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SOLUTION OF A GENERAL FAMILY OF FRACTIONAL KINETIC EQUATIONS ASSOCIATED WITH THE GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. Fractional kinetic equations are investigated in order to describe the various phenomena governed by anomalous reaction in dynamical systems with chaotic motion. Many authors have provided solutions of various families of fractional kinetic equations involving such special functions as (for example) a generalized Bessel function of the first kind [18] and the Aleph function [6]. Here, in this paper, we aim at presenting solution of a certain general family of fractional kinetic equations associated with the generalized Mittag-Leffler function. It is also shown that the result presented here includes, as its special cases, solutions of many fractional kinetic equations which were investigated in earlier works. In our investigation, we have found it to be more convenient to use (and the closed-form results derived here appear to be considerably simpler by using) the Sumudu transform instead of the classical Laplace transform.

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1. Introduction, notations and preliminaries

Fractional calculus, the differentiation and integration of arbitrary order, arises naturally in various areas of science and engineering. During the last several decades, fractional kinetic equations of different forms have been widely used in describing and solving several important problems of physics and astrophysics. Saxena *et al.* [28] introduced the solution of the generalized fractional kinetic equation associated with the generalized Mittag-Leffler function. Subsequently, Saxena *et al.* [31] developed an alternative derivation of the generalized fractional kinetic equations in terms of special functions with the Sumudu transform. More recently, Kumar *et al.* [18] gave the solution of a generalized fractional kinetic equation involving the Bessel function of the first kind; Choi and Kumar [6] presented the solution of the generalized fractional kinetic equations involving the Aleph function. For other results involving various classes of fractional kinetic equations and their solutions, one may refer to such works as (for example, [4, 10, 23, 26, 27, 28, 29, 31, 43]). In particular, Tomovski *et al.* [43] presented the *corrected* version of an obviously erroneous solution of a certain fractional kinetic equation which was given by Saxena and Kalla [26, p. 508, Eq. (3.2)] and also derived the solution of a much more general family of fractional kinetic equations (see, for details, [43, p. 813, Remark 3 *and* Theorem 10]).

Here, in this paper, we propose to investigate solution of a certain generalized fractional kinetic equation associated with the generalized Mittag-Leffler function (see [25]). It is also pointed out that the result presented here is general enough to be specialized to include many known solutions for fractional kinetic equations.

Fractional kinetic equations have gained popularity during the past decade or so due mainly to the discovery of their relation with the theory of CTRW (Continuous Time Random Walks) in [13]. These equations are investigated in order to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on (see also [12]).

Consider an arbitrary reaction characterized by a time-dependent quantity $N = N(t)$. It is possible to calculate the rate of change $\frac{dN}{dt}$ to be a balance between the destruction rate $\mathfrak{d}$ and the production rate $\mathfrak{p}$ of N, that is, $dN/dt = -\mathfrak{d} + \mathfrak{p}$. In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself, that is,

$$\mathfrak{d} = \mathfrak{d}(N) \quad \text{and} \quad \mathfrak{p} = \mathfrak{p}(N).$$

This dependence is complicated, since the destruction or the production at a time t depends not only on $N(t)$, but also on the past history $N(\eta)$ $(\eta < t)$ of the variable N. This may be formally represented by the following equation (see [10]):

$$\frac{dN}{dt} = -\mathfrak{d}\,(N_t) + \mathfrak{p}\,(N_t)\,, \tag{1.1}$$

where N_t denotes the function defined by

$$N_t\,(t^*) = N\,(t - t^*) \qquad (t^* > 0).$$

Haubold and Mathai [10] studied a special case of the equation (1.1) in the following form:

$$\frac{dN_i}{dt} = -c_i\, N_i\,(t) \tag{1.2}$$

with the initial condition that $N_i\,(t = 0) = N_0$ is the number density of species i at time $t = 0$ and the constant $c_i > 0$. This is known as a standard kinetic equation. The solution of the equation (1.2) is easily seen to be given by

$$N_i\,(t) = N_0\, e^{-c_i t}. \tag{1.3}$$

Integration gives an alternative form of the equation (1.2) as follows:

$$N\,(t) - N_0 = c \cdot \,{}_0D_t^{-1} N\,(t)\,, \tag{1.4}$$

where ${}_0D_t^{-1}$ is the standard integral operator and c is a constant.

