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COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS SATISFYING CONTRACTIVE INEQUALITIES OF INTEGRAL TYPE

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Abstract. In this paper, three results concerning the existence and uniqueness of common fixed point for four mappings satisfying contractive inequalities of integral type in metric spaces are introduced and proved. And also, we give the three examples.

1. Introduction and preliminaries

In 2008, Dutta and Choudhuty [4] proved the following result for (ψ, φ)-weakly contractive mappings in a complete metric space.

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Theorem 1.1. ([4]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{1.1}$$

where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

In 2002, Branciari [3] was the first to introduce the concept of contractive mapping of integral type and proved the following result, which generalizes the Banach fixed point theorem.

Theorem 1.2. ([3]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leqslant c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.2}$$

where $c \in (0, 1)$ is a constant and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$.

In recent years, the researchers [1, 2, 5, 6, 8-16] proved some fixed and common fixed point theorems involving a lot of (ψ, φ)-weakly contractive mappings and contractive mappings of integral type. In particular, Liu *et al.* [9] and Hosseini [6] certified the following theorems, which extend the results of Branciari [3] and Dutta and Choudhuty [4].

Theorem 1.3. ([9]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\begin{aligned} \psi\Big(\int_0^{d(Tx,Ty)} \varphi(t)dt\Big) \leqslant{}& \psi\Big(\int_0^{d(x,y)} \varphi(t)dt\Big) \\ &- \phi\Big(\int_0^{d(x,y)} \varphi(t)dt)\Big), \quad \forall x, y \in X, \end{aligned} \tag{1.3}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that $\liminf_{n\to\infty} \phi(a_n) > 0$ if and only if $\liminf_{n\to\infty} a_n > 0$ for each $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$.

Theorem 1.4. ([6]) *Let T and S be two mappings from a complete metric space (X, d) into itself satisfying*

$$\psi\Big(\int_0^{d(Tx,Sy)}\varphi(t)dt\Big)\leqslant\psi\Big(\int_0^{M(x,y)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M(x,y)}\varphi(t)dt)\Big),\quad\forall x,y\in X,\tag{1.4}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon\varphi(t)dt>0$ for each $\varepsilon>0$, $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is semi-continuous and $\phi(t)=0$ if and only if $t=0$, $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and monotone nondecreasing function with $\psi(t)=0$ if and only if $t=0$ and

$$M(x,y)=\max\Big\{d(x,y),d(x,Tx),d(y,Sy),\frac{1}{2}[d(y,Tx)+d(x,Sy)]\Big\},\quad\forall x,y\in X.\tag{1.5}$$

Then T and S have a unique common fixed point $a\in X$.

The purpose of this paper is both to study the existence and uniqueness of common fixed point for certain four mappings satisfying contractive inequalities of integral type in metric spaces, and to construct three examples to illustrate that our results are proper generalizations of Theorem 1.2 and are different from Theorems 1.1, 1.3 and 1.4.

Throughout this paper, we assume that $\mathbb{R}^+=[0,+\infty),\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}$ denotes the set of all positive integers and

$$\Phi_1=\Big\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+\text{ is Lebesgue integrable, summable on each compact subset of }\mathbb{R}^+\text{ and }\int_0^\varepsilon\varphi(t)dt>0\text{ for each }\varepsilon>0\Big\},$$

$$\Phi_2=\Big\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+\text{ satisfies that }\liminf_{n\to\infty}\varphi(a_n)>0\Leftrightarrow\liminf_{n\to\infty}a_n>0\text{ for each }\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^+\Big\},$$

$$\Phi_3=\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+\text{ is continuous }\},$$

$$\Phi_4=\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+\text{ is semi-continuous and }\varphi(t)=0\text{ if and only if }t=0\},$$

$$\Phi_5=\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+\text{ is nondecreasing and continuous and }\varphi(t)=0\Leftrightarrow t=0\}.$$

Definition 1.5. ([7]) A pair of self mappings f and g in a metric space (X, d) is said to be weakly compatible if for all $t \in X$ the equality $ft = gt$ implies $fgt = gft$.

Lemma 1.6. ([10]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^{a} \varphi(t)dt.$$

Lemma 1.7. ([9]) *Let $\varphi \in \Phi_2$. Then $\varphi(t) > 0$ if and only if $t > 0$.*

2. Common fixed point theorems

In this section, we show three common fixed point theorems for contractive mappings of integral type (2.4), (2.15) and (2.19).

Theorem 2.1. *Let A, B, S and T be mappings from a metric space (X, d) into itself such that*

$$\{A, T\} \text{ and } \{B, S\} \text{ are weakly compatible;} \tag{2.1}$$

$$T(X) \subseteq B(X) \text{ and } S(X) \subseteq A(X); \tag{2.2}$$

$$\text{one of } A(X), B(X), S(X) \text{ and } T(X) \text{ is complete}; \tag{2.3}$$

$$\begin{aligned} \psi\Big(\int_0^{d(Tx,Sy)} \varphi(t)dt\Big) \leqslant{}& \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big) \\ &- \phi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big), \quad \forall x, y \in X, \end{aligned} \tag{2.4}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and

$$\begin{aligned} &M_1(x, y) \\ &= \Big\{d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{1}{2}[d(Ax, Sy) + d(Tx, By)], \\ &\quad \frac{1 + d(Ax, By)}{1 + d(By, Sy)} d(Ax, Tx), \frac{1 + d(Ax, By)}{1 + d(Ax, Tx)} d(By, Sy) \\ &\quad \frac{d^2(Ax, Tx)}{1 + d(Tx, Sy)}, \frac{d^2(By, Sy)}{1 + d(Tx, Sy)}\Big\}, \quad \forall x, y \in X. \end{aligned} \tag{2.5}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. (2.2) guarantee that there are two sequences $\{x_n\}_{n\in\mathbb{N}_0}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X such that

$$y_{2n+1} = Bx_{2n+1} = Tx_{2n}, \quad y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.6}$$

Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Suppose that $d_{2n} < d_{2n+1}$ for some $n \in \mathbb{N}$. Because of (2.4)-(2.6), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.2, we obtain that

$$\begin{aligned}
&M_1(x_{2n}, x_{2n+1})\\
&= \max\Big\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n+1}, Sx_{2n+1}),\\
&\qquad \frac{1}{2}[d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})],\\
&\qquad \frac{1 + d(Ax_{2n}, Bx_{2n+1})}{1 + d(Bx_{2n+1}, Sx_{2n+1})} d(Ax_{2n}, Tx_{2n}),\\
&\qquad \frac{1 + d(Ax_{2n}, Bx_{2n+1})}{1 + d(Ax_{2n}, Tx_{2n})} d(Bx_{2n+1}, Sx_{2n+1}),\\
&\qquad \frac{d^2(Ax_{2n}, Tx_{2n})}{1 + d(Tx_{2n}, Sx_{2n+1})}, \frac{d^2(Bx_{2n+1}, Sx_{2n+1})}{1 + d(Tx_{2n}, Sx_{2n+1})}\Big\}\\
&= \max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}),\\
&\qquad \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})],\\
&\qquad \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+1}),\\
&\qquad \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} d(y_{2n+1}, y_{2n+2}),\\
&\qquad \frac{d^2(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n+2})}, \frac{d^2(y_{2n+1}, y_{2n+2})}{1 + d(y_{2n+1}, y_{2n+2}))}\Big\}\\
&= \max\Big\{d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2} d(y_{2n}, y_{2n+2}), \frac{1 + d_{2n}}{1 + d_{2n+1}} d_{2n},\\
&\qquad \frac{1 + d_{2n}}{1 + d_{2n}} d_{2n+1}, \frac{d_{2n}^2}{1 + d_{2n+1}}, \frac{d_{2n+1}^2}{1 + d_{2n+1}}\Big\}\\
&= \max\{d_{2n}, d_{2n+1}\} = d_{2n+1}
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d_{2n+1}} \varphi(t)dt\Big)\\
&= \psi\Big(\int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt\Big)\\
&\leqslant \psi\Big(\int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\Big)
\end{aligned}$$

$$= \psi\biggl(\int_0^{d_{2n+1}} \varphi(t)dt\biggr) - \phi\biggl(\int_0^{d_{2n+1}} \varphi(t)dt\biggr)$$
$$< \psi\biggl(\int_0^{d_{2n+1}} \varphi(t)dt\biggr),$$

which is a contradiction. Hence

$$d_{2n+1} \leqslant d_{2n} = M_1(x_{2n}, x_{2n+1}), \quad \forall n \in \mathbb{N}.$$

