\ &\quad + r(fx_{2n})[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})] \\ &\preceq k(x_{2n})q(x_{2n+1}, x_{2n+2}) \\ &\quad + l(x_{2n})[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})] \\ &\quad + r(x_{2n})[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})]. \end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned} q(x_{2n+2}, x_{2n+3}) &\preceq k(x_0)q(x_{2n+1}, x_{2n+2}) \\ &\quad + l(x_0)[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})] \\ &\quad + r(x_0)[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})]. \end{aligned} \tag{3.4}$$

It follows by adding up (3.3) and (3.4) that

$$u_{n+1} \preceq (k(x_0) + l(x_0) + r(x_0))v_n + (l(x_0) + r(x_0))u_{n+1},$$

i.e $u_{n+1} \preceq hv_n, n \in \mathbb{N}$, where $0 < h = \frac{k(x_0)+l(x_0)+r(x_0)}{1-l(x_0)-r(x_0)} < 1$, since $(k + 2l + 2r)(x) < 1$.

By a similar procedure, starting with $x = x_{2n}$ and $y = x_{2n+1}$, one can get $v_n \preceq hu_n$, $n \in \mathbb{N}$. Combining the last two inequalities, it follows that $u_{n+1} \preceq h^2 u_n$ and $v_n \preceq h^2 v_{n-1}$, and we get that $\{u_n\}$ and $\{v_n\}$ are c-sequences.

We have that $q(x_{2n}, x_{2n+1}) \preceq u_n$, $q(x_{2n+1}, x_{2n+2}) \preceq v_n$, and it follows that $q(x_n, x_{n+1}) \preceq u_n + v_n$, where $u_n + v_n$ is a c-sequence. Using Lemma 2.5(3), we get that $\{x_n\}$ is a Cauchy sequence in X. Hence $x_n \to x^* \in X(n \to \infty)$. Since f and g are continuous, it easily follows from the definition of $\{x_n\}$ that $fx^* = gx^* = x^*$. Thus, mappings f and g have a common fixed point.

Suppose that $u \in X$ is any point satisfying $fu = gu = u$. Then, (c)-(i) implies that

$$\begin{aligned} q(fu, gu) &= q(u, u) \\ &\preceq k(u)q(u, u) + l(u)[q(u, u) + q(u, u)] + r(u)[q(u, u) + q(u, u)] \\ &= (k + 2l + 2r)(u)q(u, u), \end{aligned}$$

and since $0 < (k + 2l + 2r)(x) < 1$, property (P_5) implies that $q(u, u) = \theta$. □

As corollary, we obtain common fixed point result for self-maps f and g satisfying

(i) $q(fx, gy)$
$\preceq k(x)q(x, y) + l(x)[q(x, fx) + q(x, gy)] + r(x)[q(y, fx) + q(y, gy)]$,
(ii) $q(gy, fx)$
$\preceq k(y)q(y, x) + l(y)[q(fx, x) + q(gy, x)] + r(y)[q(fx, y) + q(gy, y)]$,

for all $x, y \in X$ and $(k + 2l + 2r)(x) < 1$.

Theorem 3.3. *Let (X, d) be a complete tvs-cone metric space and q is a c-distance on X. Let $f, g : X \to X$ be two continuous self-maps and suppose that there exist mappings $k, r, l, t : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \leq k(x), r(fx) \leq r(x), l(fx) \leq l(x)$ *and* $t(fx) \leq t(x)$;
$k(gx) \leq k(x), r(gx) \leq r(x), l(gx) \leq l(x)$ *and* $t(gx) \leq t(x)$ *for all* $x \in X$;
(b) $(k + r + l + 2t)(x) < 1$ *for all* $x \in X$;
(c) (i) $q(fx, gy)$
$\preceq k(x)q(x, y)+r(x)q(fx, x)+l(x)q(gy, y)+t(x)[q(fx, y)+q(gy, x)]$
(ii) $q(gy, fx)$
$\preceq k(y)q(y, x)+r(y)q(x, fx)+l(y)q(y, gy)+t(y)[q(y, fx)+q(x, gy)]$
for all $x, y \in X$.