The fractional generalization of the equation (1.4) is given as in the following form (see [10]):

$$N\,(t) - N_0 = c^{\nu}\,{}_0D_t^{-\nu} N\,(t)\,, \tag{1.5}$$

where ${}_0D_t^{-\nu}$ is the familiar Riemann-Liouville fractional integral operator (see, *e.g.*, [14] and [20]) defined by

$${}_0D_t^{-\nu}\, f(t) = \frac{1}{\Gamma\,(\nu)} \int_0^t (t-u)^{\nu-1}\, f\,(u)\, du \quad \big(\Re\,(\nu) > 0\big). \tag{1.6}$$

In terms of the generalized Bessel function $\omega_{l,b,c}\,(t)$ of the first kind, Kumar *et al.* [18] studied the following equation:

$$N\,(t) - N_0\;\omega_{l,b,c}\,(t) = -\mathfrak{d}^{\nu}\,{}_0D_t^{-\nu}\, N\,(t)\,, \tag{1.7}$$

whose solution is given by

$$N\,(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k\,\Gamma\,(2k+l+1)}{k!\,\Gamma\left(l+k+\frac{b+1}{2}\right)} \left(\frac{t}{2}\right)^{2k+l} E_{\nu,2k+l+1}\,(-\mathfrak{d}^{\nu} t^{\nu})\,, \tag{1.8}$$

where $E_{\nu,2k+l+1}\,(\cdot)$ is the above-mentioned generalized Mittag-Leffler function (see [21, 45]; see also [36]).

Srivastava and Tomovski [42] introduced the following generalization of the Mittag-Leffler function:

$$E_{\alpha,\beta}^{\gamma,\kappa}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)}\,\frac{z^n}{n!} \tag{1.9}$$

$$\big(\alpha,\beta,\gamma\in\mathbb{C};\ \Re(\alpha)>\max\{0,\Re(\kappa)-1\};\ \min\{\Re(\beta),\Re(\kappa)\}>0\big),$$

where, in terms of the Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ (λ, $\nu\in\mathbb{C}$) is defined, in general, by (see, for details, [40] and [41]; see also [38])

$$\begin{aligned}(\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \qquad \big(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\big)\\ &= \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases}\end{aligned} \tag{1.10}$$

it being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient in (1.10) exists. Here and in the following, we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ the sets of complex numbers, real numbers, real and positive numbers, non-positive integers, and positive integers, respectively. A special case of the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ when $\kappa=\mathfrak{q}\in(0,1)\cup\mathbb{N}$ was studied earlier by Shukla and Prajapati (see [35]).

Saxena and Nishimoto [30] studied a further generalization of the generalized Mittag-Leffler function (1.9) in the following form:

$$E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);z\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\,\frac{z^n}{n!} \tag{1.11}$$

$$\Bigg(\alpha_j,\beta_j,\gamma,\kappa\in\mathbb{C};\ \min\{\Re(\kappa),\Re(\alpha_j),\Re(\beta_j)\}>0\quad (j\in\{1,\cdots,m\});$$
$$\Re\left(\sum_{j=1}^{m}\alpha_j\right)>\max\{0,\Re(\kappa)-1\}\Bigg).$$

The special case of (1.11) when $\gamma=\kappa=1$ reduces to the following multi-index Mittag-Leffler function (see [15]; see also [5]):

$$E_{1,1}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);z\right]=\sum_{n=0}^{\infty}\frac{z^n}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)} \tag{1.12}$$

$$\big(\alpha_j,\beta_j\in\mathbb{C};\ \min\{\Re(\alpha_j),\Re(\beta_j)\}>0\quad (j\in\{1,\cdots,m\})\big).$$

The Mittag-Leffler function $E_\alpha(z)$, the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$, and *all* of their aforementioned extensions and generalizations are obviously contained as special cases in the well-known Fox-Wright function ${}_p\Psi_q$ defined by (see, for details, [40, p. 21]; see also [14, p. 56])

$$\begin{aligned} {}_p\Psi_q\,[z] = {}_p\Psi_q \left[\begin{array}{c} (a_1,\alpha_1)\,,\cdots,(a_p,\alpha_p)\,; \\ \\ (b_1,\beta_1)\,,\cdots,(b_q,\beta_q)\,; \end{array} z \right] &= {}_p\Psi_q \left[\begin{array}{c} (a_i,\alpha_i)_{1,p}\,; \\ \\ (b_j,\beta_j)_{1,q}\,; \end{array} z \right] \\ = \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{p} \Gamma\left(a_i+\alpha_i n\right)}{\prod\limits_{j=1}^{q} \Gamma\left(b_j+\beta_j n\right)}\, \frac{z^n}{n!}. \end{aligned} \tag{1.13}$$

Suppose that $f(t)$ is a real- (or complex-) valued function of the (time) variable $t > 0$ and s is a real or complex parameter. The Laplace transform of the function $f(t)$ is defined by

$$\begin{aligned} F\,(s) = \mathcal{L}\left\{f\,(t):s\right\} &= \int_0^\infty e^{-st}\, f\,(t)\,dt \\ &= \lim_{\tau\to\infty} \int_0^\tau e^{-st}\, f\,(t)\,dt, \end{aligned} \tag{1.14}$$

whenever the limit exits (as a finite number). The convolution of two functions $f(t)$ and $g(t)$, which are defined for $t > 0$, plays an important rôle in a number of different physical applications. The Laplace convolution of the functions $f(t)$ and $g(t)$ is given by the following integral:

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = (g * f)(t), \tag{1.15}$$

which exists if the functions f and g are at least piecewise continuous. One of the very significant properties possessed by the convolution in connection with the Laplace transform is that the Laplace transform of the convolution of two functions is the product of their transforms (see, *e.g.*, [34]).

