**Nonlinear Functional Analysis and Applications**
**Vol. 15, No. 2 (2010), pp. 229-245**

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# EXISTENCE OF POSITIVE SOLUTIONS FOR SECOND ORDER SINGULAR IMPULSIVE STURM-LIOUVILLE BVP

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**Abstract.** By using the fixed point index theorem and the first eigenvalue, we establish the existence of one or two positive solutions for the impulsive singular Sturm-Liouville boundary value problem. In particular, we give a number of corollaries and an example to demonstrate the applications of the developed theory.

## 1. Introduction and Preliminaries

Sturm-Liouville boundary value problems play a very important role in both theory and application, which have been widely studied by many authors (see [1], [6],[8], [9],[13] and references therein). For example, Zhang and Liu [12] have established unique solution of initial value problems of nonlinear second order impulsive integral differential in Banach spaces. Sun and Zhang [8] have

[0]Received February 18, 2009. Revised October 27, 2009.
[0]2000 Mathematics Subject Classification: 34B16, 34B18, 34B24.
[0]Keywords: Singular Sturm-Liouville boundary value problem, impulsive, fixed point index, positive solution.
[0]The project is supported financially by the Natural Science Foundation of China (10871116).

applied the fixed point index theorem and the first eigenvalue to establish the existence of positive solutions.

Recently, in [7], Lin and Jiang studied the following second-order impulsive differential equation with no singularity

$$\begin{cases} u'' + f(t,u) = 0, \qquad 0 < t < 1, \\ -\Delta u'_{t=t_k} = I_k(u(t_k)), k = 1, 2, \cdots m, \\ u(0) = u(1) = 0, \end{cases}$$

and obtained two positive solutions by using the fixed point index theorems in cone.

Motivated by the work mentioned above, we study the positive solutions of nonlinear singular boundary value problems for impulsive Sturm-Liouville differential equation

$$\begin{cases} (p(t)u'(t))' + a(t)f(u(t)) = 0, \quad t \in J', \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), \\ \Delta u|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0+} p(t)u'(t) = 0, \\ \alpha_2 u(1) + \beta_2 \lim_{t\to 1-} p(t)u'(t) = 0, \end{cases} \tag{1}$$

where $J = (0,1), 0 < t_1 < t_2 < \cdots < t_m < 1, J' = J \setminus \{t_1, t_2, \cdots, t_m\}, \bar{J} = [0,1], J_0 = (0,t_1], J_1 = (t_1,t_2], \cdots, J_M = (t_m,1).I_k, \bar{I}_k \in C(\mathbb{R}^+, \mathbb{R}^+), \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-), \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), u'(t_k^+), u(t_k^+), (u'(t_k^-), u(t_k^-))$ denote the right limit (left limit) of $u'(t)$ and $u(t)$ at $t = t_k$, respectively.

$\alpha_i \geq 0, \beta_i \geq 0 \ \ (i = 1,2),\ f \in C(\mathbb{R}^+, \mathbb{R}^+), a(t) \in C(J, \mathbb{R}^+)$ is allowed to be singular at $t = 0$ or $t = 1$, $\mathbb{R}^+ = [0, +\infty), p(t) \in C([0,1], \mathbb{R}^+) \bigcap C^1(J, \mathbb{R}^+)$ and $\int_0^1 \frac{ds}{p(s)} < +\infty, \rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 \int_0^1 \frac{ds}{p(s)} > 0.$

By applying the fixed point index theorem, we shall establish the existence of one or two positive solutions for the above problems, which improve and generalize the corresponding results of papers [1]-[13].

The rest of this paper is organized as follows. In Section 1, we provide some preliminaries and establish several lemmas. In Section 2, the main results are formulated and proved and we give a number of corollaries, In Section 3, we give an example to demonstrate the application of the developed theory.

Now we denote the Green's functions for the following boundary value problems

$$\begin{cases} (p(t)u'(t))' = 0, \quad 0 \leq t \leq 1, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0+} p(t)u'(t) = 0, \\ \alpha_2 u(1) + \beta_2 \lim_{t\to 1-} p(t)u'(t) = 0, \end{cases}$$

by $G(t,s)$. It is well known that $G(t,s)$ can be written by

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\beta_1 + \alpha_1 B(0,s))\,(\beta_2 + \alpha_2 B(t,1))\,, & 0 \le s \le t \le 1, \\ (\beta_1 + \alpha_1 B(0,t))\,(\beta_2 + \alpha_2 B(s,1))\,, & 0 \le t \le s \le 1, \end{cases} \tag{2}$$

where $B(t,s) = \int_t^s \frac{d\tau}{p(\tau)}$, $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0,1)$. It is easy to verify the following properties of $G(t,s)$

(I) $G(t,s) \le G(s,s) \le \frac{1}{\rho}\,(\beta_1 + \alpha_1 B(0,1))\,(\beta_2 + \alpha_2 B(0,1)) < +\infty$,

(II) $G(t,s) \ge \sigma G(s,s)$, for any $t \in [a,b], s \in [0,1]$, where $a \in (0,t_1], b \in [t_m, 1)$ and

$$0 < \sigma = \min\left\{\frac{\beta_2 + \alpha_2 B(b,1)}{\beta_2 + \alpha_2 B(0,1)}, \frac{\beta_1 + \alpha_1 B(0,a)}{\beta_1 + \alpha_1 B(0,1)}\right\} < 1. \tag{3}$$

We denote the first eigenvalue and the corresponding eigenfunction of

$$-(p(t)\phi'(t))' = \lambda\phi(t)a(t), \alpha_1\phi(0) - \beta_1 \lim_{t\to 0+} p(t)u'(t) = 0,$$

$$\alpha_2\phi(1) + \beta_2 \lim_{t\to 1-} p(t)u'(t) = 0,$$

by $\lambda_1$, $\phi_1(t)$. It is well known that $\lambda_1 > 0$ and $\phi_1(t)$ does not change sign in (0,1) and therefore, without loss of generality, we assume that $\phi_1(t) > 0$ for $0 < t < 1$ and $\|\phi_1\| = \max_{0\le t\le 1} |\phi_1(t)| = 1$.

