Nonlinear Functional Analysis and Applications
Vol. 23, No. 3 (2018), pp. 503-525
ISSN: 1229-1595(print), 2466-0973(online)

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COMMON STATIONARY POINTS FOR MULTIVALUED CONTRACTIVE MAPPINGS OF INTEGRAL TYPE WITH δ-DISTANCE

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Abstract. The existence and uniqueness of stationary points for certain multi-valued contractive mappings of integral type with δ-distance in complete metric spaces are established and three illustrative examples are provided. The results presented in this paper extend or differ from several known results in the literature.

1. Introduction and preliminaries

Branciari [2] generalized the Banach fixed point theorem and proved the following fixed point theorem for the contractive mapping of integral type.

[0]Received January 22, 2018. Revised April 17, 2018.
[0]2010 Mathematics Subject Classification: 54H25.
[0]Keywords: Multivalued contractive mappings of integral type with δ-distance, common stationary point, bounded complete metric space.
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Theorem 1.1. ([2]) *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.1}$$

where $c \in [0, 1)$ is a constant and $\varphi \in \Phi_1 = \big\{\varphi : \ \varphi : [0, +\infty) \to [0, +\infty)$ is Lebesgue integrable, summable on each compact subset of $[0, +\infty)$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\big\}$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Afterwards the authors [1–4, 9, 10, 13–18, 20] and others continued the study of Branciari and obtained a lot of fixed point and common fixed point theorems for various single-valued and multi-valued contractive mappings of integral type. In particular, Liu *et al.* [13] extended the result of Branciari and established the following fixed point theorems for some new contractive mappings of integral type.

Theorem 1.2. ([13]) *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha(d(x, y)) \int_0^{m(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.2}$$

where $\varphi \in \Phi_1$, $\alpha \in \Theta = \{\alpha : \alpha : [0, +\infty) \to [0, 1)$ is a function with $\limsup_{s\to t} \alpha(s) < 1$ for each $t \in [0, +\infty)\}$ and

$$m(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\Big\}. \tag{1.3}$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Theorem 1.3. ([13]) *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha(m(x, y)) \int_0^{m(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.4}$$

where $\varphi \in \Phi_1$, $\alpha \in \Theta$ and m is defined by (1.3). Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Jachymski [10] proved the following fixed point theorem for the multi-valued contractive mapping of integral type with the Hausdorff metric H.

Theorem 1.4. ([10]) *Let (X,d) be a bounded complete metric space, $F : X \to K(X)$, which is the family of all nonempty compact subsets of X, be a multi-valued mapping satisfying*

$$\int_0^{H(Fx,Fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt, \quad \forall x,y \in X, \tag{1.5}$$

where $\varphi \in \Phi_1$ and $c \in [0,1)$ is a constant. Then F has a fixed point.

On the other hand, Stojakovic *et al.* [20] deduced the following fixed point theorem for the multi-valued contractive mapping of integral type with the δ-distance.

Theorem 1.5. ([20]) *Let (X,d) be a bounded complete metric space, $F : X \to B(X)$, which is the family of all nonempty bounded subsets of X, be a multi-valued mapping satisfying, for all $x,y \in X$ with $x \neq y$,*

$$\phi\bigg(\int_0^{\delta(Fx,Fy)} \varphi(t)dt\bigg) \le \alpha(d(x,y))\phi\bigg(\int_0^{d(x,y)} \varphi(t)dt\bigg), \tag{1.6}$$

where $\varphi \in \Phi_1$, $\alpha \in \Theta$ and $\phi \in \Phi_2 = \big\{\phi : \phi : [0,+\infty) \to [0,+\infty)$ is upper semicontinuous and nondecreasing and $\phi(t) < t$ for each $t > 0\big\}$. Then F has a fixed point.

Motivated and inspired by the results in [1–20], in this paper we introduce a few new multi-valued contractive mappings of integral type with δ-distance, establish the existence and uniqueness of common stationary points for these contractive mappings of integral type with δ-distance and construct three examples to illustrate that the results obtained generalize or differ from a few results in [2, 3, 7, 10–14, 17, 20].

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ = [0,+\infty)$ and
$\Phi_3 = \big\{\phi : \phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is upper semicontinuous and nondecreasing in each coordinate variable and $\phi(t,t,t,t,t) < t$ for each $t > 0\big\}$.

Let (X,d) be a metric space and $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. The Hausdorff metric $H : CB(X) \times CB(X) \to \mathbb{R}^+$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y \in B} d(y,A)\right\}, \quad \forall A,B \in CB(X),$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$. For $A,B \subseteq X$, define

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\} \text{ and } \delta(A,A) = \delta(A).$$

If A is singleton {a}, we write $\delta(A,B) = \delta(a,B)$. Let $F,G : X \to B(X)$ and $f : X \to X$. A point $x \in X$ is called a stationary point of F if $Fx = \{x\}$. Note that every stationary point of F is a fixed point of F, but not conversely. A point $x \in X$ is called a common stationary point of F and G if $Fx = Gx = \{x\}$. F and G are said to be commuting if $FGx = GFx$ for all $x \in X$. F and f are said to be commuting if $Ffx = fFx$ for all $x \in X$. Define

$$C_F = \{T : T : X \to B(X) \text{ satisfies that } T \text{ and } F \text{ are commuting } \}$$

and

$$CC_F = \{f : f : X \to X \text{ is continuous and } F \text{ and } f \text{ are commuting } \}.$$

It is clear that $C_F \supseteq \{F^n : n \in \mathbb{N}_0\}$, where $F^0 x = \{x\}$ for $x \in X$.

Definition 1.6. ([8]) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets in $B(X)$ and $A \in B(X)$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to converge to the set A if

(1) each point $a \in A$ is the limit of some convergent sequence $\{a_n\}_{n\in\mathbb{N}}$, where $a_n \in A_n$ for $n \in \mathbb{N}$;

(2) for arbitrary $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $A_n \subseteq A_\varepsilon$ for $n > k$, where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 1.7. ([14]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0$ if and only if $\lim_{n\to\infty} r_n = 0$.*

Lemma 1.8. ([19]) *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be upper semicontinuous and nondecreasing. Then for every $t > 0$, $\psi(t) < t$ if and only if $\lim_{n\to\infty} \psi^n(t) = 0$, where ψ^n denotes the composition of ψ with itself n-times.*

Lemma 1.9. ([5]) *If $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}_{n\in\mathbb{N}}$ converges to $\delta(A, B)$.*

2. Common stationary point theorems

In this section, we give five common stationary point theorems for the following multi-valued contractive mappings of integral type (2.1), (2.9), (2.11), (2.13) and (2.14).