The Laplace Convolution Theorem. *If f and g are piecewise continuous on $[0, \infty)$ and of exponential order α when $t \to \infty$, then*

$$\mathcal{L}\left\{(f * g)(t):s\right\} = \mathcal{L}\left\{f(t):s\right\} \cdot \mathcal{L}\left\{g(t):s\right\} \quad \big(\Re(s) > \alpha\big). \tag{1.16}$$

The so-called Sumudu transform is an integral transform which was defined and studied by Watugala [44] to facilitate the process of solving differential and integral equations in the time domain. The Sumudu transform has been used in various applications of system engineering and applied physics. For some fundamental properties of the Sumudu transform, one may refer to the works including (for example) [1, 2, 3, 39, 44]. It turns out that the Sumudu transform has very special properties which are useful in solving problems involving kinetic equations in science and engineering.

Let $\mathfrak{A}$ be the class of exponentially bounded functions $f : \mathbb{R} \to \mathbb{R}$, that is,

$$|f(t)| < \begin{cases} M \exp\left(-\dfrac{t}{\tau_1}\right) & (t \leqq 0) \\ \\ M \exp\left(\dfrac{t}{\tau_2}\right) & (t \geqq 0), \end{cases} \tag{1.17}$$

where M, τ_1 and τ_2 are some positive real constants. The Sumudu transform defined on the set $\mathfrak{A}$ is given by the following formula (see [44]; see also [6])

$$G\,(u) = \mathcal{S}\,[f\,(t)\,;u] := \int_0^\infty e^{-t}\, f\,(ut)\, dt \quad (-\tau_1 < u < \tau_2)\,. \tag{1.18}$$

The Sumudu transform given in (1.18) can also be derived directly from the Fourier integral. Moreover, it can be easily verified that the Sumudu transform is a linear operator and the function $G(u)$ in (1.18) keeps the same units as $f(t)$; that is, for any real or complex number λ, we have

$$\mathcal{S}[f(\lambda t); u] = G(\lambda u).$$

The Sumudu transform $G(u)$ and the Laplace transform $F(s)$ exhibit a duality relation that may be expressed as follows:

$$G\left(\frac{1}{s}\right) = s\,F\,(s) \quad \text{or} \quad G\,(u) = \frac{1}{u}\,F\left(\frac{1}{u}\right). \tag{1.19}$$

The Sumudu transform has been shown to be the theoretical dual of the Laplace transform. It is also connected to the s-multiplied Laplace transform (see [19]). The use of the convolution theorem for the Sumudu transform in (1.6) gives us the following identity:

$$\mathcal{S}\left[{}_0D_t^{-\nu} f\,(t)\,;u\right] = \mathcal{S}\left[\frac{t^{\nu-1}}{\Gamma\,(\nu)};u\right] \cdot \mathcal{S}\,[f\,(t)\,;u] = u^{\nu}\, G\,(u)\,. \tag{1.20}$$

In connection with the definition (1.18), in case the parameter u takes on negative or complex values, the dualities such as those described in (1.19) do not hold true, in general, because (after the change of variables) the contour of integration in the Laplace integral changes accordingly.

In our present investigation, we have chosen to make use of the Sumudu transform instead of the classical Laplace transform. In fact, for the various problems considered here, the Sumudu transform has not only been found to be more convenient to use, but the closed-form results derived here also appear to be remarkably simpler (see also [39]).

Throughout this paper, it is *tacitly* assumed the all involved complex powers of (for example) complex numbers take on their *principal* values.

2. Solution of generalized fractional kinetic equations by using the Laplace transform

We first find the solution of the generalized fractional kinetic equation involving the generalized Mittag-Leffler function (1.11) by applying the Laplace transform technique. We begin by stating and proving the following lemma.

Lemma 2.1. *Let* $\min\{\Re(\lambda), \Re(\rho), \Re(s)\} > 0$. *Then the following Laplace transform of* $E_{\gamma,\kappa}[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);z]$ *holds true*:

$$\mathcal{L}\left\{t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);\, t^{\rho}\right]: s\right\} = \frac{s^{-\lambda}}{\Gamma(\gamma)}\,{}_2\Psi_m\left[\begin{array}{c}(\gamma,\kappa),(\lambda,\rho);\\ (\beta_j,\alpha_j)_{1,m};\end{array}\ s^{-\rho}\right], \tag{2.1}$$

where ${}_2\Psi_m[\cdot]$ *is the Fox-Wright function given by* (1.13) *and all involved complex powers of* (*for example*) *complex numbers are assumed to take on their principal values.*

Proof. Using the definition (1.14) of the Laplace transform and (1.11), we can obtain the result (2.1). In the course of the proof, the interchange of the order of integration and summation can be justified under the stated conditions. □

For later convenience, a special case of (2.1) when $\lambda = \beta_1$ and $\rho = \alpha_1$ is given in Lemma 2.2 below.