In the same way,

$$d_{2n} \leqslant d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}.$$

That is,

$$\begin{aligned} d_{n+1} \leqslant d_n, \quad d_{2n} = M_1(x_{2n}, x_{2n+1}), \\ d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{2.7}$$

which implies that $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing sequence. Therefore, there exists $c \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = c$. Suppose that $c > 0$. In light of (2.4), (2.7), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we get that

$$\begin{aligned}
&\psi\biggl(\int_0^{c} \varphi(t)dt\biggr) \\
&= \limsup_{n\to\infty} \psi\biggl(\int_0^{d_{2n+1}} \varphi(t)dt\biggr) = \limsup_{n\to\infty} \psi\biggl(\int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt\biggr) \\
&\leqslant \limsup_{n\to\infty} \biggl[\psi\biggl(\int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\biggr) - \phi\biggl(\int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\biggr)\biggr] \\
&= \limsup_{n\to\infty} \biggl[\psi\biggl(\int_0^{d_{2n}} \varphi(t)dt\biggr) - \phi\biggl(\int_0^{d_{2n}} \varphi(t)dt\biggr)\biggr] \\
&\leqslant \limsup_{n\to\infty} \psi\biggl(\int_0^{d_{2n}} \varphi(t)dt\biggr) - \liminf_{n\to\infty} \phi\biggl(\int_0^{d_{2n}} \varphi(t)dt\biggr) \\
&< \psi\biggl(\int_0^{c} \varphi(t)dt\biggr),
\end{aligned}$$

which is absurd. Thus $c = 0$, which means that

$$\lim_{n\to\infty} d_n = 0. \tag{2.8}$$

In order to prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, according to (2.8), we need to prove that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Otherwise, $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. That is, there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there exist $2m(k), 2n(k) \in \mathbb{N}$ with $2m(k) > 2n(k) > 2k$ satisfying

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon. \tag{2.9}$$

For each $k \in \mathbb{N}$, $2m(k)$ is the least integer exceeding $2n(k)$ satisfying (2.9). It follows that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon, \quad \forall k \in \mathbb{N}, \tag{2.10}$$

which together with (2.9) and the triangle inequality give that

$$\begin{aligned} \varepsilon &\le d(y_{2n(k)}, y_{2m(k)}) \\ &\le d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\ &< \varepsilon + d_{2m(k)-2} + d_{2m(k)-1}, \quad \forall k \in \mathbb{N} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2m(k)-1}, \quad \forall k \in \mathbb{N}; \\ &|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2n(k)}, \quad \forall k \in \mathbb{N}; \\ &|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| \le d_{2n(k)}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.12}$$

Letting $k \to \infty$ in (2.11) and (2.12) and using (2.8), we have

$$\begin{aligned} \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) &= \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) \\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) \\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) \\ &= \varepsilon. \end{aligned} \tag{2.13}$$

In view of (2.4), (2.5), (2.13), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we claim that

$$\begin{aligned} &M_1(x_{2n(k)}, x_{2m(k)-1}) \\ &= \max\Big\{ d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}), \\ &\qquad d(Bx_{2m(k)-1}, Sx_{2m(k)-1}), \\ &\qquad \frac{1}{2}[d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})], \\ &\qquad \frac{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Bx_{2m(k)-1}, Sx_{2m(k)-1})} d(Ax_{2n(k)}, Tx_{2n(k)}), \\ &\qquad \frac{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Ax_{2n(k)}, Tx_{2n(k)})} d(Bx_{2m(k)-1}, Sx_{2m(k)-1}), \\ &\qquad \frac{d^2(Ax_{2n(k)}, Tx_{2n(k)})}{1 + d(Tx_{2n(k)}, Sx_{2m(k)-1})}, \frac{d^2(Bx_{2m(k)-1}, Sx_{2m(k)-1})}{1 + d(Tx_{2n(k)}, Sx_{2m(k)-1})} \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{ d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}), \\
&\qquad \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})], \\
&\qquad \frac{1 + d(y_{2n(k)}, y_{2m(k)-1})}{1 + d(y_{2m(k)-1}, y_{2m(k)})} d(y_{2n(k)}, y_{2n(k)+1}), \\
&\qquad \frac{1 + d(y_{2n(k)}, y_{2m(k)-1})}{1 + d(y_{2n(k)}, y_{2n(k)+1})} d(y_{2m(k)-1}, y_{2m(k)}), \\
&\qquad \frac{d^2(y_{2n(k)}, y_{2n(k)+1})}{1 + d(y_{2n(k)+1}, y_{2m(k)})}, \frac{d^2(y_{2m(k)-1}, y_{2m(k)})}{1 + d(y_{2n(k)+1}, y_{2m(k)})} \Big\} \\
&\to \max\Big\{ \varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon), 0, 0, 0, 0 \Big\} \\
&= \varepsilon \ \text{ as } \ k \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\psi\Big(\int_0^{\varepsilon} \varphi(t)dt \Big) &= \limsup_{k\to\infty} \psi\Big(\int_0^{d(y_{2n(k)+1}, y_{2m(k)})} \varphi(t)dt \Big) \\
&= \limsup_{k\to\infty} \psi\Big(\int_0^{d(Tx_{2n(k)}, Sx_{2m(k)-1})} \varphi(t)dt \Big) \\
&\leqslant \limsup_{k\to\infty} \Big[\psi\Big(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \Big) \\
&\qquad - \phi\Big(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \Big) \Big] \\
&\leqslant \limsup_{k\to\infty} \psi\Big(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \Big) \\
&\qquad - \liminf_{k\to\infty} \phi\Big(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \Big) \\
&< \psi\Big(\int_0^{\varepsilon} \varphi(t)dt \Big),
\end{aligned}
$$

which is impossible. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Obviously, $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Therefore, there exists $(z, w) \in A(X) \times X$ such that $\lim_{n\to\infty} y_{2n} = z = Aw$. It is easy to see that

$$
\begin{aligned}
z &= \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Bx_{2n+1} \\
&= \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Ax_{2n}.
\end{aligned} \tag{2.14}
$$

Suppose that $Tw \neq z$. Taking account of (2.4), (2.5), (2.14), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we derive that