Then f and g have a common fixed point in X. If $fu = gu = u$, then $q(u, u) = \theta$.

Proof. Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \geq 0$. Denote

$$u_n = q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n})$$

and

$$v_n = q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1}).$$

Putting $x = x_{2n+2}, y = x_{2n+1}$ in (c)-(i), we get

$$\begin{aligned} q(fx_{2n+2}, gx_{2n+1}) &= q(x_{2n+3}, x_{2n+2}) \\ &\preceq k(x_{2n+2})q(x_{2n+2}, x_{2n+1}) \\ &\quad + r(x_{2n+2})q(fx_{2n+2}, x_{2n+2}) \\ &\quad + l(x_{2n+2})q(gx_{2n+1}, x_{2n+1}) \\ &\quad + t(x_{2n+2})[q(fx_{2n+2}, x_{2n+1}) + q(gx_{2n+1}, x_{2n+2})] \\ &= k(gx_{2n+1})q(x_{2n+2}, x_{2n+1}) \\ &\quad + r(gx_{2n+1})q(x_{2n+3}, x_{2n+2}) \\ &\quad + l(gx_{2n+1})q(x_{2n+2}, x_{2n+1}) \\ &\quad + t(gx_{2n+1})[q(x_{2n+3}, x_{2n+1}) + q(x_{2n+2,}x_{2n+2})] \\ &\preceq k(x_{2n+1})q(x_{2n+2}, x_{2n+1}) \\ &\quad + r(x_{2n+1})q(x_{2n+3}, x_{2n+2}) \\ &\quad + l(x_{2n+1})q(x_{2n+2}, x_{2n+1}) \\ &\quad + t(x_{2n+1})[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})]. \end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned} q(x_{2n+3}, x_{2n+2}) \preceq\ & k(x_0)q(x_{2n+2}, x_{2n+1}) \\ & + r(x_0)q(x_{2n+3}, x_{2n+2}) + l(x_0)q(x_{2n+2}, x_{2n+1}) \\ & + t(x_0)[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})]. \end{aligned} \tag{3.5}$$

Similarly, putting $y = x_{2n+1}$ *and* $x = x_{2n+2}$ in (c)-(ii), we obtain

$$\begin{aligned} q(gx_{2n+1}, fx_{2n+2}) &= q(x_{2n+2}, x_{2n+3}) \\ &\preceq k(x_{2n+1})q(x_{2n+1}, x_{2n+2}) + r(x_{2n+1})q(x_{2n+2}, fx_{2n+2}) \\ &\quad + l(x_{2n+1})q(x_{2n+1}, gx_{2n+1}) \\ &\quad + t(x_{2n+1})[q(x_{2n+1}, fx_{2n+2}) + q(x_{2n+2}, gx_{2n+1})] \end{aligned}$$

$$\begin{aligned}
&= k(fx_{2n})q(x_{2n+1}, x_{2n+2}) + r(fx_{2n})q(x_{2n+2}, x_{2n+3}) \\
&\quad + l(fx_{2n})q(x_{2n+1}, x_{2n+2}) \\
&\quad + t(fx_{2n})[q(x_{2n+1}, x_{2n+3}) + q(x_{2n+2,}x_{2n+2})] \\
&\preceq k(x_{2n})q(x_{2n+1}, x_{2n+2}) + r(x_{2n})q(x_{2n+2}, x_{2n+3}) \\
&\quad + l(x_{2n})q(x_{2n+1}, x_{2n+2}) \\
&\quad + t(x_{2n})[q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+3})].
\end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned}
q(x_{2n+2}, x_{2n+3}) \preceq\ & k(x_0)q(x_{2n+1}, x_{2n+2}) + r(x_0)q(x_{2n+2}, x_{2n+3}) \\
& + l(x_0)q(x_{2n+1}, x_{2n+2}) + t(x_0)[q(x_{2n+1}, x_{2n+2}) \\
& + q(x_{2n+2}, x_{2n+3})].
\end{aligned} \tag{3.6}$$

By adding up (3.5) and (3.6), we get

$$u_{n+1} \preceq (k(x_0) + l(x_0) + t(x_0))v_n + (r(x_0) + t(x_0))u_{n+1},$$

i.e $u_{n+1} \preceq hv_n$ for all $n \in \mathbb{N}$, where $0 < h = \frac{k(x_0)+l(x_0)+t(x_0)}{1-r(x_0)-t(x_0)} < 1$, since $(k + r + l + 2t)(x) < 1$ for all $x \in X$.