Lemma 2.2. *The following formula holds true for* $\min\{\Re(s),\Re(\alpha_1),\Re(\beta_1)\} > 0$:

$$\mathcal{L}\left[t^{\beta_1-1}E_{\gamma,\kappa}\left\{(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);\, t^{\alpha_1}\right]: s\right\} = \frac{s^{-\beta_1}}{\Gamma(\gamma)}\,{}_1\Psi_{m-1}\left[\begin{array}{c}(\gamma,\kappa);\\ (\beta_j,\alpha_j)_{2,m};\end{array}\ s^{-\alpha_1}\right], \tag{2.2}$$

where all involved complex powers of (*for example*) *complex numbers are assumed to take on their principal values.*

Theorem 2.3. *Let* $c,\mathfrak{d},\nu,\lambda,\rho \in \mathbb{R}^+$. *Also let* $\alpha_j,\beta_j,\gamma,\kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j),\Re(\beta_j)\} > 0 \qquad (j \in \{1,\cdots,m\}),$$

$$\Re(\kappa) > 0 \quad \textit{and} \quad \Re\left(\sum_{j=1}^{m}\alpha_j\right) > \max\{0,\Re(\kappa)-1\}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m);\, \mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{2.3}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu}t^{\nu}\right)^r\, {}_2\Psi_{m+1}\left[\begin{array}{r} (\gamma,\kappa),(\lambda,\rho);\\ (\nu r+\lambda,\rho),(\beta_j,\alpha_j)_{1,m}; \end{array}\ \mathfrak{d}t^{\rho}\right]. \tag{2.4}$$

Proof. Applying the Laplace transform (1.14) to the equation (2.3) and using the identity in Lemma 2.1, we obtain

$$\mathcal{N}(s) = \frac{N_0}{1+(c/s)^{\nu}} \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{\Gamma(\rho n+\lambda)}{s^{\rho n+\lambda}}\, \frac{\mathfrak{d}^n}{n!},$$

where, just as in the definition (1.14),

$$\mathcal{N}(s) := \mathcal{L}\left\{N(t):s\right\}.$$

Using the geometric series:

$$\frac{1}{1+(c/s)^{\nu}} = \sum_{r=0}^{\infty}(-1)^r \left(\frac{c}{s}\right)^{\nu r} \quad (|s|>c),$$

we find for $|p|>c$ that

$$\mathcal{N}(s) = N_0 \sum_{r=0}^{\infty}\left(-c^{\nu}\right)^r \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{\Gamma(\rho n+\lambda)}{s^{\rho n+\nu r+\lambda}}\, \frac{\mathfrak{d}^n}{n!}. \tag{2.5}$$

Now, by inverting the Laplace transform on each side of (2.5) and using the following well-known identity:

$$\begin{aligned} &\mathcal{L}\left\{t^{\nu}:s\right\} = \frac{\Gamma(\nu+1)}{s^{\nu+1}} \\ &\Longleftrightarrow \mathcal{L}^{-1}\left(\frac{1}{s^{\nu+1}}\right) = \frac{t^{\nu}}{\Gamma(\nu+1)} \quad \big(\Re(\nu)>-1;\ \Re(s)>0\big), \end{aligned} \tag{2.6}$$

we get

$$\begin{aligned} N(t) = N_0\, t^{\lambda-1} &\sum_{r=0}^{\infty}\left(-c^{\nu}t^{\nu}\right)^r \\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\kappa n)\,\Gamma(\rho n+\lambda)}{\Gamma(\gamma)\,\Gamma(\rho n+\nu r+\lambda)\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{\left(\mathfrak{d}t^{\rho}\right)^n}{n!}, \end{aligned}$$

which, in view of the definition (1.13) of the Fox-Wright function, leads us easily to the right-hand side of (2.4). This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let* $c, \mathfrak{d}, \nu \in \mathbb{R}^+$. *Also let* $\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j), \Re(\beta_j) > 0 \qquad (j \in \{1, \cdots, m\}),$$

$$\Re(\kappa) > 0 \quad \text{and} \quad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa) - 1\}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\beta_1 - 1}\, E_{\gamma,\kappa}\left[(\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m); \mathfrak{d}t^{\alpha_1}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \quad (2.7)$$

is given by

$$N(t) = \frac{N_0\, t^{\beta_1 - 1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r\, {}_1\Psi_m \left[\begin{matrix} (\gamma, \kappa); \\ (\nu r + \beta_1, \alpha_1), (\beta_j, \alpha_j)_{2,m}; \end{matrix} \; \mathfrak{d}t^{\alpha_1}\right]. \quad (2.8)$$

Proof. Proof of the result asserted by Theorem 2.4 runs parallel to that of Theorem 2.3. Here we use (2.2) instead of (2.1). The details are, therefore, being omitted. □

Remark 2.5. For $\kappa = \mathfrak{q} \in (0,\, 1) \cup \mathbb{N}$, the results in Theorem 2.3 and Theorem 2.4 reduce to those for the generalized fractional kinetic equation involving the generalized Mittag-Leffler function studied by Saxena *et al.* [33].