$$\begin{aligned}
&M_1(w, x_{2n+1}) \\
&= \max\Big\{d(Aw, Bx_{2n+1}), d(Aw, Tw), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})], \\
&\qquad \frac{1 + d(Aw, Bx_{2n+1})}{1 + d(Bx_{2n+1}, Sx_{2n+1})} d(Aw, Tw), \\
&\qquad \frac{1 + d(Aw, Bx_{2n+1})}{1 + d(Aw, Tw)} d(Bx_{2n+1}, Sx_{2n+1}), \\
&\qquad \frac{d^2(Aw, Tw)}{1 + d(Tw, Sx_{2n+1})}, \frac{d^2(Bx_{2n+1}, Sx_{2n+1})}{1 + d(Tw, Sx_{2n+1})}\Big\} \\
&\to \max\Big\{d(Aw, z), d(Aw, Tw), d(z, z), \frac{1}{2}[d(Aw, z) + d(Tw, z)], \\
&\qquad \frac{1 + d(Aw, z)}{1 + d(z, z)} d(Aw, Tw), \frac{1 + d(Aw, z)}{1 + d(Aw, Tw)} d(z, z), \\
&\qquad \frac{d^2(Aw, Tw)}{1 + d(Tw, z)}, \frac{d^2(z, z)}{1 + d(Tw, z)}\Big\} \\
&= \max\Big\{0, d(z, Tw), 0, \frac{1}{2} d(Tw, z), d(z, Tw), 0, \frac{d^2(z, Tw)}{1 + d(Tw, z)}, 0\Big\} \\
&= d(Tw, z) \text{ as } n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\psi\bigg(\int_0^{d(Tw,z)} \varphi(t)dt\bigg) \\
&= \limsup_{n\to\infty} \psi\bigg(\int_0^{d(Tw,Sx_{2n+1})} \varphi(t)dt\bigg) \\
&\leqslant \limsup_{n\to\infty} \bigg[\psi\bigg(\int_0^{M_1(w,x_{2n+1})} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_1(w,x_{2n+1})} \varphi(t)dt\bigg)\bigg] \\
&\leqslant \limsup_{n\to\infty} \psi\bigg(\int_0^{M_1(w,x_{2n+1})} \varphi(t)dt\bigg) - \liminf_{n\to\infty} \phi\bigg(\int_0^{M_1(w,x_{2n+1})} \varphi(t)dt\bigg) \\
&< \psi\bigg(\int_0^{d(Tw,z)} \varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction. Hence $Tw = z$. Making use of (2.2), we get that there exists a point $u \in X$ with $z = Bu = Tw$. Suppose that $Su \neq z$. In terms of (2.4), (2.5), (2.14), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we deduce that

$$\begin{aligned}
&M_1(x_{2n}, u)\\
&= \max\Big\{d(Ax_{2n}, Bu), d(Ax_{2n}, Tx_{2n}), d(Bu, Su),\\
&\qquad \frac{1}{2}[d(Ax_{2n}, Su) + d(Tx_{2n}, Bu)], \frac{1 + d(Ax_{2n}, Bu)}{1 + d(Bu, Su)} d(Ax_{2n}, Tx_{2n}),\\
&\qquad \frac{1 + d(Ax_{2n}, Bu)}{1 + d(Ax_{2n}, Tx_{2n})} d(Bu, Su), \frac{d^2(Ax_{2n}, Tx_{2n})}{1 + d(Tx_{2n}, Su))},\\
&\qquad \frac{d^2(Bu, Su)}{1 + d(Tx_{2n}, Su))}\Big\}\\
&\to \max\Big\{d(z, Bu), d(z, z), d(Bu, Su), \frac{1}{2}[d(z, Su) + d(z, Bu)],\\
&\qquad \frac{1 + d(z, Bu)}{1 + d(Bu, Su)} d(z, z), \frac{1 + d(z, Bu)}{1 + d(z, z)} d(Bu, Su),\\
&\qquad \frac{d^2(z, z)}{1 + d(z, Su))}, \frac{d^2(Bu, Su)}{1 + d(z, Su)}\Big\}\\
&= \max\Big\{0, 0, d(z, Su), \frac{1}{2} d(z, Su), 0, d(z, Su), 0, \frac{d^2(z, Su)}{1 + d(z, Su)}\Big\}\\
&= d(z, Su) \ \text{ as } \ n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(z,Su)} \varphi(t)dt\Big)\\
&= \limsup_{n\to\infty} \psi\Big(\int_0^{d(Tx_{2n},Su)} \varphi(t)dt\Big)\\
&\leqslant \limsup_{n\to\infty} \Big[\psi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt\Big)\Big]\\
&\leqslant \limsup_{n\to\infty} \psi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt\Big) - \liminf_{n\to\infty} \phi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt\Big)\\
&< \psi\Big(\int_0^{d(z,Su)} \varphi(t)dt\Big),
\end{aligned}$$

which is absurd. Hence $Su = z$. By means of (2.1), we know that $Az = ATw = TAw = Tz$ and $Bz = BSu = SBu = Sz$. Suppose that $Tz \neq Sz$.

Using (2.4), (2.5), $(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$ and Lemma 1.2, we have

$$\begin{aligned}
&M_1(z,z)\\
&=\max\Big\{d(Az,Bz),d(Az,Tz),d(Bz,Sz),\frac{1}{2}[d(Az,Sz)+d(Tz,Bz)],\\
&\qquad\frac{1+d(Az,Bz)}{1+d(Bz,Sz)}d(Az,Tz),\frac{1+d(Az,Bz)}{1+d(Az,Tz)}d(Bz,Sz),\\
&\qquad\frac{d^2(Az,Tz)}{1+d(Tz,Sz))},\frac{d^2(Bz,Sz)}{1+d(Tz,Sz))}\Big\}\\
&=\max\Big\{d(Tz,Sz),0,0,\frac{1}{2}[d(Tz,Sz)+d(Tz,Sz)],0,0,0,0\Big\}\\
&=d(Tz,Sz)
\end{aligned}$$

and

$$\begin{aligned}
\psi\Big(\int_0^{d(Tz,Sz)}\varphi(t)dt\Big)&\leqslant\psi\Big(\int_0^{M_1(z,z)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M_1(z,z)}\varphi(t)dt\Big)\\
&=\psi\Big(\int_0^{d(Tz,Sz)}\varphi(t)dt\Big)-\phi\Big(\int_0^{d(Tz,Sz)}\varphi(t)dt\Big)\\
&<\psi\Big(\int_0^{d(Tz,Sz)}\varphi(t)dt\Big),
\end{aligned}$$

which is impossible. Hence $Tz=Sz$. That is, $Az=Tz=Bz=Sz$.

Suppose that $Tz\neq z$. On account of (2.4), (2.5), $(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$ and Lemma 1.2, we attain that

$$\begin{aligned}
&M_1(z,u)\\
&=\max\Big\{d(Az,Bu),d(Az,Tz),d(Bu,Su),\frac{1}{2}[d(Az,Su)+d(Tz,Bu)],\\
&\qquad\frac{1+d(Az,Bu)}{1+d(Bu,Su)}d(Az,Tz),\frac{1+d(Az,Bu)}{1+d(Az,Tz)}d(Bu,Su),\\
&\qquad\frac{d^2(Az,Tz)}{1+d(Tz,Su))},\frac{d^2(Bu,Su)}{1+d(Tz,Su))}\Big\}\\
&=\max\Big\{d(Tz,z),0,0,\frac{1}{2}[d(Tz,z)+d(Tz,z)],0,0,0,0\Big\}\\
&=d(Tz,z)
\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{d(Tz,Su)}\varphi(t)dt\Big)\\ &\leqslant \psi\Big(\int_0^{M_1(z,u)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(z,u)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big),\end{aligned}$$

which is ridiculous. Therefore, $Tz = z$, which implies that z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have a common fixed point $b \in X \setminus \{z\}$. Using (2.4), (2.5), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.2, we arrive at

$$\begin{aligned}&M_1(b,z)\\ &= \max\Big\{d(Ab,Bz), d(Ab,Tb), d(Bz,Sz), \frac{1}{2}[d(Ab,Sz)+d(Tb,Bz)],\\ &\qquad \frac{1+d(Ab,Bz)}{1+d(Bz,Sz)}d(Ab,Tb), \frac{1+d(Ab,Bz)}{1+d(Ab,Tb)}d(Bz,Sz),\\ &\qquad \frac{d^2(Ab,Tb)}{1+d(Tb,Sz))}, \frac{d^2(Bz,Sz)}{1+d(Tb,Sz))}\Big\}\\ &= \max\Big\{d(b,z), 0, 0, \frac{1}{2}[d(b,z)+d(b,z)], 0, 0, 0, 0\Big\}\\ &= d(b,z)\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{d(Tb,Sz)}\varphi(t)dt\Big)\\ &\leqslant \psi\Big(\int_0^{M_1(b,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(b,z)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big),\end{aligned}$$

which is a contradiction. Hence A, B, S and T have a unique common fixed point in X. Analogously we infer that A, B, S and T have a unique common fixed point in X if one of $B(X), S(X)$ and $T(X)$ is complete. This completes the proof. □