For convenience and simplicity in the following discussion, we denote

$$f_0 = \liminf_{x\to 0^+} \min_{t\in[a,b]} \frac{f(t,x)}{x}, \quad I_0(k) = \liminf_{x\to 0^+} \frac{I_k(x)}{x}, \quad \overline{I}_0(k) = \liminf_{x\to 0^+} \frac{\overline{I}_k(x)}{x},$$

$$f_\infty = \liminf_{x\to \infty} \min_{t\in[a,b]} \frac{f(t,x)}{x}, \quad I_\infty(k) = \liminf_{x\to \infty} \frac{I_k(x)}{x}, \quad \overline{I}_\infty(k) = \liminf_{x\to \infty} \frac{\overline{I}_k(x)}{x},$$

$$f^\infty = \limsup_{x\to \infty} \max_{t\in[a,b]} \frac{f(t,x)}{x}, \quad I^\infty(k) = \limsup_{x\to \infty} \frac{I_k(x)}{x}, \quad \overline{I}^\infty(k) = \limsup_{x\to \infty} \frac{\overline{I}_k(x)}{x},$$

$$f^0 = \limsup_{x\to 0^+} \max_{t\in[a,b]} \frac{f(t,x)}{x}, \quad I^0(k) = \limsup_{x\to 0^+} \frac{I_k(x)}{x}, \quad \overline{I}^0(k) = \limsup_{x\to 0^+} \frac{\overline{I}_k(x)}{x}.$$

$(H_1)$

$$f_0 + \frac{\sigma \sum_{k=1}^{m} (I_0(k)\phi_1(t_k) + \overline{I}_0(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a(t)dt} > \lambda_1,$$

$$f_\infty + \frac{\sigma \sum_{k=1}^{m} (I_\infty(k)\phi_1(t_k) + \overline{I}_\infty(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a(t)dt} > \lambda_1.$$

$(H_2)$

$$f^0 + \frac{\sum\limits_{k=1}^{m}(I^0(k)\phi_1(t_k) + \overline{I}^0(k)\phi_1'(t_k))p(t_k)}{\sigma\int_0^1 \phi_1(t)a(t)dt} < \lambda_1,$$

$$f^\infty + \frac{\sum\limits_{k=1}^{m}(I^\infty(k)\phi_1(t_k) + \overline{I}^\infty(k)\phi_1'(t_k))p(t_k)}{\sigma\int_0^1 \phi_1(t)a(t)dt} < \lambda_1.$$

$(H_3)$ There exist $p > 0, \eta, \eta_k, \overline{\eta}_k \geq 0$ such that for all $0 < x \leq p$ and $0 \leq t \leq 1, f(t,x) \leq \eta p, I_k(x) \leq \eta_k p, \overline{I}_k(x) \leq \overline{\eta}_k p$, and

$$\eta + \sum_{k=1}^{m}(\eta_k + \overline{\eta}_k) > 0, \eta\int_0^1 G(s,s)a(s)ds + \sum_{k=1}^{m} G(t_k,t_k)(\eta_k + \overline{\eta}_k) < 1.$$

$(H_4)$ There exist $p > 0, \lambda, \lambda_k, \overline{\lambda}_k \geq 0$ such that for all $\sigma p \leq x \leq p$ and $0 \leq t \leq 1, f(t,x) \geq \lambda p, I_k(x) \geq \lambda_k p, \overline{I}_k(x) \geq \overline{\lambda}_k p$, and

$$\lambda + \sum_{0<t_k<\frac{1}{2}}(\lambda_k + \overline{\lambda}_k) > 0, \lambda\int_a^b G(\frac{1}{2},s)a(s)ds + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2},t_k)(\lambda_k + \overline{\lambda}_k) > 1.$$

$(H_5)$ $0 < \int_0^1 G(s,s)a(s)ds < \infty$.

Let$X = C[\bar{J}, \mathbb{R}^+]$ denote the Banach space of all continuous mapping $x : \bar{J} \to \mathbb{R}^+$ with norm $\|x\| = \sup_{t\in\bar{J}}|x(t)|$, for $k = 1, 2, \cdots, m$. Let

$$\begin{aligned} PC[\bar{J}, \mathbb{R}^+] &= \{x : x \text{ is a map from } \bar{J} \text{ into } \mathbb{R}^+ \text{ s.t. } x(t) \text{ is continuous} \\ &\quad \text{at } t \neq t_k, \text{left continuous at } t = t_k \text{ and its right limit exists }\}, \end{aligned}$$

$$\begin{aligned} PC^1[\bar{J}, \mathbb{R}^+] &= \{x : x \text{ is a map from } \bar{J} \text{ into } \mathbb{R}^+ \text{ such that } x'(t) \\ &\quad \text{is continuous at } t \neq t_k, \text{left continuous at } t = t_k \\ &\quad \text{and its right limit } x'(t_k^+) \text{ exists at } t = t_k\}. \end{aligned}$$

Then they are Banach spaces with the norm $\|x\|_{PC} = \sup_{t\in\bar{J}}|x(t)|$, and $\|x\|_{PC'} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Let $K$ be a cone in $X = PC[\bar{J}, \mathbb{R}^+]$ defined by

$$K = \{x \in PC[\bar{J}, \mathbb{R}^+] : x(t) \geq 0, t \in [0,1], \text{and } x(t) \geq \sigma\|x\|_{PC}, t \in [a,b]\}$$

**Definition 1.1.** A function

$$x(t) \in PC^1[\bar{J}, \mathbb{R}^+] \cap C^2(J', \mathbb{R}), p(t)x'(t) \in C^1([0,1], \mathbb{R}),$$

is called a solution of $BVP(1)$ if it satisfies $BVP(1)$.

Next, let us define an operator $\Phi : X \to X$ by

$$\begin{aligned}\Phi(u)(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\quad + \sum_{0<t_k<t} G(t,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))), \quad t \in [0,1].\end{aligned}$$

Clearly, by $(H_1)$ and $(H_5)$, we know that the operator $\Phi$ is well defined, and $u$ is a positive solution of the $BVP(1)$ if and only if $u$ is a positive fixed point of the operator $\Phi$.