Theorem 2.1. *Let (X, d) be a bounded complete metric space, $F, G : X \to B(X)$ be continuous and commuting mappings satisfying*

$$\int_0^{\delta(F^pG^qx, F^iG^jy)} \varphi(t)dt \le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\bigg), \quad \forall x, y \in X, \quad (2.1)$$

where $p, q, i, j \in \mathbb{N}$ and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$. Then

(a) *F and G have a unique common stationary point $z \in X$;*

(b) *The sequence $\{F^nG^nx\}_{n\in\mathbb{N}}$ converges to $\{z\}$ for all $x \in X$.*

Proof. Let $M = \delta(X)$, $l_1 = \max\{p, q\}$, $l_2 = \max\{i, j\}$, $k = l_1 + l_2$, $X_n = F^nG^nX$ and $\delta_n = \delta(X_n)$ for each $n \in \mathbb{N}$. Clearly,

$$X_{n+1} \subseteq X_n, \quad \forall n \in \mathbb{N} \tag{2.2}$$

and for all $(n, D) \in \mathbb{N} \times C_{FG}$,

$$DX_n = DF^nG^nX = F^nG^nDX \subseteq F^nG^nX = X_n. \tag{2.3}$$

Let A, $B \in X$. It follows from (2.1) and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned}\int_0^{\delta(F^pG^qa, F^iG^jb)} \varphi(t)dt &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D\{a,b\})} \varphi(t)dt \bigg) \\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}}(DA\cup DB))} \varphi(t)dt \bigg), \quad \forall (a, b) \in A \times B,\end{aligned}$$

which yields that

$$\int_0^{\delta(F^pG^qA, F^iG^jB)} \varphi(t)dt \le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}}(DA\cup DB))} \varphi(t)dt \bigg). \tag{2.4}$$

Put $n \in \mathbb{N}$. It is clear that there exist $a_n, b_n \in \mathbb{N}_0$ with $0 \le b_n < k$ satisfying

$$n = ka_n + b_n \quad \text{and} \quad n \to \infty \quad \text{if and only if} \quad a_n \to \infty. \tag{2.5}$$

It follows from (2.1)-(2.5) and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned}\delta_n &= \delta(F^{l_1}G^{l_1}(F^{l_2+b_n}G^{l_2+b_n}X_{k(a_n-1)}), F^{l_2}G^{l_2}(F^{l_1+b_n}G^{l_1+b_n}X_{k(a_n-1)})) \\ &= \delta(F^pG^q(F^{l_1+l_2+b_n-p}G^{l_1+l_2+b_n-q}X_{k(a_n-1)}), \\ &\quad F^iG^j(F^{l_1+l_2+b_n-i}G^{l_1+l_2+b_n-j}X_{k(a_n-1)})),\end{aligned}$$

$$\begin{aligned}&\delta(\cup_{D\in C_{FG}} D(F^{l_1+l_2+b_n-p}G^{l_1+l_2+b_n-q}X_{k(a_n-1)} \\ &\qquad \cup F^{l_1+l_2+b_n-i}G^{l_1+l_2+b_n-j}X_{k(a_n-1)})) \\ &= \delta(\cup_{D\in C_{FG}} F^{k(a_n-1)}G^{k(a_n-1)}(DF^{l_1+l_2+b_n-p}G^{l_1+l_2+b_n-q}X \\ &\qquad \cup DF^{l_1+l_2+b_n-i}G^{l_1+l_2+b_n-j}X)) \\ &\le \delta(X_{k(a_n-1)}) = \delta_{k(a_n-1)}\end{aligned}$$

and

$$\int_0^{\delta_n} \varphi(t)dt = \int_0^{N_1} \varphi(t)dt \le \phi\bigg(\int_0^{N_2} \varphi(t)dt \bigg) \le \phi\bigg(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt \bigg),$$

where

$$\begin{aligned}N_1 = \delta(&F^pG^q(F^{l_1+l_2+b_n-p}G^{l_1+l_2+b_n-q}X_{k(a_n-1)}), \\ &F^iG^j(F^{l_1+l_2+b_n-i}G^{l_1+l_2+b_n-j}X_{k(a_n-1)}))\end{aligned}$$

and

$$N_2 = \delta(\cup_{D\in C_{FG}} D(F^{l_1+l_2+b_n-p}G^{l_1+l_2+b_n-q}X_{k(a_n-1)} \\ \cup F^{l_1+l_2+b_n-i}G^{l_1+l_2+b_n-j}X_{k(a_n-1)})),$$

which implies that

$$\int_0^{\delta_n}\varphi(t)dt \le \phi\Big(\int_0^{\delta_{k(a_n-1)}}\varphi(t)dt\Big) \le \phi^2\Big(\int_0^{\delta_{k(a_n-2)}}\varphi(t)dt\Big) \\ \le \cdots \le \phi^{a_n-1}\Big(\int_0^{\delta_k}\varphi(t)dt\Big) \le \phi^{a_n}\Big(\int_0^{M}\varphi(t)dt\Big).$$

That is,

$$\int_0^{\delta_n}\varphi(t)dt \le \phi^{a_n}\Big(\int_0^{M}\varphi(t)dt\Big), \tag{2.6}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is defined by (2.5).