By setting $m = 1$ in (2.3), we get an interesting generalized fractional kinetic equation with its solution given by the following corollary.

Corollary 2.6. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$. *Also let* $\alpha, \beta, \gamma, \kappa \in \mathbb{C}$ *with*

$$\Re(\alpha) > \max\{0, \Re(\kappa) - 1\} \quad \text{and} \quad \min\{\Re(\beta), \Re(\kappa)\} > 0.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda - 1}\, E_{\alpha,\beta}^{\gamma,\kappa}\left[\mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \quad (2.9)$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda - 1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r\, {}_2\Psi_2 \left[\begin{matrix} (\gamma, \kappa), (\lambda, \rho); \\ (\nu r + \lambda, \rho), (\beta, \alpha); \end{matrix} \; \mathfrak{d}t^{\rho}\right], \quad (2.10)$$

where $E_{\alpha,\beta}^{\gamma,\kappa}[z]$ *is the generalized Mittag-Leffler function defined by* (1.9).

In its *further* special case when $\kappa = \mathfrak{q} \in (0,\,1) \cup \mathbb{N}$, Corollary 2.6 would reduce immediately to Corollary 2.7 below.

Corollary 2.7. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$. *Also let* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $\min\{\Re(\alpha), \Re(\beta)\} > 0$. *Suppose that* $\mathfrak{q} \in (0,\,1) \cup \mathbb{N}$. *Then the solution of the following generalized fractional kinetic equation*

$$N(t) - N_0\, t^{\lambda-1}\, E_{\alpha,\beta}^{\gamma,\mathfrak{q}}\left[\mathfrak{d}t^{\rho}\right] = -c^{\nu}\,{}_0D_t^{-\nu}\, N(t) \tag{2.11}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-1}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r\, {}_2\Psi_2 \left[\begin{array}{c} (\gamma, \mathfrak{q}), (\lambda, \rho); \\ (\nu r + \lambda, \rho), (\beta, \alpha); \end{array} \mathfrak{d}t^{\rho} \right], \tag{2.12}$$

where $E_{\alpha,\beta}^{\gamma,\mathfrak{q}}[z]$ *is the above-mentioned special case of the generalized Mittag-Leffler function in* (1.9) *when* $\kappa = \mathfrak{q} \in (0,\,1) \cup \mathbb{N}$.

Remark 2.8. The result asserted by Theorem 2.4 can also be suitably specialized to deduce solutions of certain generalized fractional kinetic equations analogous to those which are dealt with in Corollary 2.6 and Corollary 2.7.

3. Solution of generalized fractional kinetic equations by using the Sumudu transform

In this section we propose to investigate the solution of the generalized fractional kinetic equation involving the generalized Mittag-Leffler function (1.11) by applying the Sumudu transform technique. The following lemmas will be required in our derivations.

Lemma 3.1. *Let* $\min\{\Re(\lambda), \Re(\rho), \Re(u)\} > 0$. *Then the following Sumudu transform holds true*:

$$\begin{aligned} &\mathcal{S}\left[t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m); t^{\rho}\right]; u\right] \\ &= \frac{u^{\lambda-1}}{\Gamma(\gamma)}\, {}_2\Psi_m \left[\begin{array}{c} (\gamma, \kappa), (\lambda, \rho); \\ (\beta_j, \alpha_j)_{1,m}; \end{array} u^{\rho} \right]. \end{aligned} \tag{3.1}$$

Proof. By using (1.11), we readily have

$$\begin{aligned}\mathfrak{S} &:= \mathcal{S}\left[t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1)\,,\cdots,(\alpha_m,\beta_m)\,;\,t^{\rho}\right];u\right] \\ &= \int_0^{\infty} e^{-t}\,\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod\limits_{j=1}^{m}\Gamma\left(n\alpha_j+\beta_j\right)}\,\frac{(ut)^{\rho n+\lambda-1}}{n!}\,dt \\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod\limits_{j=1}^{m}\Gamma\left(n\alpha_j+\beta_j\right)}\,\frac{u^{\rho n+\lambda-1}}{n!}\,\int_0^{\infty} e^{-t}\,t^{\rho n+\lambda-1}\,dt. \qquad (3.2)\end{aligned}$$

This last integral in (3.2) can be evaluated by means of Euler's Gamma-function integral:

$$\int_0^{\infty} e^{-t}\,t^{z-1}\,dt=\Gamma\left(z\right) \qquad \big(\Re\left(z\right)>0\big). \qquad (3.3)$$

We thus find that

$$\begin{aligned}\mathfrak{S} &= \sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}\,\Gamma\left(\rho n+\lambda\right)}{\prod\limits_{j=1}^{m}\Gamma\left(n\alpha_j+\beta_j\right)}\,\frac{u^{\rho n+\lambda-1}}{n!} \\ &= \frac{u^{\lambda-1}}{\Gamma\left(\gamma\right)}\,\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+\kappa n\right)\Gamma\left(\rho n+\lambda\right)}{\prod\limits_{j=1}^{m}\Gamma\left(n\alpha_j+\beta_j\right)}\,\frac{u^{\rho n}}{n!},\end{aligned}$$

which, in view of (1.13), leads us to the right-hand side of (3.1). □

Remark 3.2. We find it to be convenient to record here a special case of (3.1) when $\lambda=\beta_1$ and $\rho=\alpha_1$ as Lemma 3.3 below.