**Lemma 1.1.** $\Phi(K) \subset K$. $K$ is a cone defined before.

*Proof.* We show that for any $u \in K$

$$\Phi u(t) \geq \sigma \|\Phi u(t)\|_{PC}, t \in [a,b].$$

For any $u \in K$, from the property (**I**) of $G(t,s)$, we know that

$$\|\Phi\|_{PC} \leq \int_0^1 G(s,s)a(s)f(u(s))ds + \sum_{0<t_k<t} G(t_k,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))). \tag{4}$$

On the other hand, by the property (**II**) of $G(t,s)$, for any $t \in [a,b]$, we have

$$\begin{aligned}&\Phi u(t) \\ &= \int_0^1 G(t,s)a(s)f(u(s))ds + \sum_{0<t_k<t} G(t,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\ &\geq \sigma \int_0^1 G(s,s)a(s)f(u(s))ds + \sigma \sum_{0<t_k<t} G(t_k,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\ &= \sigma \left\{ \int_0^1 G(s,s)a(s)f(u(s))ds + \sum_{0<t_k<t} G(t_k,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \right\}.\end{aligned} \tag{5}$$

It follows from (4) and (5) that for any $u \in K$,

$$\Phi u(t) \geq \sigma \|\Phi u(t)\|_{PC}, t \in [a,b].$$

Thus, $\Phi u \in K$. Therefore, $\Phi(K) \subset K$. □

**Lemma 1.2.** $\Phi : K \to K$ is a completely continuous operator.

*Proof.* For any $n \geq 2$, we defined a continuous function $a_n$ by

$$a_n(t) = \begin{cases} \inf\left\{a(t), a(\frac{1}{n})\right\}, & 0 < t \leq \frac{1}{n}, \\ a(t), & \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, \\ \inf\left\{a(t), a(1-\frac{1}{n})\right\}, & 1 - \frac{1}{n} \leq t \leq 1. \end{cases}$$

Next, for $n \geq 2$, we define an operator $\Phi_n : K \to K$ by

$$\begin{aligned}\Phi_n u(t) &= \int_0^1 G(t,s)a_n(s)f(u(s))ds \\ &\quad + \sum_{0<t_k<t} G(t,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))), \quad t \in [0,1].\end{aligned}$$

Obviously, for any $n \geq 2$, $\Phi_n$ is completely continuous on $K$ by an application of the Ascoli-Arzela theorem (see [3]). Then $\|\Phi_n - \Phi\|_{PC} \to 0$, as $n \to +\infty$. In fact, for any $u \in B_1 = \{u \in K : \|u\|_{PC} \leq 1\}$, from $(H_1)$, $(H_5)$ and the property (**I**) of $G(t,s)$, we obtain

$$\begin{aligned}\|\Phi_n u - \Phi u\|_{PC} &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)[a(s) - a_n(s)]f(u(s))ds \right| \\ &\leq \int_0^{\frac{1}{n}} G(s,s)|a(s) - a_n(s)|f(u(s))ds \\ &\quad + \int_{1-\frac{1}{n}}^1 G(s,s)|a(s) - a_n(s)|f(u(s))ds \\ &\leq M \int_0^{\frac{1}{n}} G(s,s)|a(s) - a_n(s)|ds \\ &\quad + M \int_{1-\frac{1}{n}}^1 G(s,s)|a(s) - a_n(s)|ds \\ &\to 0, \ n \to +\infty,\end{aligned}$$

where $M = \max\limits_{0\leq x\leq 1} f(x)$. Hence $\|\Phi_n - \Phi\|_{PC} \to 0$, as $n \to +\infty$. Therefore, $\Phi$ is completely continuous. This completes the proof of Lemma 2.2. $\square$

For $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$ and $\partial K_r = \{x \in K : \|x\| = r\}$. The following lemma is needed in this paper.

**Lemma 1.3.** [3] Let $\Phi : K \to K$ be a completely continuous operator, assume $\Phi x \neq x$ for every $x \in \partial K_r$. Then the following conclusions hold.

(i) if $\|x\| \leq \|\Phi x\|$ for $x \in \partial K_r$, then $i(\Phi, K_r, K) = 0$;

(ii) if $\|x\| \geq \|\Phi x\|$ for $x \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.

**Lemma 1.4.** ([2],[3]) Let $\Phi : K \to K$ be a completely continuous mapping and $\mu\Phi x \neq x$ for $x \in \partial K_r$ and $0 < \mu \leq 1$. Then $i(\Phi, K_r, K) = 1$.

**Lemma 1.5.** ([2],[3]) Let $\Phi : K \to K$ be a completely continuous mapping. Suppose the following two conditions are satisfied.

(i) $\inf_{x\in\partial K_r} \|\Phi x\| > 0$

(ii) $\mu\Phi x \neq x$ for every $x \in \partial K_r$ and $\mu \geq 1$.
Then $i(\Phi, K_r, K) = 0$.

## 2. Existence of positive solutions and some corollaries

**Theorem 2.1** Suppose that $(H_1)$ and $(H_3)$ hold. Then $BVP(1)$ has at least two positive solutions $u_1$ and $u_2$, such that $0 \leq \|u_1\|_{pc} \leq p \leq \|u_2\|_{pc}$.

*Proof.* The first step, suppose that $(H_3)$ holds, then $i(\Phi, K_p, K) = 1$.
In fact let $u \in K$ with $\|u\|_{pc} = p$. From $(H_3)$ we have

$$\begin{aligned}
\|\Phi u\|_{pc} &\leq \int_0^1 G(s,s)a(s)f(s,u(s))ds + \sum_{k=1}^m G(t_k,t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\
&\leq p\left(\eta \int_0^1 G(s,s)a(s)ds + \sum_{k=1}^m G(t_k,t_k)(\eta_k) + \bar{\eta}_k)\right) \\
&< p \\
&= \|u\|_{pc}.
\end{aligned}$$

That is $\|u\|_{pc} \geq \|\Phi u\|_{pc}$ for $u \in \partial K_p$. Therefore, by Lemma 1.3, we obtain

$$i(\Phi, K_p, K) = 1. \tag{6}$$

The second step, we prove that there exists $0 < r < p$ such that

$$i(\Phi, K_r, K) = 0.$$

We first show $\inf_{u \in \partial K_r} \|\Phi u\| > 0$. By $(H_1)$, there exists $0 < \varepsilon_0 < 1$ such that

$$\begin{aligned}
(1-\varepsilon_0)\left(f_0 + \frac{\sigma \sum\limits_{k=1}^{\infty}(I_0(k)\phi_1(t_k) + \overline{I}_0(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a(t)dt}\right) > \lambda_1, \\
(1-\varepsilon_0)\left(f_\infty + \frac{\sigma \sum\limits_{k=1}^{m}(I_\infty(k)\phi_1(t_k) + \overline{I}_\infty(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a(t)dt}\right) > \lambda_1.
\end{aligned} \tag{7}$$