Choose $x_n \in X_n$ for each $n \in \mathbb{N}$. By (2.2), (2.5) and (2.6), we get that

$$\int_0^{d(x_n,x_m)}\varphi(t)dt \le \int_0^{\delta(X_n,X_m)}\varphi(t)dt \le \int_0^{\delta_n}\varphi(t)dt \\ \le \phi^{a_n}\Big(\int_0^{M}\varphi(t)dt\Big), \quad \forall m,n\in\mathbb{N} \text{ with } m>n. \tag{2.7}$$

Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence by (2.7), Lemmas 1.7 and 1.8. Since X is complete, it follows that there exists a point z in X such that $x_n \to z$ as $n\to\infty$. From (2.2), we have

$$\begin{aligned}\delta(z,X_n) &\le d(z,x_m)+\delta(x_m,X_n)\\ &\le d(z,x_m)+\delta(X_m,X_n)\\ &\le d(z,x_m)+\delta_n, \quad \forall m,n\in\mathbb{N} \text{ with } m>n.\end{aligned}$$

Letting m tend to infinity, we obtain that

$$\delta(z,X_n)\le\delta_n, \quad \forall n\in\mathbb{N}.$$

Since F and G are continuous and $x_n\to z$ as $n\to\infty$, it follows that $\{Fx_n\}_{n\in\mathbb{N}}$ and $\{Gx_n\}_{n\in\mathbb{N}}$ converge to $\{Fz\}$ and $\{Gz\}$, respectively. Note that

$$Fx_n \subseteq F(F^nG^nX) = F^nG^n(FX)\subseteq X_n, \quad \forall n\in\mathbb{N},$$

$$Gx_n \subseteq G(F^nG^nX) = F^nG^n(GX)\subseteq X_n, \quad \forall n\in\mathbb{N},$$

which yield that

$$\max\{\delta(z,Fx_n),\delta(z,Gx_n)\}\le\delta(z,X_n)\le\delta_n, \quad \forall n\in\mathbb{N}. \tag{2.8}$$

By virtue of (2.5), (2.6), (2.8), Lemmas 1.8 and 1.9, we have

$$\begin{aligned} 0 &\le \int_0^{\max\{\delta(z,Fx_n),\delta(z,Gx_n)\}} \varphi(t)dt \le \int_0^{\delta_n} \varphi(t)dt \\ &\le \phi^{a_n}\Big(\int_0^M \varphi(t)dt\Big) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which together with Lemma 1.7 and the continuity of F and G means that

$$\int_0^{\max\{\delta(z,Fz),\delta(z,Gz)\}} \varphi(t)dt = 0,$$

that is, $\max\{\delta(z,Fz),\delta(z,Gz)\} = 0$. Consequently, we conclude immediately that $Fz = Gz = \{z\}$.

Suppose that F and G have a second common stationary point $\omega \in X - \{z\}$. Obviously $\{u\} = F^n G^n u \subseteq X_n$ for each $u \in \{z, \omega\}$ and $n \in \mathbb{N}$. In view of (2.2), (2.5), (2.6), Lemmas 1.7 and 1.8, we infer that

$$0 \le \int_0^{d(z,\omega)} \varphi(t)dt \le \int_0^{\delta_n} \varphi(t)dt \le \phi^{a_n}\Big(\int_0^M \varphi(t)dt\Big) \to 0 \quad \text{as } n \to \infty,$$

which yields that $z = \omega$. Hence F and G have a unique common stationary point z.

Choose $y_n \in F^n G^n x$ for each $(x,n) \in X \times \mathbb{N}$. By means of (2.2), (2.5), (2.6), Definition 1.6, Lemmas 1.7 and 1.8, we have

$$\begin{aligned} 0 &\le \int_0^{d(y_n,z)} \varphi(t)dt \le \int_0^{\delta(F^nG^nx,z)} \varphi(t)dt \le \int_0^{\delta(X_n,z)} \varphi(t)dt \\ &\le \int_0^{\delta_n} \varphi(t)dt \le \phi^{a_n}\Big(\int_0^M \varphi(t)dt\Big) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which means that $\{F^n G^n x\}_{n\in\mathbb{N}}$ converges to $\{z\}$. This completes the proof. $\square$

Theorem 2.2. *Let (X,d) be a bounded complete metric space, $F, G : X \to B(X)$ be continuous and commuting mappings satisfying*

$$\int_0^{\delta(F^pG^qx,F^iy)} \varphi(t)dt \le \phi\Big(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\Big), \quad \forall x, y \in X, \qquad (2.9)$$

where $p, q, i \in \mathbb{N}$ and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$. Then (a) *and* (b) *of Theorem* 2.1 *hold.*

Proof. Let $M = \delta(X)$, $l = \max\{p,q\}$, $k = l+i$, $X_n = F^n G^n X$ and $\delta_n = \delta(X_n)$ for each $n \in \mathbb{N}$. Clearly, (2.2), (2.3) and (2.5) hold. Let A, $B \in X$. It follows from (2.9) and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned}\int_0^{\delta(F^pG^qa, F^ib)} \varphi(t)dt &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D\{a,b\})} \varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} (DA\cup DB))} \varphi(t)dt\bigg), \quad \forall (a,b) \in A\times B,\end{aligned}$$

which yields that

$$\int_0^{\delta(F^pG^qA, F^iy)} \varphi(t)dt) \le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} (DA\cup DB))} \varphi(t)dt\bigg). \tag{2.10}$$

In view of (2.2), (2.3), (2.5), (2.9), (2.10) and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$, we get that

$$\begin{aligned}\delta_n &= \delta(F^lG^l(F^{i+b_n}G^{i+b_n}X_{k(a_n-1)}), F^i(F^{l+b_n}G^{l+i+b_n}X_{k(a_n-1)}))\\ &= \delta(F^pG^q(F^{l+i+b_n-p}G^{l+i+b_n-q}X_{k(a_n-1)}), F^i(F^{l+b_n}G^{l+i+b_n}X_{k(a_n-1)})),\end{aligned}$$

$$\begin{aligned}&\delta(\cup_{D\in C_{FG}} D(F^{l+i+b_n-p}G^{l+i+b_n-q}X_{k(a_n-1)} \cup F^{l+b_n}G^{l+i+b_n}X_{k(a_n-1)}))\\ &= \delta(\cup_{D\in C_{FG}} F^{k(a_n-1)}G^{k(a_n-1)}(DF^{l+i+b_n-p}G^{l+i+b_n-q}X \cup DF^{l+b_n}G^{l+i+b_n}X))\\ &\le \delta_{k(a_n-1)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\delta_n} \varphi(t)dt\\ &= \int_0^{\delta(F^pG^q(F^{l+i+b_n-p}G^{l+i+b_n-q}X_{k(a_n-1)}), F^i(F^{l+b_n}G^{l+i+b_n}X_{k(a_n-1)}))} \varphi(t)dt\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D(F^{l+i+b_n-p}G^{l+i+b_n-q}X_{k(a_n-1)}\cup F^{l+b_n}G^{l+i+b_n}X_{k(a_n-1)}))} \varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt\bigg),\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^{\delta_n} \varphi(t)dt &\le \phi\bigg(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt\bigg) \le \phi^2\bigg(\int_0^{\delta_{k(a_n-2)}} \varphi(t)dt\bigg)\\ &\le \cdots \le \phi^{a_n-1}\bigg(\int_0^{\delta_k} \varphi(t)dt\bigg) \le \phi^{a_n}\bigg(\int_0^{M} \varphi(t)dt\bigg).\end{aligned}$$