Theorem 2.2. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (2.1)-(2.3) *and*

$$\psi\Big(\int_0^{d(Tx,Sy)}\varphi(t)dt\Big) \leqslant \psi\Big(\int_0^{M_2(x,y)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_2(x,y)}\varphi(t)dt\Big), \quad \forall x, y \in X, \tag{2.15}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ *and*

$$\begin{aligned} M_2(x,y) = \Big\{ & d(Ax,By), d(Ax,Tx), d(By,Sy), \\ & \frac{1}{2}[d(Ax,Sy)+d(Tx,By)], \frac{1+d(Ax,Sy)}{2+d(Tx,Sy)}d(Tx,By), \\ & \frac{1+d(Tx,By)}{2+d(Tx,Sy)}d(Ax,Sy), \frac{1+d(Ax,Sy)}{1+2d(Tx,Sy)}d(Ax,By), \\ & \frac{1+d(Tx,By)}{1+2d(Tx,Sy)}d(Ax,By)\Big\}, \quad \forall x, y \in X. \end{aligned} \tag{2.16}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in \mathbb{X}$. By virtue of (2.2), we get that there are two sequences $\{x_n\}_{n\in\mathbb{N}_0}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X such that (2.6) holds. Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Suppose that $d_{2n} < d_{2n+1}$ for some $n \in \mathbb{N}$. Apparently,

$$d_{2n+1} - \frac{d(y_{2n}, y_{2n+2})}{2+d_{2n+1}} \geqslant \frac{2d_{2n+1}+d_{2n+1}^2-d_{2n}-d_{2n+1}}{2+d_{2n+1}} > 0 \tag{2.17}$$

and

$$d_{2n+1} - \frac{1+d(y_{2n}, y_{2n+2})}{1+2d_{2n+1}}d_{2n} \geqslant \frac{d_{2n+1}+2d_{2n+1}^2-d_{2n}-d_{2n}d_{2n+1}-d_{2n}^2}{1+2d_{2n+1}} > 0,$$

which together with (2.6), (2.15)-(2.17), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.2 ensure that

$$
\begin{aligned}
& M_2(x_{2n}, x_{2n+1}) \\
&= \max\Big\{ d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})] \\
&\qquad \frac{1 + d(Ax_{2n}, Sx_{2n+1})}{2 + d(Tx_{2n}, Sx_{2n+1})} d(Tx_{2n}, Bx_{2n+1}), \\
&\qquad \frac{1 + d(Tx_{2n}, Bx_{2n+1})}{2 + d(Tx_{2n}, Sx_{2n+1})} d(Ax_{2n}, Sx_{2n+1}), \\
&\qquad \frac{1 + d(Ax_{2n}, Sx_{2n+1})}{1 + 2d(Tx_{2n}, Sx_{2n+1})} d(Ax_{2n}, Bx_{2n+1}), \\
&\qquad \frac{1 + d(Tx_{2n}, Bx_{2n+1})}{1 + 2d(Tx_{2n}, Sx_{2n+1})} d(Ax_{2n}, Bx_{2n+1}) \Big\} \\
&= \max\Big\{ d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\
&\qquad \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})], \frac{1 + d(y_{2n}, y_{2n+2})}{2 + d(y_{2n+1}, y_{2n+2})} d(y_{2n+1}, y_{2n+1}), \\
&\qquad \frac{1 + d(y_{2n+1}, y_{2n+1})}{2 + d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+2}), \frac{1 + d(y_{2n}, y_{2n+2})}{1 + 2d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+1}), \\
&\qquad \frac{1 + d(y_{2n+1}, y_{2n+1})}{1 + 2d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+1}) \Big\} \\
&= \max\Big\{ d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2} d(y_{2n}, y_{2n+2}), 0, \frac{d(y_{2n}, y_{2n+2})}{2 + d_{2n+1}} \\
&\qquad \frac{1 + d(y_{2n}, y_{2n+2})}{1 + 2d_{2n+1}} d_{2n}, \frac{d_{2n}}{1 + 2d_{2n+1}} \Big\} \\
&= \max\{d_{2n}, d_{2n+1}\} = d_{2n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi\Big(\int_0^{d_{2n+1}} \varphi(t)dt \Big) \\
&= \psi\Big(\int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt \Big) \\
&\leqslant \psi\Big(\int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt \Big) - \phi\Big(\int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt \Big)
\end{aligned}
$$

$$= \psi\bigg(\int_0^{d_{2n+1}}\varphi(t)dt\bigg) - \phi\bigg(\int_0^{d_{2n+1}}\varphi(t)dt\bigg)$$
$$< \psi\bigg(\int_0^{d_{2n+1}}\varphi(t)dt\bigg),$$

which is impossible. Hence

$$d_{2n+1} \leqslant d_{2n} = M_2(x_{2n}, x_{2n+1}), \quad \forall n \in \mathbb{N}.$$

In the same manner, we get that

$$d_{2n} \leqslant d_{2n-1} = M_2(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}.$$

That is,

$$\begin{aligned} d_{n+1} \leqslant d_n, \quad d_{2n} = M_2(x_{2n}, x_{2n+1}),\\ d_{2n-1} = M_2(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{2.18}$$

which implies that $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing sequence. Observe that there exists $c \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = c$. Suppose that $c > 0$. In virtue of (2.15), (2.18), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we claim that

$$\begin{aligned}
&\psi\bigg(\int_0^{c}\varphi(t)dt\bigg)\\
&= \limsup_{n\to\infty}\psi\bigg(\int_0^{d_{2n+1}}\varphi(t)dt\bigg) = \limsup_{n\to\infty}\psi\bigg(\int_0^{d(Tx_{2n}, Sx_{2n+1})}\varphi(t)dt\bigg)\\
&\leqslant \limsup_{n\to\infty}\bigg[\psi\bigg(\int_0^{M_2(x_{2n}, x_{2n+1})}\varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_2(x_{2n}, x_{2n+1})}\varphi(t)dt\bigg)\bigg]\\
&= \limsup_{n\to\infty}\bigg[\psi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg) - \phi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg)\bigg]\\
&\leqslant \limsup_{n\to\infty}\psi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg) - \liminf_{n\to\infty}\phi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg)\\
&< \psi\bigg(\int_0^{c}\varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction. Thus $c = 0$, that is, (2.8) holds.

Now we assert that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Due to (2.8), we need to prove that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. On the contrary, $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. That is, there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there exist $2m(k), 2n(k) \in \mathbb{N}$ with $2m(k) > 2n(k) > 2k$ satisfying (2.9)-(2.13). It follows from (2.13), (2.15), (2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1