By the definitions of $f_0, I_0$ and $\overline{I}_0$, there exists $0 < r_0 < p$ such that for any $t \in [a,b], 0 \leq x \leq r_0$,

$$f(t,x) \geq f_0(1-\varepsilon_0)x, \quad I_k(x) \geq I_0(k)(1-\varepsilon_0)x, \quad \overline{I}_k(x) \geq \overline{I}_0(k)(1-\varepsilon_0)x. \tag{8}$$

Let $r \in (0, r_0)$. Then for $u \in \partial K_r$, we have

$$r_0 > \|u\|_{pc} \geq u(t) \geq \sigma\|u\|_{pc} = \sigma r > 0, \quad t \in [a,b]. \tag{9}$$

So, by (8)and (9) we get

$$\begin{aligned} \|\Phi u\|_{pc} &\geq \Phi u(\tfrac{1}{2}) = \int_0^1 G(\tfrac{1}{2},s)a(s)f(s,u(s))ds \\ &\quad + \sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2},t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\ &\geq \int_a^b G(\tfrac{1}{2},s)a(s)f(s,u(s))ds \\ &\quad + \sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2},t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\ &\geq f_0(1-\varepsilon_0)\int_a^b G(\tfrac{1}{2},s)a(s)u(s)ds \\ &\quad +(1-\varepsilon_0)\sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2},t_k)\,(I_0(k)(u(t_k)) \\ &\quad +\bar{I}_0(k)(u(t_k))) \\ &\geq (1-\varepsilon_0)\sigma r(f_0\int_a^b G(\tfrac{1}{2},s)a(s)ds \\ &\quad + \sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2},t_k)(I_0(k) + \bar{I}_0(k))) > 0, \end{aligned} \tag{10}$$

this implies that $\inf_{u\in\partial K_r}\|\Phi u\| > 0$.

Next we show $\mu\Phi u \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$. If it is not true, then there exist $u_0 \in \partial K_r$ and $\mu_0 \geq 1$ such that $\mu_0\Phi u_0 = u_0$. It is easy to see that $u_0(t)$ satisfies

$$\begin{cases} (p(t)u_0'(t))' + \mu_0 a(t)f(u_0(t)) = 0, \quad t \in J', \\ -\Delta u_0'|_{t=t_k} = I_k(u_0(t_k)), \\ \Delta u_0|_{t=t_k} = \bar{I}_k(u_0(t_k)), \quad k = 1,2,\cdots,m, \\ \alpha_1 u_0(0) - \beta_1 \lim_{t\to 0+} p(t)u_0'(t) = 0, \\ \alpha_2 u_0(1) + \beta_2 \lim_{t\to 1-} p(t)u_0'(t) = 0. \end{cases} \tag{11}$$

Multiplying $(p(t)u_0'(t))'$ by $\phi_1(t)$ and integrating the product from $a$ to $b$ with respect to $t$, then we get

$$\begin{aligned} &\int_a^b (p(t)u_0'(t))'\phi_1(t)dt \\ = \ &\int_a^{t_1} \phi_1(t)d(p(t)u_0'(t)) + \sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}} \phi_1(t)d(p(t)u_0'(t)) \\ &+ \int_{t_m}^b \phi_1(t)d(p(t)u_0'(t)) \\ = \ &\phi_1(t_1)p(t_1)u_0'(t_1-0) - \int_a^{t_1}\phi_1'(t)p(t)u_0'(t)dt \\ &+ \sum_{k=1}^{m-1}(\phi_1(t_{k+1})p(t_{k+1})u_0'(t_{k+1}-0) - \phi_1(t_k)p(t_k)u_0'(t_k+0)) \\ &- \sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}}\phi_1'(t)p(t)u_0'(t)dt - \phi_1(t_m)p(t_m)u_0'(t_m+0) \\ &- \int_{t_m}^b \phi_1'(t)p(t)u_0'(t)dt \\ = \ &-\sum_{k=1}^{m}\Delta u_0'(t_k)\phi_1(t_k)p(t_k) - \int_a^b p(t)u_0'(t)\phi_1'(t)dt. \end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned}&\int_a^b p(t)u_0'(t)\phi_1'(t)dt\\&=\int_a^{t_1}\phi_1'(t)p(t)du_0(t)+\sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}}\phi_1'(t)p(t)du_0(t)+\int_{t_m}^b\phi_1'(t)p(t)du_0(t)\\&=-\sum_{k=1}^{m}\Delta u_0(t_k)\phi_1'(t_k)p(t_k)-\int_a^b(p(t)\phi_1'(t))'u_0(t)dt\\&=-\sum_{k=1}^{m}\Delta u_0(t_k)\phi_1'(t_k)p(t_k)+\lambda_1\int_a^b a(t)\phi_1(t)u_0(t)dt.\end{aligned}\tag{13}$$

Then, from (12) and (13), we get

$$\begin{aligned}&\int_a^b(p(t)u_0'(t))'\phi_1(t)dt\\=\;&-\sum_{k=1}^{m}\Delta u_0'(t_k)\phi_1(t_k)p(t_k)+\sum_{k=1}^{m}\Delta u_0(t_k)\phi_1'(t_k)p(t_k)\\&-\lambda_1\int_a^b a(t)\phi_1(t)u_0(t)dt\\=\;&\mu_0\sum_{k=1}^{m}\left(I_k(u_0(t_k))\phi_1(t_k)+\overline{I}_k(u_0(t_k))\phi_1'(t_k)\right)p(t_k)\\&-\lambda_1\int_a^b a(t)\phi_1(t)u_0(t)dt.\end{aligned}\tag{14}$$

From (11), we obtain $(p(t)u_0'(t))'=-\mu_0 a(t)f(u_0(t))$, so

$$\int_a^b(p(t)u_0'(t))'\phi_1(t)dt=-\mu_0\int_a^b\phi_1(t)a(t)f(u_0(t))dt.\tag{15}$$

Then, from (14) and (15), we get

$$\begin{aligned}\lambda_1\int_a^b a(t)\phi_1(t)u_0(t)dt&=\mu_0\sum_{k=1}^{m}\left(I_k(u_0(t_k))\phi_1(t_k)+\overline{I}_k(u_0(t_k))\phi_1'(t_k)\right)p(t_k)\\&\quad+\mu_0\int_a^b\phi_1(t)a(t)f(u_0(t))dt\\&\ge(1-\varepsilon_0)\sum_{k=1}^{m}\left(I_0(k)\phi_1(t_k)+\overline{I}_0(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k)\\&\quad+(1-\varepsilon_0)f_0\int_a^b\phi_1(t)a(t)u_0(t)dt.\end{aligned}\tag{16}$$