That is,

$$\int_0^{\delta_n} \varphi(t)dt \le \phi^{a_n}\bigg(\int_0^{M} \varphi(t)dt\bigg),$$

where $\{a_n\}_{n\in\mathbb{N}}$ is defined by (2.5). That is, (2.6) holds. The remaining portion of the proof can be derived as in that of Theorem 2.1. This completes the proof. $\square$

Theorem 2.3. *Let (X,d) be a bounded complete metric space, $F,G : X \to B(X)$ be continuous and commuting mappings satisfying*

$$\int_0^{\delta(F^p x, G^j y)} \varphi(t)dt \le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\bigg), \quad \forall x,y \in X, \tag{2.11}$$

where $p,j \in \mathbb{N}$ and $(\varphi,\phi) \in \Phi_1 \times \Phi_2$. Then (a) *and* (b) *of Theorem* 2.1 *hold.*

Proof. Let $M = \delta(X)$, $k = p+j$, $X_n = F^n G^n X$ and $\delta_n = \delta(X_n)$ for each $n \in \mathbb{N}$. Clearly, (2.2), (2.3) and (2.5) hold. Let A, $B \in X$. It follows from (2.11) and $(\varphi,\phi) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned}\int_0^{\delta(F^p a, G^j b)} \varphi(t)dt &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D\{a,b\})} \varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} (DA\cup DB))} \varphi(t)dt\bigg), \quad \forall (a,b) \in A\times B,\end{aligned}$$

which yields that

$$\int_0^{\delta(F^p A, G^j B)} \varphi(t)dt) \le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} (DA\cup DB))} \varphi(t)dt\bigg). \tag{2.12}$$

By virtue of (2.2), (2.3), (2.5), (2.11) and (2.12), we deduce that

$$\begin{aligned}&\delta(\cup_{D\in C_{FG}} D(F^{j+b_n}G^{k+b_n}X_{k(a_n-1)} \cup F^{k+b_n}G^{p+b_n}X_{k(a_n-1)}))\\ &= \delta(\cup_{D\in C_{FG}} F^{k(a_n-1)}G^{k(a_n-1)}(DF^{j+b_n}G^{k+b_n}X \cup DF^{k+b_n}G^{p+b_n}X))\\ &\le \delta_{k(a_n-1)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\delta_n} \varphi(t)dt\\ &= \int_0^{\delta(F^p(F^{j+b_n}G^{k+b_n}X_{k(a_n-1)}), G^j(F^{k+b_n}G^{p+b_n}X_{k(a_n-1)}))} \varphi(t)dt\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}} D(F^{j+b_n}G^{k+b_n}X_{k(a_n-1)}\cup F^{k+b_n}G^{p+b_n}X_{k(a_n-1)}))} \varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt\bigg),\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^{\delta_n}\varphi(t)dt &\le \phi\bigg(\int_0^{\delta_{k(a_n-1)}}\varphi(t)dt\bigg) \le \phi^2\bigg(\int_0^{\delta_{k(a_n-2)}}\varphi(t)dt\bigg)\\ &\le \cdots \le \phi^{a_n-1}\bigg(\int_0^{\delta_k}\varphi(t)dt\bigg) \le \phi^{a_n}\bigg(\int_0^{M}\varphi(t)dt\bigg).\end{aligned}$$

That is,

$$\int_0^{\delta_n}\varphi(t)dt \le \phi^{a_n}\bigg(\int_0^{M}\varphi(t)dt\bigg),$$

where $\{a_n\}_{n\in\mathbb{N}}$ is defined by (2.5). That is, (2.6) holds. The remaining portion of the proof can be derived as in that of Theorem 2.1. This completes the proof. □

As in the arguments of Theorems 2.1, 2.2 and 2.3, we conclude similarly the following result and omit its proof.

Theorem 2.4. *Let (X, d) be a bounded complete metric space, $F : X \to B(X)$ be a continuous mapping satisfying*

$$\int_0^{\delta(F^px,F^iy)}\varphi(t)dt \le \phi\bigg(\int_0^{\delta(\cup_{D\in C_F}D\{x,y\})}\varphi(t)dt\bigg), \quad \forall x, y \in X, \tag{2.13}$$

where $p, i \in \mathbb{N}$ and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$. Then

(c) *F has a unique stationary point $z \in X$;*
(d) *The sequence $\{F^n x\}_{n\in\mathbb{N}}$ converges to $\{z\}$ for all $x \in X$.*

Now we give a common fixed point theorem for two pairs of single and multi-valued contractive mappings of integral type in metric spaces.

Theorem 2.5. *Let (X, d) be a bounded complete metric space, $F, G : X \to B(X)$ be commuting, $f, g : X \to X$ satisfy that $f, g \in CC_F \cap CC_G$ and*

$$\begin{aligned}&\int_0^{\delta(F^px,G^qy)}\varphi(t)dt\\ &\le \phi\bigg(\int_0^{\delta(fx,F^px)}\varphi(t)dt, \int_0^{\delta(gy,G^qy)}\varphi(t)dt, \int_0^{\delta(fx,G^qy)}\varphi(t)dt,\\ &\qquad \int_0^{\delta(gy,F^px)}\varphi(t)dt, \int_0^{d(fx,gy)}\varphi(t)dt\bigg), \quad \forall x, y \in X,\end{aligned} \tag{2.14}$$

where $p, q \in \mathbb{N}$ and $(\varphi, \phi) \in \Phi_1\times\Phi_3$. Then (b) of Theorem 2.1 and the following (e) and (f) hold:

(e) *f, g, F^p and G^q have a unique common fixed point $z \in X$;*

(f) $F^p z = G^q z = \{z\}$.