Lemma 3.3. *Let* $\min\{\Re\left(\alpha_1\right),\Re\left(\beta_1\right),\Re\left(u\right)\}>0$. *Then the following Sumudu transform holds true*:

$$\begin{aligned}&\mathcal{S}\left[t^{\beta_1-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1)\,,\cdots,(\alpha_m,\beta_m)\,;\,t^{\alpha_1}\right];u\right] \\ &= \frac{u^{\beta_1-1}}{\Gamma\left(\gamma\right)}\,{}_1\Psi_{m-1}\left[\begin{matrix}(\gamma,\kappa)\,; \\ (\beta_j,\alpha_j)_{2,m}\,;\end{matrix}\; u^{\alpha_1}\right]. \qquad (3.4)\end{aligned}$$

Theorem 3.4. *Let* $c,\mathfrak{d},\nu,\lambda,\rho\in\mathbb{R}^{+}$ *and* $\Re\left(u\right)>0$ *with* $|u|<c^{-1}$ $(c\neq\mathfrak{d})$. *Also let* $\alpha_j,\beta_j,\gamma,\kappa\in\mathbb{C}$ *with*

$$\min\{\Re\left(\alpha_j\right),\Re\left(\beta_j\right)\}>0 \qquad \left(j\in\{1,\cdots,m\}\right),$$

$$\Re(\kappa) > 0 \quad \textit{and} \quad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\left[0, \Re(\kappa) - 1\right].$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda-1}\, E_{\gamma,\kappa}\left[(\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m); \mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{3.5}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r\, {}_2\Psi_{m+1}\left[\begin{array}{r} (\gamma, \kappa), (\lambda, \rho); \\ (\nu r + \lambda - 1, \rho), (\beta_j, \alpha_j)_{1,m}; \end{array} \mathfrak{d}t^{\rho}\right]. \tag{3.6}$$

Proof. Taking the Sumudu transform on both sides of (3.5) and using Lemma 3.1 and (1.20), we find that

$$\mathfrak{N}(u) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{\mathfrak{d}^n u^{\rho n + \lambda - 1}}{n!} - c^{\nu} u^{\nu}\, \mathfrak{N}(u), \tag{3.7}$$

where

$$\mathfrak{N}(u) := \mathcal{S}\left[N(t); u\right]. \tag{3.8}$$

Equivalently, we can write (3.7) as follows:

$$\mathfrak{N}(u) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{u^{\rho n + \lambda - 1}}{1 + c^{\nu} u^{\nu}} \frac{\mathfrak{d}^n}{n!}. \tag{3.9}$$

Using the binomial series expansion of $(1 + c^{\nu} u^{\nu})^{-1}$ in (3.9) and inverting the Sumudu transform on both sides of the resulting equation, we get

$$N(t) = N_0 \sum_{r=0}^{\infty} (-c^{\nu})^r \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\, \Gamma(\rho n + \lambda)}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)} \frac{\mathfrak{d}^n}{n!}\, \mathcal{S}^{-1}\left\{u^{\rho n + \nu r + \lambda - 1}\right\}.$$

Finally, we make use of the following formula:

$$\mathcal{S}^{-1}\{u^{\nu}\} = \frac{t^{\nu-1}}{\Gamma(\nu)} \qquad \big(\min\{\Re(\nu), \Re(u)\} > 0\big).$$

After some simplification, we thus find that

$$N(t) = N_0\, t^{\lambda-2} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r \times \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\kappa n)\,\Gamma(\rho n+\lambda)}{\Gamma(\gamma)\,\Gamma(\rho n+\nu r+\lambda-1)\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\, \frac{\mathfrak{d}^n t^{\rho n}}{n!},$$

which, in view of (1.13), leads us to the right-hand side of (3.6). This complete the proof of Theorem 3.4. □

Theorem 3.5. *Let* $c, \mathfrak{d}, \nu \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$. *Also let* $\alpha_j, \beta_j, \gamma, \kappa \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha_j), \Re(\beta_j)\} > 0 \qquad (j \in \{1, \cdots, m\}),$$

$$\Re(\kappa) > 0 \quad \text{and} \quad \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa)-1\}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\beta_1-1}\, E_{\gamma,\kappa}\left[(\alpha_1,\beta_1), \cdots, (\alpha_m,\beta_m);\, \mathfrak{d}t^{\alpha_1}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{3.10}$$

is given by

$$N(t) = \frac{N_0\, t^{\beta_1-2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} (-c^{\nu} t^{\nu})^r\, {}_1\Psi_m \left[\begin{matrix} (\gamma,\kappa); \\ (\nu r+\beta_1-1, \alpha_1), (\beta_j,\alpha_j)_{2,m}; \end{matrix}\ \mathfrak{d}t^{\rho}\right]. \tag{3.11}$$

Proof. Our demonstration of Theorem 3.5 would run parallel to that of Theorem 3.4. Here, in this case, we use (3.4) instead of (3.1). We, therefore, omit the details involved. □

Upon setting $m = 1$ in Theorem 3.4, we can deduce the following simpler result.