Since $u_0(t)\ge\sigma\|u\|_{pc}>0$ for all $t\in[a,b]$, we have

$$\int_a^b\phi_1(t)a(t)u_0(t)dt>0$$

and

$$\sum_{k=1}^{m}\left(I_0(k)\phi_1(t_k)+\overline{I}_0(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k)>0.$$

By (16) we know $\lambda_1 > (1-\varepsilon_0)f_0$, and hence we obtain

$$\begin{aligned}
&(\lambda_1-(1-\varepsilon_0)f_0)\int_a^b \phi_1(t)a(t)\|u_0(t)\|dt\\
&\geq (\lambda_1-(1-\varepsilon_0)f_0)\int_a^b \phi_1(t)a(t)u_0(t)dt\\
&\geq (1-\varepsilon_0)\sum_{k=1}^{m}\left(I_0(k)\phi_1(t_k)+\overline{I}_0(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k)\\
&\geq (1-\varepsilon_0)\sigma\|u_0(t)\|\sum_{k=1}^{m}\left(I_0(k)\phi_1(t_k)+\overline{I}_0(k)\phi_1'(t_k)\right)p(t_k).
\end{aligned} \tag{17}$$

This implies that

$$(\lambda_1-(1-\varepsilon_0)f_0)\int_a^b \phi_1(t)a(t)dt \geq (1-\varepsilon_0)\sigma\sum_{k=1}^{m}\left(I_0(k)\phi_1(t_k)+\overline{I}_0(k)\phi_1'(t_k)\right)p(t_k).$$

So,

$$\lambda_1 \geq (1-\varepsilon_0)\left(f_0+\frac{\sigma\sum_{k=1}^{m}\left(I_0(k)\phi_1(t_k)+\overline{I}_0(k)\phi_1'(t_k)\right)p(t_k)}{\int_a^b \phi_1(t)a(t)dt}\right)$$

which is a contradiction with (7). So we obtain $\mu\Phi u \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$. Hence, by Lemma 2.5, we get

$$i(\Phi, K_r, K) = 0. \tag{18}$$

The third step, we prove that there exists large enough $R$ such that

$$i(\Phi, K_R, K) = 0.$$

Firstly, we show $\inf_{u\in\partial K_R}\|\Phi u\| > 0$. From the definitions of $f_\infty, I_\infty$ and $\overline{I}_\infty$, there exists $H > p > 0$ such that for any $t \in [a,b]$ and $x \geq H$,

$$f(t,x) \geq f_\infty(1-\varepsilon_0)x, \quad I_k(x) \geq I_\infty(k)(1-\varepsilon_0)x, \quad \overline{I}_k(x) \geq \overline{I}_\infty(k)(1-\varepsilon_0)x. \tag{19}$$

Let

$$\begin{aligned}
c \;=\; &\max_{0\leq x\leq H}\max_{a\leq t\leq b}|f(t,x)-f_\infty(1-\varepsilon_0)x|\\
&+\sum_{k=1}^{m}\max_{0\leq x\leq H}|I_k(x)-I_\infty(k)(1-\varepsilon_0)x|\\
&+\sum_{k=1}^{m}\max_{0\leq x\leq H}|\overline{I}_k(x)-\overline{I}_\infty(k)(1-\varepsilon_0)x|.
\end{aligned} \tag{20}$$

Then, from (19) and (20), we have

$$\begin{aligned}
&f(t,x) \geq f_\infty(1-\varepsilon_0)x-c, \quad I_k(x) \geq I_\infty(k)(1-\varepsilon_0)x-c,\\
&\overline{I}_k(x) \geq \overline{I}_\infty(k)(1-\varepsilon_0)x-c, \quad \text{for all } \; t\in[a,b], x>0.
\end{aligned} \tag{21}$$

Choose $R > R_0 = \max\{\frac{H}{\sigma}, p\}$. Let $u \in \partial K_R$. Then $u(t) \geq \sigma\|u\|_{pc} = \sigma R > H$ for all $t \in [a,b]$, by (19) and (**II**) we have

$$f(t,x) \geq f_\infty(1-\varepsilon_0)\sigma R, \quad I_k(x) \geq I_\infty(k)(1-\varepsilon_0)\sigma R, \quad \overline{I}_k(x) \geq \overline{I}_\infty(k)(1-\varepsilon_0)\sigma R.$$

Proceeding as in second step, we can get $\inf_{u\in\partial K_R}\|\Phi u\|>0$.

Secondly, we show that if $R$ is large enough, then we have $\mu\Phi u\neq u$ for every $u\in\partial K_R$ and $\mu\geq 1$. In fact, if it is not true, then there exist $u_0\in\partial K_R$ and $\mu_0\geq 1$ such that $\mu_0\Phi u_0=u_0$. It is easy to see that $u_0(t)$ satisfies (11), and similar to the analysis in second step, by (24), we obtain

$$\begin{aligned}\lambda_1\int_a^b a(t)\phi_1(t)u_0(t)dt &= \mu_0\sum_{k=1}^m\left(I_k(u_0(t_k))\phi_1(t_k)+\overline{I}_k(u_0(t_k))\phi_1'(t_k)\right)p(t_k)\\ &\quad+\mu_0\int_a^b\phi_1(t)a(t)f(u_0(t))dt\\ &\geq(1-\varepsilon_0)\sum_{k=1}^m\left(I_\infty(k)\phi_1(t_k)+\overline{I}_\infty(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k)\\ &\quad+(1-\varepsilon_0)f_\infty\int_a^b\phi_1(t)a(t)u_0(t)dt\\ &\quad-c\left(\sum_{k=1}^m\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b\phi_1(t)a(t)dt\right).\end{aligned}$$

(I) If $(1-\varepsilon_0)f_\infty\leq\lambda_1$, then

$$\begin{aligned}&(\lambda_1-(1-\varepsilon_0)f_\infty)\int_a^b\phi_1(t)a(t)u_0(t)dt+c\left(\sum_{k=1}^m\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)\right.\\ &\left.+\int_a^b\phi_1(t)a(t)dt\right)\geq(1-\varepsilon_0)\sum_{k=1}^m\left(I_\infty(k)\phi_1(t_k)+\overline{I}_\infty(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k),\end{aligned}$$

such that

$$\begin{aligned}&\|u_0\|_{pc}(\lambda_1-(1-\varepsilon_0)f_\infty)\int_a^b\phi_1(t)a(t)dt+c\Big(\sum_{k=1}^m\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)\\ &+\int_a^b\phi_1(t)a(t)dt\Big)\geq(1-\varepsilon_0)\sigma\|u_0\|_{pc}\sum_{k=1}^m\left(I_\infty(k)\phi_1(t_k)+\overline{I}_\infty(k)\phi_1'(t_k)\right)p(t_k).\end{aligned}$$