Proof. Let $r(t) = \phi(t,t,t,t,t)$ for each $t \in \mathbb{R}^+$, $M = \delta(X)$, $k = p+q$, $X_n = F^n G^n X$ and $\delta_n = \delta(X_n)$ for each $n \in \mathbb{N}$. Clearly, (2.2) and (2.5) hold. As in the proof of Theorem 2.1, we infer that by (2.14) and $(\varphi, \phi) \in \Phi_1 \times \Phi_3$

$$\begin{aligned}
&\int_0^{\delta(F^pA,G^qB)} \varphi(t)dt \\
&\le \phi\bigg(\int_0^{\delta(fA,F^pA)} \varphi(t)dt, \int_0^{\delta(gB,G^qB)} \varphi(t)dt, \int_0^{\delta(fA,G^qB)} \varphi(t)dt, \\
&\qquad \int_0^{\delta(gB,F^pA)} \varphi(t)dt, \int_0^{\delta(fA,gB)} \varphi(t)dt \bigg), \quad \forall A, B \in B(X).
\end{aligned} \tag{2.15}$$

Note that $f, g \in CC_F \cap CC_G$ and F and G are commuting. It follows from (2.2), (2.5), (2.14), (2.15) and $\varphi \in \Phi_3$ that

$$\begin{aligned}
&\int_0^{\delta_n} \varphi(t)dt \\
&= \int_0^{\delta(F^p(F^{q+b_n}G^{k+b_n}X_{k(a_n-1)}),G^q(F^{k+b_n}G^{p+b_n}X_{k(a_n-1)}))} \varphi(t)dt \\
&\le \phi\bigg(\int_0^{\delta(fF^{q+b_n}G^{k+b_n}X_{k(a_n-1)},F^{k+b_n}G^{k+b_n}X_{k(a_n-1)})} \varphi(t)dt, \\
&\qquad \int_0^{\delta(gF^{k+b_n}G^{p+b_n}X_{k(a_n-1)},F^{k+b_n}G^{k+b_n}X_{k(a_n-1)})} \varphi(t)dt, \\
&\qquad \int_0^{\delta(fF^{q+b_n}G^{k+b_n}X_{k(a_n-1)},F^{k+b_n}G^{k+b_n}X_{k(a_n-1)})} \varphi(t)dt, \\
&\qquad \int_0^{\delta(gF^{k+b_n}G^{p+b_n}X_{k(a_n-1)},F^{k+b_n}G^{k+b_n}X_{k(a_n-1)})} \varphi(t)dt, \\
&\qquad \int_0^{\delta(fF^{q+b_n}G^{k+b_n}X_{k(a_n-1)},gF^{k+b_n}G^{p+b_n}X_{k(a_n-1)})} \varphi(t)dt \bigg) \\
&\le \phi\bigg(\int_0^{\delta(X_{k(a_n-1)},X_n)} \varphi(t)dt, \int_0^{\delta(X_{k(a_n-1)},X_n)} \varphi(t)dt, \int_0^{\delta(X_{k(a_n-1)},X_n)} \varphi(t)dt, \\
&\qquad \int_0^{\delta(X_{k(a_n-1)},X_n)} \varphi(t)dt, \int_0^{\delta(X_{k(a_n-1)},X_{k(a_n-1)})} \varphi(t)dt \bigg) \\
&\le \phi\bigg(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt, \int_0^{\delta_{k(a_n-1)}} \varphi(t)dt, \int_0^{\delta_{k(a_n-1)}} \varphi(t)dt, \\
&\qquad \int_0^{\delta_{k(a_n-1)}} \varphi(t)dt, \int_0^{\delta_{k(a_n-1)}} \varphi(t)dt \bigg)
\end{aligned}$$

$$= r\Big(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt\Big),$$

which implies that

$$\begin{aligned}\int_0^{\delta_n} \varphi(t)dt &\le r\Big(\int_0^{\delta_{k(a_n-1)}} \varphi(t)dt\Big) \le r^2\Big(\int_0^{\delta_{k(a_n-2)}} \varphi(t)dt\Big)\\ &\le \cdots \le r^{a_n-1}\Big(\int_0^{\delta_k} \varphi(t)dt\Big) \le r^{a_n}\Big(\int_0^{M} \varphi(t)dt\Big).\end{aligned}$$

That is,

$$0 \le \int_0^{\delta_n} \varphi(t)dt \le r^{a_n}\Big(\int_0^{M} \varphi(t)dt\Big), \tag{2.16}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is defined by (2.5). Obviously, (2.5), (2.16) and Lemmas 1.7 and 1.8 ensure that

$$\lim_{n\to\infty} \delta_n = 0. \tag{2.17}$$

For each $n \in \mathbb{N}$, choose a point $x_n \in X_n$. It follows that

$$fx_n \in fF^nG^nX = F^nG^nfX \subseteq F^nG^nX, \quad \forall n \in \mathbb{N}. \tag{2.18}$$

Similarly, $gx_n \in F^nG^nX$ for each $n \in \mathbb{N}$. From (2.2), (2.5) and (2.16), we get that

$$\begin{aligned}0 &\le \int_0^{d(x_n,x_m)} \varphi(t)dt \le \int_0^{\delta(F^nG^nX,G^nF^nG^{m-n}F^{m-n}X)} \varphi(t)dt\\ &\le \int_0^{\delta_n} \varphi(t)dt \le r^{a_n}\Big(\int_0^{M} \varphi(t)dt\Big), \quad \forall m,n \in \mathbb{N} \text{ with } m > n.\end{aligned} \tag{2.19}$$

In terms of (2.19), Lemmas 1.7 and 1.8, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, it follows that there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. The continuity of f and g ensures that $fx_n \to fz$ and $gx_n \to gz$ as $n \to \infty$. Consequently, it follows from (2.18) that

$$\begin{aligned}0 \le d(fz, gz) &\le d(fz, fx_n) + d(fx_n, gx_n) + d(gx_n, gz)\\ &\le d(fz, fx_n) + \delta_n + d(gx_n, gz), \quad \forall n \in \mathbb{N}.\end{aligned}$$

Letting n tend to infinity and using (2.17), we obtain that $d(fz, gz) = 0$, that is, $fz = gz$.

We next show that z is a common fixed point of f, g, F^p and G^q. Clearly, (2.18) yields that

$$\begin{aligned}0 \le d(z, gz) &\le d(z, x_n) + d(x_n, gx_n) + d(gx_n, gz)\\ &\le d(z, x_n) + \delta_n + d(gx_n, gz), \quad \forall n \in \mathbb{N}.\end{aligned}$$

As $n \to \infty$ we conclude that $d(z, gz) = 0$, that is, $z = gz$. Similarly, $z = fz$.