that

$$\begin{aligned}
&M_2(x_{2n(k)}, x_{2m(k)-1})\\
&= \max\Big\{d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}),\\
&\qquad d(Bx_{2m(k)-1}, Sx_{2m(k)-1}),\\
&\qquad \frac{1}{2}[d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})],\\
&\qquad \frac{1 + d(Ax_{2n(k)}, Sx_{2m(k)-1})}{2 + d(Tx_{2n(k)}, Sx_{2m(k)-1})} d(Tx_{2n(k)}, Bx_{2m(k)-1}),\\
&\qquad \frac{1 + d(Tx_{2n(k)}, Bx_{2m(k)-1})}{2 + d(Tx_{2n(k)}, Sx_{2m(k)-1})} d(Ax_{2n(k)}, Sx_{2m(k)-1}),\\
&\qquad \frac{1 + d(Ax_{2n(k)}, Sx_{2m(k)-1})}{1 + 2d(Tx_{2n(k)}, Sx_{2m(k)-1})} d(Ax_{2n(k)}, Bx_{2m(k)-1}),\\
&\qquad \frac{1 + d(Tx_{2n(k)}, Bx_{2m(k)-1})}{1 + 2d(Tx_{2n(k)}, Sx_{2m(k)-1})} d(Ax_{2n(k)}, Bx_{2m(k)-1})\Big\}\\
&= \max\Big\{d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}),\\
&\qquad \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})],\\
&\qquad \frac{1 + d(y_{2n(k)}, y_{2m(k)})}{2 + d(y_{2n(k)+1}, y_{2m(k)})} d(y_{2n(k)+1}, y_{2m(k)-1}),\\
&\qquad \frac{1 + d(y_{2n(k)+1}, y_{2m(k)-1})}{2 + d(y_{2n(k)+1}, y_{2m(k)})} d(y_{2n(k)}, y_{2m(k)}),\\
&\qquad \frac{1 + d(y_{2n(k)}, y_{2m(k)})}{1 + 2d(y_{2n(k)+1}, y_{2m(k)})} d(y_{2n(k)}, y_{2m(k)-1}),\\
&\qquad \frac{1 + d(y_{2n(k)+1}, y_{2m(k)-1})}{1 + 2d(y_{2n(k)+1}, y_{2m(k)})} d(y_{2n(k)}, y_{2m(k)-1})\Big\}\\
&\to \max\left\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon), \frac{1+\varepsilon}{2+\varepsilon}\varepsilon, \frac{1+\varepsilon}{2+\varepsilon}\varepsilon, \frac{1+\varepsilon}{1+2\varepsilon}\varepsilon, \frac{1+\varepsilon}{1+2\varepsilon}\varepsilon\right\}\\
&= \varepsilon \text{ as } k \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(\int_0^{\varepsilon} \varphi(t)dt\right)\\
&= \limsup_{k\to\infty} \psi\left(\int_0^{d(y_{2n(k)+1}, y_{2m(k)})} \varphi(t)dt\right)
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{k\to\infty} \psi\bigg(\int_0^{d(Tx_{2n(k)}, Sx_{2m(k)-1})} \varphi(t)dt \bigg) \\
&\leqslant \limsup_{k\to\infty} \bigg[\psi\bigg(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \bigg) \\
&\qquad\qquad - \phi\bigg(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \bigg) \bigg] \\
&\leqslant \limsup_{k\to\infty} \psi\bigg(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \bigg) \\
&\qquad\qquad - \liminf_{k\to\infty} \phi\bigg(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt \bigg) \\
&< \psi\bigg(\int_0^{\varepsilon} \varphi(t)dt \bigg),
\end{aligned}
$$

which is absurd. Therefore, $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. It is clear that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Therefore, there exists $(z, w) \in A(X)\times X$ with $\lim_{n\to\infty} y_{2n} = z = Aw$. It is obvious that (2.14) holds. Suppose that $Tw \neq z$. Notice that (2.14)-(2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1 yield that

$$
\begin{aligned}
&M_2(w, x_{2n+1}) \\
&= \max\bigg\{ d(Aw, Bx_{2n+1}), d(Aw, Tw), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})], \frac{1 + d(Aw, Sx_{2n+1})}{2 + d(Tw, Sx_{2n+1})} d(Tw, Bx_{2n+1}), \\
&\qquad \frac{1 + d(Tw, Bx_{2n+1})}{2 + d(Tw, Sx_{2n+1})} d(Aw, Sx_{2n+1}), \frac{1 + d(Aw, Sx_{2n+1})}{1 + 2d(Tw, Sx_{2n+1})} d(Aw, Bx_{2n+1}), \\
&\qquad \frac{1 + d(Tw, Bx_{2n+1})}{1 + 2d(Tw, Sx_{2n+1})} d(Aw, Bx_{2n+1}) \bigg\} \\
&\to \max\bigg\{ d(Aw, z), d(Aw, Tw), d(z, z), \frac{1}{2}[d(Aw, z) + d(Tw, z)], \\
&\qquad \frac{1 + d(Aw, z)}{2 + d(Tw, z)} d(Tw, z), \frac{1 + d(Tw, z)}{2 + d(Tw, z)} d(Aw, z), \\
&\qquad \frac{1 + d(Aw, z)}{1 + 2d(Tw, z)} d(Aw, z), \frac{1 + d(Tw, z)}{1 + 2d(Tw, z)} d(Aw, z) \bigg\} \\
&= \max\bigg\{ 0, d(z, Tw), 0, \frac{1}{2} d(Tw, z), \frac{d(Tw, z)}{2 + d(Tw, z)}, 0, 0, 0 \bigg\} \\
&= d(Tw, z) \ \text{ as } \ n \to \infty
\end{aligned}
$$

and

$$\begin{aligned}
&\psi\bigg(\int_0^{d(Tw,z)}\varphi(t)dt\bigg)\\
&=\limsup_{n\to\infty}\psi\bigg(\int_0^{d(Tw,Sx_{2n+1})}\varphi(t)dt\bigg)\\
&\leqslant\limsup_{n\to\infty}\bigg[\psi\bigg(\int_0^{M_2(w,x_{2n+1})}\varphi(t)dt\bigg)-\phi\bigg(\int_0^{M_2(w,x_{2n+1})}\varphi(t)dt\bigg)\bigg]\\
&\leqslant\limsup_{n\to\infty}\psi\bigg(\int_0^{M_2(w,x_{2n+1})}\varphi(t)dt\bigg)-\liminf_{n\to\infty}\phi\bigg(\int_0^{M_2(w,x_{2n+1})}\varphi(t)dt\bigg)\\
&<\psi\bigg(\int_0^{d(Tw,z)}\varphi(t)dt\bigg),
\end{aligned}$$

which is absurd. Thus $Tw = z$. By means of (2.2), we gain that there exists a point $u \in X$ with $z = Bu = Tw$. Suppose that $Su \neq z$. Taking advantage of (2.14)-(2.16), $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we derive that

$$\begin{aligned}
&M_2(x_{2n},u)\\
&=\max\bigg\{d(Ax_{2n},Bu),d(Ax_{2n},Tx_{2n}),d(Bu,Su),\\
&\qquad\frac{1}{2}[d(Ax_{2n},Su)+d(Tx_{2n},Bu)],\\
&\qquad\frac{1+d(Ax_{2n},Su)}{2+d(Tx_{2n},Su)}d(Tx_{2n},Bu),\frac{1+d(Tx_{2n},Bu)}{2+d(Tx_{2n},Su)}d(Ax_{2n},Su),\\
&\qquad\frac{1+d(Ax_{2n},Su)}{1+2d(Tx_{2n},Su)}d(Ax_{2n},Bu),\frac{1+d(Tx_{2n},Bu)}{1+2d(Tx_{2n},Su)}d(Ax_{2n},Bu)\bigg\}\\
&\to\max\bigg\{d(z,Bu),d(z,z),d(Bu,Su),\frac{1}{2}[d(z,Su)+d(z,Bu)],\\
&\qquad\frac{1+d(z,Su)}{2+d(z,Su)}d(z,Bu),\frac{1+d(z,Bu)}{2+d(z,Su)}d(z,Su),\\
&\qquad\frac{1+d(z,Su)}{1+2d(z,Su)}d(z,Bu),\frac{1+d(z,Bu)}{1+2d(z,Su)}d(z,Bu)\bigg\}\\
&=\max\bigg\{0,0,d(z,Su),\frac{1}{2}d(z,Su),0,\frac{d(z,Su)}{1+d(z,Su)},0,0\bigg\}\\
&=d(z,Su)\ \text{ as }\ n\to\infty
\end{aligned}$$

and

$$
\begin{aligned}
&\psi\Big(\int_0^{d(z,Su)}\varphi(t)dt\Big)\\
&=\limsup_{n\to\infty}\psi\Big(\int_0^{d(Tx_{2n},Su)}\varphi(t)dt\Big)\\
&\leqslant\limsup_{n\to\infty}\Big[\psi\Big(\int_0^{M_2(x_{2n},u)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M_2(x_{2n},u)}\varphi(t)dt\Big)\Big]\\
&\leqslant\limsup_{n\to\infty}\psi\Big(\int_0^{M_2(x_{2n},u)}\varphi(t)dt\Big)-\liminf_{n\to\infty}\phi\Big(\int_0^{M_2(x_{2n},u)}\varphi(t)dt\Big)\\
&<\psi\Big(\int_0^{d(z,Su)}\varphi(t)dt\Big),
\end{aligned}
$$