This implies

$$\|u_0\|_{pc}\leq\frac{c\left(\sum_{k=1}^m\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b\phi_1(t)a(t)dt\right)}{(1-\varepsilon_0)\sigma\sum_{k=1}^m\left(I_\infty(k)\phi_1(t_k)+\overline{I}_\infty(k)\phi_1'(t_k)\right)p(t_k)-(\lambda_1-(1-\varepsilon_0)f_\infty)\int_a^b\phi_1(t)a(t)dt}=:R_1.$$

(II) If $(1-\varepsilon_0)f_\infty>\lambda_1$, then

$$\begin{aligned}&c\left(\sum_{k=1}^m\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b\phi_1(t)a(t)dt\right)\\ &\geq\left((1-\varepsilon_0)f_\infty-\lambda_1\right)\int_a^b\phi_1(t)a(t)u_0(t)dt\\ &\geq((1-\varepsilon_0)f_\infty-\lambda_1)\sigma\|u_0\|_{pc}\int_a^b\phi_1(t)a(t)dt.\end{aligned}$$

Thus

$$\|u_0\|_{pc}\leq\frac{c\left(\sum_{k=1}^m\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b\phi_1(t)a(t)dt\right)}{((1-\varepsilon_0)f_\infty-\lambda_1)\sigma\int_a^b\phi_1(t)a(t)dt}=:R_2.$$

Let $R > \max\{R_0, R_1, R_2\}$. Then for all $u \in \partial K_R$ and $\mu \geq 1$, $\mu\Phi u \neq u$. Hence, by Lemma 2.5, we have

$$i(\Phi, K_R, K) = 0. \tag{22}$$

By (6), (18), (25) and the property of the fixed points index, we obtain $i(\Phi, K_R \backslash \overline{K}_P, K) = -1$, $i(\Phi, K_p \backslash \overline{K}_r, K) = 1$. Thus, $BVP(1)$ has at least two positive solutions $u_1$ and $u_2$ satisfying $0 \leq \|u_1\|_{pc} \leq p \leq \|u_2\|_{pc}$. □

**Theorem 2.2.** Suppose that $(H_2)$ and $(H_4)$ hold. Then $BVP(1)$ has at least two positive solutions $u_1$ and $u_2$ satisfying $0 \leq \|u_1\|_{pc} \leq p \leq \|u_2\|_{pc}$.

*Proof.* The first step, suppose that $(H_4)$ holds, then $i(\Phi, K_p, K) = 0$.

Let $u \in K$ with $\|u\|_{pc} = p$. From $(H_4)$ we have

$$\begin{aligned}
\|\Phi u\|_{pc} &\geq \Phi u(\tfrac{1}{2}) \\
&= \int_0^1 G(\tfrac{1}{2}, s)a(s)f(s, u(s))ds + \sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2}, t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\
&\geq \int_a^b G(\tfrac{1}{2}, s)a(s)f(s, u(s))ds + \sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2}, t_k)(I_k(u(t_k)) + \bar{I}_k(u(t_k))) \\
&\geq p \left( \lambda \int_a^b G(\tfrac{1}{2}, s)a(s)ds + \sum_{0<t_k<\frac{1}{2}} G(\tfrac{1}{2}, t_k) \left(\lambda_k + \overline{\lambda}_k\right) \right) \\
&> p \\
&= \|u\|_{pc}.
\end{aligned}$$

This implies that for any $u \in \partial K_p$, we have $\|u\| \leq \|\Phi u\|$. Therefore, by Lemma 1.3, we obtain

$$i(\Phi, K_p, K) = 0. \tag{23}$$

The second step, suppose that $0 < r < p$ holds. Then $i(\Phi, K_r, K) = 1$. In fact, by $(H_2)$, there exists $0 < \varepsilon_1 < \min\{\lambda_1 - f^0, \lambda_1 - f^\infty\}$ such that

$$\begin{aligned}
(\lambda_1 - \varepsilon_1 - f^0)\sigma \int_a^b \phi_1(t)a(t)dt &> \sum_{k=1}^m \Big( \left(I^0(k) + \varepsilon_1\right) \phi_1(t_k) \\
&\quad + \left(\overline{I}^0(k) + \varepsilon_1\right) \phi_1'(t_k)\Big) p(t_k) \\
(\lambda_1 - \varepsilon_1 - f^\infty)\sigma \int_a^b \phi_1(t)a(t)dt &> \sum_{k=1}^m \left( \left(I^\infty(k) + \varepsilon_1\right) \phi_1(t_k) \right. \\
&\quad \left. + \left(\overline{I}^\infty(k) + \varepsilon_1\right) \phi_1'(t_k)\right) p(t_k).
\end{aligned} \tag{24}$$

By the definitions of $f^0, I^0$ and $\overline{I}^0$, there exists $0 < r_0 < p$ such that for any $t \in [a, b], 0 \leq x \leq r_0$, we have

$$f(t, x) \leq (f^0 + \varepsilon_1)x, \quad I_k(x) \leq (I^0(k) + \varepsilon_1)x, \quad \overline{I}_k(x) \leq (\overline{I}^0(k) + \varepsilon_1)x. \tag{25}$$

Let $r \in (0, r_0)$. We now show that $\mu\Phi u \neq u$ for $u \in \partial K_r$ and $0 < \mu \leq 1$. If this is not true, then there exist $u_0 \in \partial K_r$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \Phi u_0 = u_0$. Then $u_0(t)$ satisfies $BVP(11)$. From (25), multiplying $(p(t)u_0'(t))'$ by $\phi_1(t)$ and integrating the product from $a$ to $b$ with respect to $t$, and proceeding as in the second step of proof of theorem 2.1, we have

$$\begin{aligned}
&\lambda_1 \int_a^b a(t)\phi_1(t)u_0(t)dt \\
&= \mu_0 \sum_{k=1}^{m} \left( I_k(u_0(t_k))\phi_1(t_k) + \overline{I}_k(u_0(t_k))\phi_1'(t_k) \right) p(t_k) \\
&\quad + \mu_0 \int_a^b \phi_1(t)a(t)f(u_0(t))dt \\
&\leq \sum_{k=1}^{m} \Big( (I^0(k) + \varepsilon_1)\phi_1(t_k) + (\overline{I}^0(k) + \varepsilon_1)\phi_1'(t_k) \Big) u_0(t_k)p(t_k) \\
&\quad + (f^0 + \varepsilon_1) \int_a^b \phi_1(t)a(t)u_0(t)dt.
\end{aligned}$$