We now assert that $\delta(z, G^q z) = 0$. Otherwise $\delta(z, G^q z) > 0$. From (2.2) and (2.18), we have

$$\begin{aligned}\delta(z, G^q z) &\le d(z, gx_m) + \delta(gx_m, G^q z)\\ &\le d(z, gx_m) + \delta(F^m G^m X, G^q z)\\ &\le d(z, gx_m) + \delta(F^n G^n X, G^q z), \quad \forall m, n \in \mathbb{N} \text{ with } m > n.\end{aligned}$$

Letting m tend to infinity, we obtain that

$$\delta(z, G^q z) \le \delta(F^n G^n X, G^q z), \quad \forall n \in \mathbb{N}. \tag{2.20}$$

It follows from (2.15), (2.18), (2.20), $f, g \in CC_F \cap CC_G$, $(\varphi, \phi) \in \Phi_1 \times \Phi_3$ and $z = gz$ that

$$\begin{aligned}\delta(fF^{n-p}G^n X, G^q z) &\le \delta(F^{n-p}G^{n-p}fG^p X, gx_{n-p}) + d(gx_{n-p}, z) + \delta(z, G^q z)\\ &\le \delta_{n-p} + d(gx_{n-p}, z) + \delta(z, G^q z),\end{aligned}$$

$$\delta(gz, F^p F^{n-p} G^n X) \le d(gz, gx_n) + \delta(gx_n, F^n G^n X) \le d(z, gx_n) + \delta_n,$$

$$\begin{aligned}\delta(fF^{n-p}G^n X, gz) &\le \delta(F^{n-p}G^{n-p}fG^p X, gx_{n-p}) + d(gx_{n-p}, gz)\\ &\le \delta_{n-p} + d(gx_{n-p}, z)\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\delta(z,G^q z)} \varphi(t)dt \le \int_0^{\delta(F^n G^n X, G^q z)} \varphi(t)dt\\ &\le \phi\Bigg(\int_0^{\delta(fF^{n-p}G^n X, F^p F^{n-p} G^n X)} \varphi(t)dt,\\ &\qquad \int_0^{\delta(gz, G^q z)} \varphi(t)dt, \int_0^{\delta(fF^{n-p}G^n X, G^q z)} \varphi(t)dt,\\ &\qquad \int_0^{\delta(gz, F^p F^{n-p} G^n X)} \varphi(t)dt, \int_0^{\delta(fF^{n-p}G^n X, gz)} \varphi(t)dt\Bigg)\\ &\le \phi\Bigg(\int_0^{\delta_{n-p}} \varphi(t)dt, \int_0^{\delta(z,G^q z)} \varphi(t)dt, \int_0^{\delta_{n-p}+d(gx_{n-p},z)+\delta(z,G^q z)} \varphi(t)dt,\\ &\qquad \int_0^{d(z,gx_n)+\delta_n} \varphi(t)dt, \int_0^{\delta_{n-p}+d(gx_{n-p},z)} \varphi(t)dt\Bigg), \quad \forall n > p.\end{aligned}$$

Letting n tend to infinity and using (2.17) and $\phi \in \Phi_3$, we get that

$$\begin{aligned}\int_0^{\delta(z,G^q z)} \varphi(t)dt &\le \phi\Bigg(0, \int_0^{\delta(z,G^q z)} \varphi(t)dt, \int_0^{\delta(z,G^q z)} \varphi(t)dt, 0, 0\Bigg)\\ &\le r\Bigg(\int_0^{\delta(z,G^q z)} \varphi(t)dt\Bigg) < \int_0^{\delta(z,G^q z)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence $\delta(z, G^q z) = 0$. Consequently, $G^q z = \{z\}$. Similarly, $F^p z = \{z\}$. That is, $F^p z = G^q z = \{z\}$. For each $(x, n) \in X \times \mathbb{N}$, choose $y_n \in F^n G^n x$. It follows from (2.16) that

$$\begin{aligned}\int_0^{d(y_n,z)} \varphi(t)dt \le \int_0^{\delta(F^n G^n x,z)} \varphi(t)dt \le \int_0^{\delta(X_n,z)} \varphi(t)dt \\ \le \int_0^{\delta_n} \varphi(t)dt \le r^{a_n}\Big(\int_0^M \varphi(t)dt\Big),\end{aligned}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is defined by (2.5). Letting n tend to infinity and using Definition 1.6, Lemmas 1.7 and 1.8, we conclude that $\{F^n G^n x\}_{n\in\mathbb{N}}$ converges to $\{z\}$.

We finally show that z is the unique common fixed point of f, g, F^p and G^q. Suppose that f, g, F^p and G^q have a second common fixed point $\omega \in X - \{z\}$. If $\delta(F^p\omega, G^q\omega) > 0$, from (2.14) and $\phi \in \Phi_3$, we have

$$\begin{aligned}&\int_0^{\delta(F^p\omega,G^q\omega)} \varphi(t)dt \\ &\le \phi\Big(\int_0^{\delta(f\omega,F^p\omega)} \varphi(t)dt, \int_0^{\delta(g\omega,G^q\omega)} \varphi(t)dt, \int_0^{\delta(f\omega,G^q\omega)} \varphi(t)dt, \\ &\qquad \int_0^{\delta(g\omega,F^p\omega)} \varphi(t)dt, \int_0^{d(f\omega,g\omega)} \varphi(t)dt\Big) \\ &\le \phi\Big(\int_0^{\delta(G^q\omega,F^p\omega)} \varphi(t)dt, \int_0^{\delta(F^p\omega,G^q\omega)} \varphi(t)dt, \\ &\qquad \int_0^{\delta(F^p\omega,G^q\omega)} \varphi(t)dt, \int_0^{\delta(G^q\omega,F^p\omega)} \varphi(t)dt, 0\Big) \\ &\le r\Big(\int_0^{\delta(F^p\omega,G^q\omega)} \varphi(t)dt\Big) < \int_0^{\delta(F^p\omega,G^q\omega)} \varphi(t)dt,\end{aligned}$$

which is impossible. Therefore $\delta(F^p\omega, G^q\omega) = 0$. Note that $\omega \in F^p\omega \cap G^q\omega$. Consequently, $F^p\omega = G^q\omega = \{\omega\}$. Using (2.14) and $\phi \in \Phi_3$, we get that