which is a contradiction. Hence $Su = z$. Using (2.1), we know that $Az = ATw = TAw = Tz$ and $Bz = BSu = SBu = Sz$. Suppose that $Tz \neq Sz$. It follows from (2.15), (2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.2 that

$$
\begin{aligned}
&M_2(z,z)\\
&=\max\Big\{d(Az,Bz),d(Az,Tz),d(Bz,Sz),\frac{1}{2}[d(Az,Sz)+d(Tz,Bz)],\\
&\qquad\frac{1+d(Az,Sz)}{2+d(Tz,Sz)}d(Tz,Bz),\frac{1+d(Tz,Bz)}{2+d(Tz,Sz)}d(Az,Sz),\\
&\qquad\frac{1+d(Az,Sz)}{1+2d(Tz,Sz)}d(Az,Bz),\frac{1+d(Tz,Bz)}{1+2d(Tz,Sz)}d(Az,Bz)\Big\}\\
&=\max\Big\{d(Tz,Sz),0,0,\frac{1}{2}[d(Tz,Sz)+d(Tz,Sz)],d(Tz,Sz),\\
&\qquad\frac{1+d(Tz,Sz)}{2+d(Tz,Sz)}d(Tz,Sz),\frac{1+d(Tz,Sz)}{2+d(Tz,Sz)}d(Tz,Sz),\\
&\qquad\frac{1+d(Tz,Sz)}{1+2d(Tz,Sz)}d(Tz,Sz),\frac{1+d(Tz,Sz)}{1+2d(Tz,Sz)}d(Tz,Sz)\Big\}\\
&=d(Tz,Sz)
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi\Big(\int_0^{d(Tz,Sz)}\varphi(t)dt\Big)\\
&\leqslant\psi\Big(\int_0^{M_2(z,z)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M_2(z,z)}\varphi(t)dt\Big)
\end{aligned}
$$

$$= \psi\left(\int_0^{d(Tz,Sz)} \varphi(t)dt\right) - \phi\left(\int_0^{d(Tz,Sz)} \varphi(t)dt\right)$$
$$< \psi\left(\int_0^{d(Tz,Sz)} \varphi(t)dt\right),$$

which is absurd. Hence $Tz = Sz$, which yields that $Az = Bz = Tz = Sz$.

Suppose that $Tz \neq z$. Taking account of (2.15), (2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.2, we have

$$\begin{aligned}
&M_2(z,u)\\
&= \max\Big\{d(Az,Bu), d(Az,Tz), d(Bu,Su), \frac{1}{2}[d(Az,Su)+d(Tz,Bu)],\\
&\qquad \frac{1+d(Az,Su)}{2+d(Tz,Su)}d(Tz,Bu), \frac{1+d(Tz,Bu)}{2+d(Tz,Su)}d(Az,Su),\\
&\qquad \frac{1+d(Az,Su)}{1+2d(Tz,Su)}d(Az,Bu), \frac{1+d(Tz,Bu)}{1+2d(Tz,Su)}d(Az,Bu)\Big\}\\
&= \max\Big\{d(Tz,z), 0, 0, \frac{1}{2}[d(Tz,z)+d(Tz,z)],\\
&\qquad \frac{1+d(Tz,z)}{2+d(Tz,z)}d(Tz,z), \frac{1+d(Tz,z)}{2+d(Tz,z)}d(Tz,z),\\
&\qquad \frac{1+d(Tz,z)}{1+2d(Tz,z)}d(Tz,z), \frac{1+d(Tz,z)}{1+2d(Tz,z)}d(Tz,z)\Big\}\\
&= d(Tz,z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(\int_0^{d(Tz,z)} \varphi(t)dt\right)\\
&= \psi\left(\int_0^{d(Tz,Su)} \varphi(t)dt\right)\\
&\leqslant \psi\left(\int_0^{M_2(z,u)} \varphi(t)dt\right) - \phi\left(\int_0^{M_2(z,u)} \varphi(t)dt\right)\\
&= \psi\left(\int_0^{d(Tz,z)} \varphi(t)dt\right) - \phi\left(\int_0^{d(Tz,z)} \varphi(t)dt\right)\\
&< \psi\left(\int_0^{d(Tz,z)} \varphi(t)dt\right),
\end{aligned}$$

which is ridiculous. Therefore, $Tz = z$, that is, z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have another common fixed point $b \in \mathbb{X} \setminus \{z\}$. Making use of (2.15), (2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.2, we claim that

$$\begin{aligned}
&M_2(b,z)\\
&= \max\Big\{d(Ab,Bz), d(Ab,Tb), d(Bz,Sz), \frac{1}{2}[d(Ab,Sz)+d(Tb,Bz)],\\
&\qquad \frac{1+d(Ab,Sz)}{2+d(Tb,Sz)}d(Tb,Bz), \frac{1+d(Tb,Bz)}{2+d(Tb,Sz)}d(Ab,Sz),\\
&\qquad \frac{1+d(Ab,Sz)}{1+2d(Tb,Sz)}d(Ab,Bz), \frac{1+d(Tb,Bz)}{1+2d(Tb,Sz)}d(Ab,Bz)\Big\}\\
&= \max\Big\{d(b,z), 0, 0, \frac{1}{2}[d(b,z)+d(b,z)], d(b,z),\\
&\qquad \frac{1+d(b,z)}{2+d(b,z)}d(b,z), \frac{1+d(b,z)}{2+d(b,z)}d(b,z),\\
&\qquad \frac{1+d(b,z)}{1+2d(b,z)}d(b,z), \frac{1+d(b,z)}{1+2d(b,z)}d(b,z)\Big\}\\
&= d(b,z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)\\
&= \psi\Big(\int_0^{d(Tb,Sz)}\varphi(t)dt\Big)\\
&\leqslant \psi\Big(\int_0^{M_2(b,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_2(b,z)}\varphi(t)dt\Big)\\
&= \psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)\\
&< \psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big),
\end{aligned}$$

which is a contradiction. Hence z is a unique common fixed point of A, B, S and T in X.

Analogously we conclude that A, B, S and T have a unique common fixed point in X if one of $B(X), S(X)$ and $T(X)$ is complete. This completes the proof. □

Similar to the proof of Theorems 2.1 and 2.2, we have the following result and omit its proof.

Theorem 2.3. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (2.1) − (2.3) *and*

$$\psi\Big(\int_0^{d(Tx,Sy)}\varphi(t)dt\Big) \leqslant \psi\Big(\int_0^{M_3(x,y)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_3(x,y)}\varphi(t)dt\Big), \quad \forall x, y \in X, \tag{2.19}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ *and*

$$\begin{aligned} &M_3(x,y)\\ &= \Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)],\\ &\quad \frac{1+d(Ax,Sy)}{1+2d(Tx,Sy)}d(By,Sy), \frac{1+d(Tx,By)}{1+2d(Tx,Sy)}d(Ax,Tx),\\ &\quad \frac{1+d(Ax,Sy)d(Tx,By)}{1+d(Ax,By)d(Tx,Sy)}d(Ax,Tx),\\ &\quad \frac{1+d(Ax,Sy)d(Tx,By)}{1+d(Ax,By)d(Tx,Sy)}d(By,Sy)\Big\}, \quad \forall x, y \in X. \end{aligned} \tag{2.20}$$

Then A, B, S and T have a unique common fixed point in X.

3. Remark and examples

Remark 3.1. Theorems 2.1-2.3 generalize Theorem 1.2. Examples 3.1-3.3 show that Theorems 2.1-2.3 extend substantially Theorem 1.2, and differ from Theorems 1.1, 1.3 and 1.4.