Corollary 3.6. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$. *Also let* $\alpha, \beta, \gamma, \kappa \in \mathbb{C}$ *with*

$$\Re(\alpha) > \max\{0, \Re(\kappa)-1\} \quad \text{and} \quad \min\{\Re(\beta), \Re(\kappa)\} > 0.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda-1}\, E_{\alpha,\beta}^{\gamma,\kappa}\left[\mathfrak{d}t^{\rho}\right] = -c^{\nu}\, {}_0D_t^{-\nu}\, N(t) \tag{3.12}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^{\nu r} {}_2\Psi_2 \left[\begin{array}{r} (\gamma, \kappa), (\lambda, \rho); \\ (\nu r + \lambda - 1, \rho), (\beta, \alpha); \end{array} \mathfrak{d} t^{\rho} \right]. \tag{3.13}$$

If we set $m = 1$ *and* $\kappa = \mathfrak{q} \in (0,1) \cup \mathbb{N}$ in Theorem 3.4, we are led easily to Corollary 3.7 below, which would follow also as a *further* special case of Corollary 3.6 when $\kappa = \mathfrak{q} \in (0,1) \cup \mathbb{N}$.

Corollary 3.7. *Let* $c, \mathfrak{d}, \nu, \lambda, \rho \in \mathbb{R}^+$ *and* $\Re(u) > 0$ *with* $|u| < c^{-1}$ $(c \neq \mathfrak{d})$. *Also let* $\alpha, \beta, \gamma \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha), \Re(\beta)\} > 0 \quad \textit{and} \quad \mathfrak{q} \in (0,1) \cup \mathbb{N}.$$

Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0\, t^{\lambda-1} E_{\alpha,\beta}^{\gamma,\mathfrak{q}} [\mathfrak{d} t^{\rho}] = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{3.14}$$

is given by

$$N(t) = \frac{N_0\, t^{\lambda-2}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^{r} {}_2\Psi_2 \left[\begin{array}{r} (\gamma, \mathfrak{q}), (\lambda, \rho); \\ (\nu r + \lambda - 1, \rho), (\beta, \alpha); \end{array} \mathfrak{d} t^{\rho} \right]. \tag{3.15}$$

We conclude this paper by remarking that the results presented here are general enough to yield, as their special cases, solutions of a number of known or new fractional kinetic equations involving such other special functions as (for example) those considered by Haubold and Mathai [10] and Saxena *et al.* [27, 28, 31]. Moreover, in our investigation here, our choice to make use of the Sumudu transform instead of the classical Laplace transform is prompted by the various problems considered here and also by the fact that the closed-form results derived here happen to be remarkably simpler (see also [39]).

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References

[1] M.A. Asiru, *Sumudu transform and the solution of integral equations of convolution type,* Internat. J. Math. Ed. Sci. Tech., **32** (2001), 906–910.

[2] F.B. M. Belgacem and A.A. Karaballi, *Sumudu transform fundamental properties investigations and applications,* J. Appl. Math. Stochast. Anal., **2006** (2006), Article ID 91083, 1–23.

[3] F.B.M. Belgacem, A.A. Karaballi and S.L. Kalla, *Analytical investigations of the Sumudu transform and applications to integral production equations,* Math. Probl. Engrg., **3** (2003), 103–118.

[4] V.B.L. Chaurasia and D. Kumar, *On the solution of generalized fractional kinetic equation,* Adv. Stud. Theoret. Phys., **4** (2010), 773–780.

[5] J. Choi and P. Agarwal, *A note on fractional integral operator associated with multiindex Mittag-Leffler functions,* Filomat, **30**(7) (2016), 1931–1939.

[6] J. Choi and D. Kumar, *Solutions of generalized fractional kinetic equations involving Aleph functions,* Math. Commun., **20** (2015), 113–123.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1954.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. **III**, McGraw-Hill Book Company, New York, Toronto and London, 1955.

[9] R. Gorenflo, Y. Luchko and F. Mainardi, *Analytic properties and applications of the Wright function,* Fract. Calc. Appl. Anal., **2** (1999), 383–414.

[10] H.J. Haubold and A.M. Mathai, *The fractional kinetic equation and thermonuclear functions,* Astrophys. Space Sci., **273** (2000), 53–63.

[11] H.J. Haubold, A.M. Mathai and R.K. Saxena, *Mittag-Leffler functions and their applications,* J. Appl. Math., **2011** (2011), Article ID 298628, 1–51.