$$\begin{aligned}\int_0^{\delta(z,\omega)} \varphi(t)dt &= \int_0^{\delta(F^p z,G^q\omega)} \varphi(t)dt \\ &\le \phi\Big(\int_0^{\delta(fz,F^p z)} \varphi(t)dt, \int_0^{\delta(g\omega,G^q\omega)} \varphi(t)dt, \int_0^{\delta(fz,G^q\omega)} \varphi(t)dt, \\ &\qquad \int_0^{\delta(g\omega,F^p z)} \varphi(t)dt, \int_0^{d(fz,g\omega)} \varphi(t)dt\Big)\end{aligned}$$

$$\leq \phi\Big(0, 0, \int_0^{\delta(z,\omega)} \varphi(t)dt, \int_0^{\delta(z,\omega)} \varphi(t)dt, \int_0^{\delta(z,\omega)} \varphi(t)dt\Big)$$
$$\leq r\Big(\int_0^{\delta(z,\omega)} \varphi(t)dt\Big) < \int_0^{\delta(z,\omega)} \varphi(t)dt,$$

which is a contradiction. Therefore z is the unique common fixed point of f, g, F^p and G^q. This completes the proof. □

3. Remarks and illustrative examples

In this section, we construct three examples to show that Theorems 2.1-2.5 generalize or are different from some results in [2, 3, 7, 10–14, 17, 20].

Remark 3.1. Theorem 2.1 extends Theorem 2.2 in [12]. The following example manifests that Theorem 2.1 differs from Theorems 1.4 and 1.5 in the first section.

Example 3.2. Let $X = [0,1] \cup \{2\}$ be endowed with the Euclidean metric $d = |\cdot|$. Define $F, G : X \to CB(X)$ and $\varphi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Fx = \begin{cases} [0, \frac{x}{2}], & \forall x \in [0,1], \\ \{1\}, & x = 2, \end{cases} \qquad Gx = \{x\}, \quad \forall x \in X,$$

$$\varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+ \quad \text{and} \quad \phi(t) = \frac{1}{2}t, \quad \forall t \in \mathbb{R}^+.$$

Take $p = j = 2$ and $q = i = 3$. Obviously, (X, d) is a bounded complete metric space, F and G are continuous and commuting, $(\varphi, \phi) \in \Phi_1 \times \Phi_2$ and

$$F^2x = \cup_{y\in Fx} Fy = \cup_{y\in[0,\frac{x}{2}]} \Big[0, \frac{y}{2}\Big] = \Big[0, \frac{x}{4}\Big], \quad F^3x = \Big[0, \frac{x}{8}\Big], \quad \forall x \in [0,1],$$
$$F^2 2 = F1 = \Big[0, \frac{1}{2}\Big], \quad F^3 2 = \Big[0, \frac{1}{4}\Big].$$

Put $x, y \in X$. In order to verify (2.1), we need to consider four possible cases as follows:

Case 1. $x, y \in [0, 1]$. It follows that

$$\begin{aligned}
\int_0^{\delta(F^2G^3x, F^3G^2y)} \varphi(t)dt &= \int_0^{\delta([0,\frac{x}{4}],[0,\frac{y}{8}])} \varphi(t)dt \\
&= \int_0^{\max\{\frac{x}{4},\frac{y}{8}\}} 2tdt = \max\Big\{\frac{x^2}{16}, \frac{y^2}{64}\Big\} \\
&\leq \frac{1}{2}\max\{x^2, y^2\} = \phi\Big(\int_0^{\delta([0,\frac{x}{4}]\cup[0,\frac{y}{4}]\cup\{x,y\})} \varphi(t)dt\Big) \\
&\leq \phi\Big(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\Big);
\end{aligned}$$

Case 2. $x = y = 2$. It is clear that

$$\begin{aligned}\int_0^{\delta(F^2G^3x,F^3G^2y)}\varphi(t)dt &= \int_0^{\delta([0,\frac{1}{2}],[0,\frac{1}{4}])}\varphi(t)dt = \int_0^{\frac{1}{2}}2tdt = \frac{1}{4}\\ &< 2 = \phi\bigg(\int_0^{\delta([0,\frac{1}{2}]\cup\{2\})}\varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}}D\{x,y\})}\varphi(t)dt\bigg);\end{aligned}$$

Case 3. $x \in [0,1], y = 2$. It is easy to see that

$$\begin{aligned}\int_0^{\delta(F^2G^3x,F^3G^2y)}\varphi(t)dt &= \int_0^{\delta([0,\frac{x}{4}],[0,\frac{1}{4}])}\varphi(t)dt = \int_0^{\frac{1}{4}}2tdt = \frac{1}{16}\\ &< 2 = \phi\bigg(\int_0^{\delta([0,\frac{x}{4}]\cup[0,\frac{1}{2}]\cup\{x,2\})}\varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}}D\{x,y\})}\varphi(t)dt\bigg);\end{aligned}$$

Case 4. $x = 2, y \in [0,1]$. It is easy to verify that

$$\begin{aligned}\int_0^{\delta(F^2G^3x,F^3G^2y)}\varphi(t)dt &= \int_0^{\delta([0,\frac{1}{2}],[0,\frac{y}{8}])}\varphi(t)dt = \int_0^{\frac{1}{2}}2tdt = \frac{1}{4}\\ &< 2 = \phi\bigg(\int_0^{\delta([0,\frac{1}{2}]\cup[0,\frac{y}{4}]\cup\{2,y\})}\varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\delta(\cup_{D\in C_{FG}}D\{x,y\})}\varphi(t)dt\bigg).\end{aligned}$$

Hence, (2.1) holds. That is, the conditions of Theorem 2.1 are satisfied. It follows from Theorem 2.1 that F and G have a unique common stationary point $0 \in X$ and the sequence $\{F^nG^nx\}_{n\in\mathbb{N}}$ converges to $\{0\}$ for all $x \in X$.