Since $u_0(t) \geq \sigma \|u_0\|_{pc} = \sigma r$ for $t \in [a, b]$, we have

$$\begin{aligned}
r(\lambda_1 - f^0 - \varepsilon_1) \int_a^b \sigma a(t)\phi_1(t)dt \quad &\leq (\lambda_1 - f^0 - \varepsilon_1) \int_a^b \phi_1(t)a(t)u_0(t)dt \\
&\leq \sum_{k=1}^{m} \big( (I^0(k) + \varepsilon_1)\phi_1(t_k) \\
&\quad + (\overline{I}^0(k) + \varepsilon_1)\phi_1'(t_k) \Big) u_0(t_k)p(t_k) \\
&\leq r \sum_{k=1}^{m} \Big( (I^0(k) + \varepsilon_1)\phi_1(t_k) + \overline{I}^0(k) \\
&\quad + \varepsilon_1 \phi_1'(t_k) \big) p(t_k).
\end{aligned}$$

This is a contradiction with (24). Hence, by Lemma 1.4, we have

$$i(\Phi, K_r, K) = 1. \tag{26}$$

Thirdly, we prove $i(\Phi, K_R, K) = 1$. From the definitions of $f^\infty, I^\infty$ and $\overline{I}^\infty$, there exists $H > p > 0$ such that for any $t \in [a, b]$ and $x \geq H$,

$$f(t, x) \leq (f^\infty + \varepsilon_1)x, \quad I_k(x) \leq (I^\infty(k) + \varepsilon_1)x, \quad \overline{I}_k(x) \leq (\overline{I}^\infty(k) + \varepsilon_1)x. \tag{27}$$

Proceeding as in the third step of proof of theorem 2.1, for any $t \in [a, b]$ and $x \geq H$, let

$$\begin{aligned}
c \;&= \max_{0 \leq x \leq H} \max_{a \leq t \leq b} |f(t, x) - (f^\infty + \varepsilon_1)x| \\
&\quad + \sum_{k=1}^{m} \max_{0 \leq x \leq H} |I_k(x) - (I^\infty(k) + \varepsilon_1)x| \\
&\quad + \sum_{k=1}^{m} \max_{0 \leq x \leq H} |\overline{I}_k(x) - (\overline{I}^\infty(k) + \varepsilon_1)x|.
\end{aligned} \tag{28}$$

Then, from (27), for all $t \in [a, b], x > 0$, we have

$$\begin{aligned}
&f(t, x) \leq (f^\infty + \varepsilon_1)x + c, \quad I_k(x) \leq (I^\infty(k) + \varepsilon_1)x + c, \\
&\overline{I}_k(x) \leq (\overline{I}^\infty(k) + \varepsilon_1)x + c.
\end{aligned} \tag{29}$$

Then we show that if $R$ is large enough, we have $\mu\Phi u \neq u$ for every $u \in \partial K_R$ and $0 < \mu \leq 1$. In fact, if it is not true, then there exist $u_0 \in \partial K_R$ and $\mu_0 \geq 1$ such that $\mu_0 \Phi u_0 = u_0$. It is easy to see that $u_0(t)$ satisfies (11), and similar to the proof of the third step of Theorem 2.1, we obtain

$$\begin{aligned}
&\|u_0\|_{pc}(\lambda_1 - f^\infty - \varepsilon_1) \int_a^b \sigma a(t)\phi_1(t)dt \\
&\leq (\lambda_1 - f^\infty - \varepsilon_1) \int_a^b \phi_1(t)a(t)u_0(t)dt \\
&\leq \sum_{k=1}^{m} \left((I^\infty(k) + \varepsilon_1)\phi_1(t_k) + (\overline{I}^\infty(k) + \varepsilon_1)\phi_1'(t_k)\right) u_0(t_k)p(t_k) \\
&\quad + c\left(\sum_{k=1}^{m} (\phi_1(t_k) + \phi_1'(t_k))\, p(t_k) + \int_a^b \phi_1(t)a(t)dt\right) \\
&\leq \|u_0\|_{pc} \sum_{k=1}^{m} \left((I^\infty(k) + \varepsilon_1)\phi_1(t_k) + (\overline{I}^\infty(k) + \varepsilon_1)\phi_1'(t_k)\right) p(t_k) \\
&\quad + c\left(\sum_{k=1}^{m} (\phi_1(t_k) + \phi_1'(t_k))\, p(t_k) + \int_a^b \phi_1(t)a(t)dt\right).
\end{aligned}$$

So,

$$\begin{aligned}
\|u_0\|_{pc} &\leq \frac{c\left(\sum_{k=1}^{m} \left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k) + \int_a^b \phi_1(t)a(t)dt\right)}{(\lambda_1 - f^\infty - \varepsilon_1)\int_a^b \sigma a(t)\phi_1(t)dt - \sum_{k=1}^{m}\left((I^\infty(k)+\varepsilon_1)\phi_1(t_k) + (\overline{I}^\infty(k)+\varepsilon_1)\phi_1'(t_k)\right)p(t_k)} \\
&=: R_1.
\end{aligned}$$

Let $R > \max\{H, R_1\}$. Then for all $u \in \partial K_R$ and $0 < \mu \leq 1$, $\mu\Phi u \neq u$. Hence, by Lemma 2.5, we have

$$i(\Phi, K_R, K) = 1. \tag{30}$$

By (23), (26), (30) and the property of the fixed points index, we obtain $i(\Phi, K_R \backslash \overline{K}_P, K) = 1$, $i(\Phi, K_p \backslash \overline{K}_r, K) = -1$. Thus, $BVP(1)$ has at least two positive solutions $u_1$ and $u_2$ satisfying $0 \leq \|u_1\|_{pc} \leq p \leq \|u_2\|_{pc}$. □

**Corollary 2.1.**. Suppose that $(A_1)$

$$f^0 + \frac{\sum_{k=1}^{m} (I^0(k)\phi_1(t_k) + \overline{I}^0(k)\phi_1'(t_k))p(t_k)}{\sigma \int_0^1 \phi_1(t)a(t)dt} < \lambda_1,$$

$$f^\infty + \frac{\sum_{k=1}^{m} (I^\infty(k)\phi_1(t_k) + \overline{I}^\infty(k)\phi_1'(t_k))p(t_k)}{\sigma \int_0^1 \phi_1(t)a(t)dt} < \lambda_1.$$

Then $BVP(1)$ has at least one positive solution.