By a similar procedure, starting with $x = x_{2n}$ and $y = x_{2n+1}$, one can get $v_n \preceq hu_n$, $n \in \mathbb{N}$. Combining the last two inequalities, it follows that $u_{n+1} \preceq h^2u_n$ and $v_n \preceq h^2v_{n-1,}$ and we obtain that $\{u_n\}$ and $\{v_n\}$ are c-sequences.

We have that $q(x_{2n}, x_{2n+1}) \preceq u_n$, $q(x_{2n+1}, x_{2n+2}) \preceq v_n$, and it follows that $q(x_n, x_{n+1}) \preceq u_n + v_n$, where $u_n + v_n$ is a c-sequence. Using Lemma 2.5(3), we get that $\{x_n\}$ is a Cauchy sequence in X. Hence $x_n \to x^* \in X(n \to \infty)$. Since f and g are continuous, it easily follows from the definition of $\{x_n\}$ that $fx^* = gx^* = x^*$. Thus, mappings f and g have a common fixed point.

Suppose that $u \in X$ is another point satisfying $fu = gu = u$. Then, (c)-(i) implies that

$$\begin{aligned}
q(fu, gu) &\preceq k(u)q(u, u) + r(u)q(fu, u) + l(u)q(gu, u) \\
&\quad + t(u)[q(fu, u) + q(gu, u)] \\
&= (k + r + l + 2t)(u)q(u, u)
\end{aligned}$$

and since $0 < (k+r+l+2t)(x) < 1$, property (P_5) implies that $q(u, u) = \theta$. □

Now, we present the following example to demonstrate the use of Theorem 3.1.

Example 3.4. Let $E = \mathbb{R}$, $P = \{x \in E : x \geq 0\}, X = [0, 1]$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y$ for all $x, y \in X$. Then q is c-distance on X.

Define the mappings $f, g : X \to X$ by $f(x) = \frac{x^2}{16}$ and $g(x) = \frac{x}{16}$ for all $x \in X$. Take $k(x) = \left(\frac{2x+3}{16}\right)$ and $l(x) = \left(\frac{3x+2}{16}\right)$ for all $x \in X$.

We observe that

(a) $k(fx) = k(\frac{x^2}{16}) = \left(\frac{2(\frac{x^2}{16})+3}{16}\right) = \frac{1}{16}\left(\frac{x^2}{8}+3\right) = \frac{1}{8}\left(\frac{x^2+24}{16}\right) \leq \frac{2x+3}{16} = k(x)$.

Similarly we can get $l(fx) \leq l(x), k(gx) \leq k(x)$ and $l(gx) \leq l(x)$ for all $x \in X$.

(b) $(k+2l)(x) = k(x) + 2l(x) = \left(\frac{2x+3}{16}\right) + 2\left(\frac{3x+2}{16}\right) = \frac{8x+7}{16} < 1$ for all $x \in X$.

(c) $$q(fx, gy) = gy = \frac{y}{16} \preceq \left(\frac{(2x+3)y}{16}\right) = \left(\frac{(2x+3)}{16}y\right) = k(x)q(x,y)$$
$$\preceq k(x)q(x,y) + l(x)[q(fx,y) + q(x,gy)], \text{ for all } x \in X.$$

Therefore, the conditions of Theorem 3.1 (with c-distance metric version) are satisfied. Hence f and g have a common fixed point $u = 0$ and that $q(u,u) = q(0,0) = \theta$.

This example can be easily modified to the tvs-cone metric case. We define tvs-cone metric on X by $d(x,y)(t) = |x-y|\emptyset(t)$ with fixed $\emptyset \in P = \{f \in C[0,1] : f(t) \geq 0 \text{ for } t \in [0,1]\}$ and take c-distance $q_1(x,y)(t) = y.\emptyset(t)$.

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