Example 3.2. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $A, B, S, T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Ax = 2x,\ Bx = x^2,\ Sx = 0, \quad \forall x \in X,$$

$$Tx = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{8}\}, \\ \frac{1}{16}, & x = \frac{1}{8}, \end{cases}$$

$$\psi(t) = \log_2(t+1), \quad \varphi(t) = 2^t \ln 2, \quad \forall t \in \mathbb{R}^+,$$

$$\phi(t) = \begin{cases} \log_{256}(t+1), & \forall t \in [0,\ \sqrt{2}-1], \\ \frac{1}{16}, & \forall t \in (\sqrt{2}-1,\ +\infty). \end{cases}$$

It is easy to see that (2.1)-(2.3) hold, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, ψ is increasing, $\sup \phi(\mathbb{R}^+) \leqslant \frac{1}{16}$ and $\psi(t) \geqslant \frac{1}{2} > \frac{1}{16} = \phi(t)$ for each $t \in (\sqrt{2}-1,\ +\infty)$, that is,

$\psi(t) \geqslant \phi(t)$ for each $t \in \mathbb{R}^+$. Let $x, y \in X$. In order to prove (2.1), we need to consider two possible cases as follows:
Case 1. $x \in X \setminus \{\frac{1}{8}\}$. Obviously

$$\psi\Big(\int_0^{d(Tx,Sy)} \varphi(t)dt\Big) = 0 \leqslant \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big);$$

Case 2. $x = \frac{1}{8}$. Clearly

$$M_1(x,y) \geqslant d(Ax,Tx) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

and

$$\begin{aligned}\psi\Big(\int_0^{d(Tx,Sy)} \varphi(t)dt\Big) &= \psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) = \psi\big(2^{\frac{1}{16}} - 1\big) \\ &= \log_2 2^{\frac{1}{16}} = \frac{1}{16} < \frac{3}{16} - \frac{1}{16} \\ &\leqslant \psi\Big(\int_0^{\frac{3}{16}} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big) \\ &\leqslant \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big).\end{aligned}$$

Thus, (2.1) holds. It follows from Theorem 2.1 that the mappings A, B, S and T have a unique common fixed point $0 \in X$. But Theorems 1.1-1.4 are useless in proving the existence of fixed points of T and common fixed points of T and S in X.

Suppose that there exist ϕ and $\psi \in \Phi_5$ satisfy the conditions of Theorem 1.1. It follows from (1.1) that

$$\begin{aligned}\psi\Big(\frac{1}{16}\Big) &= \psi\Big(d\Big(T\frac{1}{8}, T\frac{1}{16}\Big)\Big) \leqslant \psi\Big(d\Big(\frac{1}{8}, \frac{1}{16}\Big)\Big) - \phi\Big(d\Big(\frac{1}{8}, \frac{1}{16}\Big)\Big) \\ &= \psi\Big(\frac{1}{16}\Big) - \phi\Big(\frac{1}{16}\Big) < \psi\Big(\frac{1}{16}\Big),\end{aligned}$$

which is absurd.

Suppose that there exist $c \in (0, 1)$ and $\varphi \in \Phi_1$ satisfy the conditions of Theorem 1.2. By (1.2), we get that

$$0 < \int_0^{\frac{1}{16}} \varphi(t)dt = \int_0^{d(T\frac{1}{8}, T\frac{1}{16})} \varphi(t)dt \leqslant c\int_0^{d(\frac{1}{8}, \frac{1}{16})} \varphi(t)dt < \int_0^{\frac{1}{16}} \varphi(t)dt,$$

which is a contradiction.

Suppose that there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_5$ satisfies the conditions of Theorem 1.3. Using (1.3), we gain that

$$\begin{aligned}\psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) &= \psi\Big(\int_0^{d(T\frac{1}{8}, T\frac{1}{16})} \varphi(t)dt\Big) \\ &\leqslant \psi\Big(\int_0^{d(\frac{1}{8}, \frac{1}{16})} \varphi(t)dt\Big) - \phi\Big(\int_0^{d(\frac{1}{8}, \frac{1}{16})} \varphi(t)dt\Big) \\ &= \psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) \\ &< \psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big),\end{aligned}$$

which is impossible.

Suppose that there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_4 \times \Phi_5$ satisfies the conditions of Theorem 1.4. It follows from (1.4) and (1.5) that

$$\begin{aligned}M\Big(\frac{1}{8}, \frac{1}{16}\Big) &= \max\Big\{d\Big(\frac{1}{8}, \frac{1}{16}\Big), d\Big(\frac{1}{8}, T\frac{1}{8}\Big), d\Big(\frac{1}{16}, S\frac{1}{16}\Big), \\ &\qquad \frac{1}{2}\Big[d\Big(\frac{1}{16}, T\frac{1}{8}\Big) + d\Big(\frac{1}{8}, S\frac{1}{16}\Big)\Big]\Big\} \\ &= \frac{1}{16}\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) &= \psi\Big(\int_0^{d(T\frac{1}{8}, S\frac{1}{16})} \varphi(t)dt\Big) \\ &\leqslant \psi\Big(\int_0^{M(\frac{1}{8}, \frac{1}{16})} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(\frac{1}{8}, \frac{1}{16})} \varphi(t)dt\Big) \\ &= \psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big) \\ &< \psi\Big(\int_0^{\frac{1}{16}} \varphi(t)dt\Big),\end{aligned}$$

which is absurd.

Example 3.3. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $A, B, S, T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Ax = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{4}\}, \\ \frac{1}{2}, & \forall x = \frac{1}{4}, \end{cases} \qquad Tx = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{4}\}, \\ \frac{1}{8}, & \forall x = \frac{1}{4}, \end{cases}$$

$$Bx = x, \quad Sx = 0, \quad \forall x \in X,$$
$$\psi(t) = 64t, \quad \varphi(t) = 2t, \quad \phi(t) = \frac{t}{t+1}, \quad \forall t \in \mathbb{R}^+.$$

It is easy to see that (2.1)-(2.3) hold, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, ψ is increasing, $\psi(t) \geqslant \phi(t)$ and $\phi(t) < 1$ for each $t \in \mathbb{R}^+$. Let $x, y \in X$. In order to prove (2.15), we need to consider two possible cases as follows:

Case 1. $x \in X \setminus \{\frac{1}{4}\}$. It is easy to see that

$$\psi\bigg(\int_0^{d(Tx,Sy)} \varphi(t)dt\bigg) = 0 \leqslant \psi\bigg(\int_0^{M_2(x,y)} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_2(x,y)} \varphi(t)dt\bigg);$$

Case 2. $x = \frac{1}{4}$. It is clear that

$$M_2(x,y) \geqslant d(Ax, Tx) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

and

$$\begin{aligned}\psi\bigg(\int_0^{d(Tx,Sy)} \varphi(t)dt\bigg) &= \psi\bigg(\int_0^{\frac{1}{8}} \varphi(t)dt\bigg) = 1 < 9 - 1 \\ &< \psi\bigg(\int_0^{\frac{3}{8}} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_2(x,y)} \varphi(t)dt\bigg) \\ &\leqslant \psi\bigg(\int_0^{M_2(x,y)} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_2(x,y)} \varphi(t)dt\bigg).\end{aligned}$$

Thus, (2.15) holds. It follows from Theorem 2.2 that the mappings A, B, S and T have a unique common fixed point $0 \in X$. But Theorems 1.1-1.4 are useless in proving the existence of fixed points of T and common fixed points of T and S in X.

Suppose that there exist ϕ and $\psi \in \Phi_5$ satisfy the conditions of Theorem 1.1. It follows from (1.1) that

$$\begin{aligned}\psi\Big(\frac{1}{8}\Big) &= \psi\Big(d\Big(T\frac{1}{4}, T\frac{1}{8}\Big)\Big) \leqslant \psi\Big(d\Big(\frac{1}{4}, \frac{1}{8}\Big)\Big) - \phi\Big(d\Big(\frac{1}{4}, \frac{1}{8}\Big)\Big) \\ &= \psi\Big(\frac{1}{8}\Big) - \phi\Big(\frac{1}{8}\Big) < \psi\Big(\frac{1}{8}\Big),\end{aligned}$$

which is absurd.