[12] R. Hilfer, Fractional time evolution, in *Applications of Fractional Calculus in Physics* (R. Hilfer, Editor), World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000, pp. 87-130.

[13] R. Hilfer and L. Anton, *Fractional master equations and fractal time random walks,* Phys. Rev. E, **51** (1995), R848–R851.

[14] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[15] V. Kiryakova, *Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus,* J. Comput. Appl. Math., **118** (2000), 214–259.

[16] D. Kumar and S. Kumar, *Fractional integrals and derivatives of the generalized Mittag-Leffler type function,* Internat. Scholarly Res. Notice, **2014** (2014), Article ID 907432, 1–5.

[17] D. Kumar and S.D. Purohit, *Fractional differintegral operators of the generalized Mittag-Leffler type function,* Malaya J. Mat., **2** (2014), 419–425.

[18] D. Kumar, S.D. Purohit, A. Secer and A. Atangana, *On generalized fractional kinetic equations involving generalized Bessel function of the first kind,* Math. Probl. Engrg., **2015** (2015), Article ID 289387, 1–7.

[19] N.W. McLachlan, *Modern Operational Calculus with Applications in Technical Mathematics*, Macmillan, London, 1948.

[20] K.S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore, 1993.

[21] G.M. Mittag-Leffler, *Sur la nouvelle function* $E_\alpha(x)$*,* C. R. Acad. Sci. Paris, **137** (1903), 554–558.

[22] T.R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in kernel,* Yokohama Math. J., **19** (1971), 7–15.

[23] A. Saichev and M. Zaslavsky, *Fractional kinetic equations: solutions and applications,* Chaos, **7** (1997), 753–764.

[24] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives*: *Theory and Applications*, Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania), 1993.

[25] R.K. Saxena, J.P. Chauhan, R.K. Jana and A.K. Shukla, *Further results on the generalized Mittag-Leffler function operator,* J. Inequal. Appl., **75** (2015), 1–12.

[26] R.K. Saxena and S.L. Kalla, *On the solutions of certain fractional kinetic equations,* Appl. Math. Comput., **199** (2008), 504–511.

[27] R.K. Saxena, A.M. Mathai and H.J. Haubold, *On fractional kinetic equations,* Astrophys. Space Sci., **282** (2002), 281–287.

[28] R. K. Saxena, A. M. Mathai and H. J. Haubold, *On generalized fractional kinetic equations,* Physica A, **344** (2004), 653–664.

[29] R.K. Saxena, A.M. Mathai and H.J. Haubold, *Unified fractional kinetic equation and a fractional diffusion equation,* Astrophys. Space Sci., **290** (2004), 299–310.

[30] R.K. Saxena and K. Nishimoto, *N-Fractional calculus of generalized Mittag-Leffler functions,* J. Fract. Calc., **37** (2010), 43–52.

[31] R.K. Saxena, J. Ram and D. Kumar, *Alternative derivation of generalized kinetic equations, J. Fract. Calc. Appl.,* **4** (2013), 322–334.

[32] R.K. Saxena, J. Ram and D. Kumar, *Sumudu and Laplace transforms of the Aleph-function,* Caspian J. Appl. Math. Ecology Econ., **1**(2) (2013), 19–28.

[33] R.K. Saxena, J. Ram and M. Vishnoi, *Fractional differentiation and fractional integration of the generalized Mittag-Leffler function,* J. Indian Acad. Math., **32** (2010), 153–162.

[34] J.L. Schiff, *The Laplace Transform*: *Theory and Applications*, Springer-Verlag, Berlin, Heidelberg and New York, 1999.

[35] A.K. Shukla and J.C. Prajapati, *On a generalization of Mittag-Leffler function and its properties,* J. Math. Anal. Appl., **336** (2007), 797–811.

[36] H.M. Srivastava, *On an extension of the Mittag-Leffler function,* Yokohama Math. J., **16** (1968), 77–88.

[37] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[38] H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[39] H.M. Srivastava, A.K. Golmankhaneh, D. Baleanu and X.-J. Yang, *Local fractional Sumudu transform with applications to IVPs on Cantor sets,* Abstr. Appl. Anal., **2014** (2014), Article ID 620529, 1–7.

[40] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[41] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[42] H.M. Srivastava and Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel,* Appl. Math. Comput., **211** (2009), 198–210.

[43] Ž. Tomovski, R. Hilfer and H.M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions,* Integral Transforms Spec. Funct., **21** (2010), 797–814.

[44] G.K. Watugala, *Sumudu transform: A new integral transform to solve differential equations and control engineering problems,* Math. Engrg. Industr., **4** (1998), 319–329.

[45] A. Wiman, *Über de fundamental satz in der theorie der funktionen $E_\alpha(x)$,* Acta Math., **29** (1905), 191–201.

[46] X.-J. Yang, D. Baleanu and H.M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press (Elsevier Science Publishers), Amsterdam, Heidelberg, London and New York, 2016.

[47] G.M. Zaslavsky, *Fractional kinetic equation for Hamiltonian chaos,* Physica D, **78** (1994), 110–122.