**Corollary 2.2.** Suppose that $(A_2)$

$$f_0 + \frac{\sigma \sum_{k=1}^{m} (I_0(k)\phi_1(t_k) + \overline{I}_0(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a(t)dt} > \lambda_1,$$

$$f_\infty + \frac{\sigma \sum_{k=1}^{m} (I_\infty(k)\phi_1(t_k) + \overline{I}_\infty(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a(t)dt} > \lambda_1.$$

Then $BVP(1)$ has at least one positive solution.

**Corollary 2.3.** Theorem 2.1 is valid if $(H_1)$ is replaced by

$$f_0 = \infty, \quad \text{or} \quad \sum_{k=1}^{m} I_0(k)\phi_1(t_k)p(t_k) = \infty, \quad \text{or} \quad \sum_{k=1}^{m} \overline{I}_0(k)\phi_1'(t_k)p(t_k) = \infty,$$

and

$$f_\infty = \infty, \quad \text{or} \quad \sum_{k=1}^{m} I_\infty(k)\phi_1(t_k)p(t_k) = \infty, \quad \text{or} \quad \sum_{k=1}^{m} \overline{I}_\infty(k)\phi_1'(t_k)p(t_k) = \infty.$$

**Corollary 2.4.** Theorem 2.2 is valid if $(H_2)$ is replaced by

$$f^0 = 0, \quad I^0(k) = 0, \quad \overline{I}^0(k) = 0, \text{or} \quad f^\infty = 0,$$

$$I^\infty(k) = 0, \quad \overline{I}^\infty(k) = 0, \quad k = 1, 2, \cdots, m.$$

**Corollary 2.5.** Corollary 2.1 is valid if $(A_1)$ is replaced by

$$f_0 = \infty, \quad \text{or} \quad \sum_{k=1}^{m} I_0(k)\phi_1(t_k)p(t_k) = \infty, \quad \text{or} \quad \sum_{k=1}^{m} \overline{I}_0(k)\phi_1'(t_k)p(t_k) = \infty,$$

and

$$f^\infty = 0, \quad I^\infty(k) = 0, \quad \overline{I}^\infty(k) = 0, \quad k = 1, 2, \cdots, m.$$

**Corollary 2.6.** Corollary 2.2 is valid if $(A_2)$ is replaced by

$$f_\infty = \infty, \text{or} \sum_{k=1}^{m} I_\infty(k)\phi_1(t_k)p(t_k) = \infty, \text{or} \sum_{k=1}^{m} \overline{I}_\infty(k)\phi_1'(t_k)p(t_k) = \infty,$$

and

$$f^0 = 0, \quad I^0(k) = 0, \quad \overline{I}^0(k) = 0, \quad k = 1, 2, \cdots, m.$$

## 3. Example

**Example 3.1** Now we give an example

$$\begin{cases} ((t-2)^8 u'(t))' + u^\alpha + u^\beta = 0, t \in J', 0 < \alpha < 1 < \beta, \\ -\Delta u'|_{t=t_k} = c_k u(t_k), c_k \geq 0, k = 1, 2, \cdots, m, \\ \Delta u|_{t=t_k} = d_k u(t_k), d_k \geq 0, k = 1, 2, \cdots, m, \\ u(0) = 0, \\ \frac{12}{7} u(1) + \frac{1}{2} u'(1) = 0. \end{cases} \tag{31}$$

Then $BVP(31)$ has at least two positive solutions $u_1$ and $u_2$, satisfying

$$0 \leq \|u_1\|_{pc} \leq p \leq \|u_2\|_{pc}.$$

Where

$$\beta_1 = 0, \quad \alpha_1 = \beta_2 = \frac{1}{2}, \quad \alpha_2 = \frac{12}{7}, \quad \sum_{k=1}^{m} G(t_k, t_k) c_k < \frac{1}{5}, \quad \sum_{k=1}^{m} G(t_k, t_k) d_k < \frac{1}{5}. \tag{32}$$

By (32), choose $\eta > 0$ such that

$$2 < \eta < 5 \left( 1 - \sum_{k=1}^{m} G(t_k, t_k) c_k - \sum_{k=1}^{m} G(t_k, t_k) c_k \right).$$

Since $f(t, u) = u^\alpha + u^\beta$, $f_0 = \infty, f_\infty = \infty$, so $(H_1)$ holds.

From $B(t, s) = \int_t^s \frac{d\tau}{p(\tau)}$, and $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0, 1)$,

$$\begin{aligned} G(s, s) &= \frac{1}{\rho} (\beta_1 + \alpha_1 B(0, s)) (\beta_2 + \alpha_2 B(s, 1)) \\ &\leq \frac{(\beta_1 + \alpha_1 B(0, 1)) (\beta_2 + \alpha_2 B(0, 1))}{\alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0, 1)} \\ &= \frac{1}{6}. \end{aligned}$$

So, we have

$$\int_0^1 G(s, s) ds \leq \int_0^1 \frac{(\beta_1 + \alpha_1 B(0, 1)) (\beta_2 + \alpha_2 B(0, 1))}{\alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0, 1)} ds = \frac{1}{6}.$$

Let $\eta_k = c_k, \overline{\eta}_k = d_k$ such that $\eta, \eta_k, \overline{\eta}_k$ satisfying

$$\eta + \sum_{k=1}^{m} (\eta_k + \overline{\eta}_k) > 0, \eta \int_0^1 G(s, s) a(s) ds + \sum_{k=1}^{m} G(t_k, t_k)(\eta_k + \overline{\eta}_k) < 1.$$

Let $p = 1$ for every $0 < u \leq p$. Then, we have

$$f(t,u) = u^{\alpha} + u^{\beta} \leq p^{\alpha} + p^{\beta} = 2 < \eta p,$$
$$I_k(u) = c_k u = \eta_k u \leq \eta_k p,$$
$$\overline{I}_k(u) = d_k u = \overline{\eta}_k) u \leq \overline{\eta}_k p.$$

So $(H_3)$ hlods. From Theorem 2.1, the conclusion is established.

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