Suppose that there exist $c \in (0, 1)$ and $\varphi \in \Phi_1$ satisfy the conditions of Theorem 1.2. By (1.2), we get that

$$0 < \int_0^{\frac{1}{8}} \varphi(t)dt = \int_0^{d(T\frac{1}{4}, T\frac{1}{8})} \varphi(t)dt \leqslant c\int_0^{d(\frac{1}{4}, \frac{1}{8})} \varphi(t)dt < \int_0^{\frac{1}{8}} \varphi(t)dt,$$

which is a contradiction.

Suppose that there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_5$ satisfies the conditions of Theorem 1.3. Using (1.3), we gain that

$$\begin{aligned}\psi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big) &= \psi\Big(\int_0^{d(T\frac{1}{4}, T\frac{1}{8})} \varphi(t)dt\Big) \\ &\leqslant \psi\Big(\int_0^{d(\frac{1}{4}, \frac{1}{8})} \varphi(t)dt\Big) - \phi\Big(\int_0^{d(\frac{1}{4}, \frac{1}{8})} \varphi(t)dt\Big) \\ &= \psi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big) \\ &< \psi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big),\end{aligned}$$

which is impossible.

Suppose that there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_4 \times \Phi_5$ satisfies the conditions of Theorem 1.4. It follows from (1.4) and (1.5) that

$$\begin{aligned}M\Big(\frac{1}{4}, \frac{1}{8}\Big) &= \max\Big\{d\Big(\frac{1}{4}, \frac{1}{8}\Big), d\Big(\frac{1}{4}, T\frac{1}{4}\Big), d\Big(\frac{1}{8}, S\frac{1}{8}\Big), \\ &\qquad \frac{1}{2}\Big[d\Big(\frac{1}{8}, T\frac{1}{4}\Big) + d\Big(\frac{1}{4}, S\frac{1}{8}\Big)\Big]\Big\} \\ &= \frac{1}{8}\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big) &= \psi\Big(\int_0^{d(T\frac{1}{4}, S\frac{1}{8})} \varphi(t)dt\Big) \\ &\leqslant \psi\Big(\int_0^{M(\frac{1}{4}, \frac{1}{8})} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(\frac{1}{4}, \frac{1}{8})} \varphi(t)dt\Big) \\ &= \psi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big) \\ &< \psi\Big(\int_0^{\frac{1}{8}} \varphi(t)dt\Big),\end{aligned}$$

which is absurd.

Example 3.4. Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $A, B, S, T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Tx = \begin{cases} \frac{2}{3}, & \forall x \in [0, \frac{1}{3}], \\ 1, & \forall x \in (\frac{1}{3}, 1], \end{cases}$$

$$Ax = x^2, \quad Bx = x, \quad Sx = 1, \quad \forall x \in X,$$
$$\psi(t) = 9t, \quad \varphi(t) = 1, \quad \phi(t) = \frac{t}{10t+1}, \quad \forall t \in \mathbb{R}^+.$$

It is easy to see that (2.1)-(2.3) hold, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, ψ is increasing, $\psi(t) \geqslant \phi(t)$ for each $t \in \mathbb{R}^+$ and $\sup \phi(\mathbb{R}^+) \leqslant \frac{1}{10}$. Let $x, y \in X$. In order to prove (2.19), we need to consider two possible cases as follows:

Case 1. $x \in [0, \frac{1}{3}]$. It follows that

$$M_3(x,y) \geqslant d(Ax, Tx) = \left| x^2 - \frac{2}{3} \right| \geqslant \frac{5}{9}$$

and

$$\begin{aligned}
\psi\bigg(\int_0^{d(Tx,Sy)} \varphi(t)dt\bigg) &= \psi\bigg(\int_0^{\frac{1}{3}} \varphi(t)dt\bigg) = 3 < 5 - \frac{1}{10} \\
&\leqslant \psi\bigg(\int_0^{\frac{5}{9}} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_3(x,y)} \varphi(t)dt\bigg) \\
&\leqslant \psi\bigg(\int_0^{M_3(x,y)} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_3(x,y)} \varphi(t)dt\bigg);
\end{aligned}$$

Case 2. $x \in (\frac{1}{3}, 1]$. Clearly

$$\psi\bigg(\int_0^{d(Tx,Sy)} \varphi(t)dt\bigg) = 0 \leqslant \psi\bigg(\int_0^{M_3(x,y)} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_3(x,y)} \varphi(t)dt\bigg).$$

Thus, (2.19) holds. It follows from Theorem 2.3 that the mappings A, B, S and T have a unique common fixed point $1 \in X$. But Theorems 1.1-1.4 are useless in proving the existence of fixed points of T and common fixed points of T and S in X.

Suppose that there exist ϕ and $\psi \in \Phi_5$ satisfy the conditions of Theorem 1.1. It follows from (1.1) that

$$\begin{aligned}
\psi\Big(\frac{1}{3}\Big) &= \psi\Big(d\Big(T\frac{2}{3}, T\frac{1}{3}\Big)\Big) \leqslant \psi\Big(d\Big(\frac{2}{3}, \frac{1}{3}\Big)\Big) - \phi\Big(d\Big(\frac{2}{3}, \frac{1}{3}\Big)\Big) \\
&= \psi\Big(\frac{1}{3}\Big) - \phi\Big(\frac{1}{3}\Big) < \psi\Big(\frac{1}{3}\Big),
\end{aligned}$$

which is absurd.

Suppose that there exist $c \in (0, 1)$ and $\varphi \in \Phi_1$ satisfy the conditions of Theorem 1.2. By (1.2), we get that

$$0 < \int_0^{\frac{1}{3}} \varphi(t)dt = \int_0^{d(T\frac{2}{3}, T\frac{1}{3})} \varphi(t)dt \leqslant c \int_0^{d(\frac{2}{3}, \frac{1}{3})} \varphi(t)dt < \int_0^{\frac{1}{3}} \varphi(t)dt,$$

which is a contradiction.

Suppose that there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_5$ satisfies the conditions of Theorem 1.3. Using (1.3), we gain that

$$\begin{aligned}\psi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big) &= \psi\Big(\int_0^{d(T\frac{2}{3}, T\frac{1}{3})} \varphi(t)dt\Big)\\ &\leqslant \psi\Big(\int_0^{d(\frac{2}{3}, \frac{1}{3})} \varphi(t)dt\Big) - \phi\Big(\int_0^{d(\frac{2}{3}, \frac{1}{3})} \varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big),\end{aligned}$$

which is impossible.

Suppose that there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_4 \times \Phi_5$ satisfies the conditions of Theorem 1.4. It follows from (1.4) and (1.5) that

$$\begin{aligned}M\Big(\frac{1}{3}, \frac{2}{3}\Big) &= \max\Big\{d\Big(\frac{1}{3}, \frac{2}{3}\Big), d\Big(\frac{1}{3}, T\frac{1}{3}\Big), d\Big(\frac{2}{3}, S\frac{2}{3}\Big),\\ &\qquad \frac{1}{2}\Big[d\Big(\frac{2}{3}, T\frac{1}{3}\Big) + d\Big(\frac{1}{3}, S\frac{2}{3}\Big)\Big]\Big\}\\ &= \frac{1}{3}\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big) &= \psi\Big(\int_0^{d(T\frac{1}{3}, S\frac{2}{3})} \varphi(t)dt\Big)\\ &\leqslant \psi\Big(\int_0^{M(\frac{1}{3}, \frac{2}{3})} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(\frac{1}{3}, \frac{2}{3})} \varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{\frac{1}{3}} \varphi(t)dt\Big),\end{aligned}$$

which is absurd.

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