However, we don't invoke Theorems 1.4 and 1.5 in the first section to show the existence of fixed points of F in X. Suppose that F satisfies the conditions of Theorem 1.4. That is, there exist $\varphi \in \Phi_1$ and a constant $c \in [0,1)$ satisfying (1.5). It follows from (1.5) that

$$\begin{aligned}\int_0^1\varphi(t)dt &= \int_0^{H([0,\frac{1}{2}],1)}\varphi(t)dt = \int_0^{H(F1,F2)}\varphi(t)dt\\ &\le c\int_0^{d(1,2)}\varphi(t)dt = c\int_0^1\varphi(t)dt < \int_0^1\varphi(t)dt,\end{aligned}$$

which is a contradiction. Suppose that F satisfies the conditions of Theorem 1.5. That is, there exist $(\varphi,\phi) \in \Phi_1 \times \Phi_2$ and a function $\alpha \in \Theta$ satisfying

(1.6). By virtue of (1.6) and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$, we deduce that

$$\begin{aligned}\phi\Big(\int_0^1 \varphi(t)dt\Big) &= \phi\Big(\int_0^{\delta([0,\frac{1}{2}],1)} \varphi(t)dt\Big) = \phi\Big(\int_0^{\delta(F1,F2)} \varphi(t)dt\Big)\\ &\le \alpha(d(1,2))\phi\Big(\int_0^{d(1,2)} \varphi(t)dt\Big)\\ &= \alpha(1)\phi\Big(\int_0^1 \varphi(t)dt\Big) < \phi\Big(\int_0^1 \varphi(t)dt\Big),\end{aligned}$$

which is impossible.

Remark 3.3. The following example reveals that Theorem 2.3 differs from Theorems 1.1, 1.2 and 1.3 in the first section, Theorem 3.1 in [14] and Theorem 2 in [17].

Example 3.4. Let $X = \big[0, \frac{3}{2}\big]$ be endowed with the Euclidean metric $d = |\cdot|$. Define $F, G : X \to CB(X)$ and $\varphi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Fx = \begin{cases}\{\frac{x}{2}\}, & \forall x \in [0,1],\\ \{x - \frac{1}{2}\}, & \forall x \in (1, \frac{3}{2}],\end{cases} \qquad Gx = \begin{cases}\{\frac{x}{3}\}, & \forall x \in [0,1],\\ \{\frac{2}{3}(x - \frac{1}{2})\}, & \forall x \in (1, \frac{3}{2}],\end{cases}$$

$$\varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+ \quad \text{and} \quad \phi(t) = \frac{1}{2}t, \quad \forall t \in \mathbb{R}^+.$$

Take $p = j = 1$. Obviously, (X, d) is a bounded complete metric space, F and G are continuous and commuting, $(\varphi, \phi) \in \Phi_1 \times \Phi_2$ and

$$FGx = \begin{cases}\frac{x}{6}, & \forall x \in [0,1],\\ \frac{1}{3}(x - \frac{1}{2}), & \forall x \in (1, \frac{3}{2}].\end{cases}$$

Put $x, y \in X$. In order to verify (2.11), we need to consider four possible cases as follows:
Case 1. $x, y \in [0, 1]$. It follows that

$$\begin{aligned}\int_0^{d(Fx,Gy)} \varphi(t)dt &= \int_0^{|\frac{x}{2}-\frac{y}{3}|} 2tdt = \Big(\frac{x}{2} - \frac{y}{3}\Big)^2\\ &\le \frac{1}{2}\max\Big\{\Big(x - \frac{y}{6}\Big)^2, \Big(y - \frac{x}{6}\Big)^2\Big\}\\ &= \frac{1}{2}\int_0^{\max\{x-\frac{y}{6}, y-\frac{x}{6}\}} 2tdt \le \phi\Big(\int_0^{\delta(\{x,y,\frac{x}{6},\frac{y}{6}\})} \varphi(t)\Big)dt\\ &\le \phi\Big(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\Big);\end{aligned}$$

Case 2. $x, y \in (1, \frac{3}{2}]$. It is clear that

$$\begin{aligned}
&\int_0^{d(Fx,Gy)} \varphi(t)dt = \int_0^{|(x-\frac{1}{2})-\frac{2}{3}(y-\frac{1}{2})|} 2tdt = \Big(\Big(x-\frac{1}{2}\Big)-\frac{2}{3}\Big(y-\frac{1}{2}\Big)\Big)^2\\
&\le \frac{1}{2}\max\Big\{\Big(x-\frac{1}{3}\Big(y-\frac{1}{2}\Big)\Big)^2, \Big(y-\frac{1}{3}\Big(x-\frac{1}{2}\Big)\Big)^2\Big\}\\
&= \frac{1}{2}\int_0^{\max\{x-\frac{1}{3}(y-\frac{1}{2}), y-\frac{1}{3}(x-\frac{1}{2})\}} 2tdt \le \phi\Big(\int_0^{\delta(\{x,y,\frac{1}{3}(x-\frac{1}{2}),\frac{1}{3}(y-\frac{1}{2})\})} \varphi(t)dt\Big)\\
&\le \phi\Big(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\Big);
\end{aligned}$$

Case 3. $x \in [0, 1], y \in (1, \frac{3}{2}]$. It is easy to see that

$$\begin{aligned}
&\int_0^{d(Fx,Gy)} \varphi(t)dt = \int_0^{|\frac{x}{2}-\frac{2}{3}(y-\frac{1}{2})|} 2tdt = \Big(\frac{1}{2}x-\frac{2}{3}\Big(y-\frac{1}{2}\Big)\Big)^2\\
&\le \frac{1}{2}\max\Big\{\Big(x-\frac{1}{3}\Big(y-\frac{1}{2}\Big)\Big)^2, \Big(y-\frac{1}{6}x\Big)^2\Big\}\\
&= \frac{1}{2}\int_0^{\max\{x-\frac{1}{3}(y-\frac{1}{2}), y-\frac{1}{6}x\}} 2tdt \le \phi\Big(\int_0^{\delta(\{x,y,\frac{1}{6}x,\frac{1}{3}(y-\frac{1}{2})\})} \varphi(t)dt\Big)\\
&\le \phi\Big(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\Big);
\end{aligned}$$

Case 4. $x \in (1, \frac{3}{2}], y \in [0, 1]$. It is easy to verify that

$$\begin{aligned}
&\int_0^{d(Fx,Gy)} \varphi(t)dt = \int_0^{(x-\frac{1}{2})-\frac{y}{3}} 2tdt = \Big(\Big(x-\frac{1}{2}\Big)-\frac{y}{3}\Big)^2\\
&\le \frac{1}{2}\Big(x-\frac{y}{6}\Big)^2 = \frac{1}{2}\int_0^{x-\frac{1}{6}y} 2tdt \le \phi\Big(\int_0^{\delta(\{x,y,\frac{1}{3}(x-\frac{1}{2}),\frac{1}{6}y\})} \varphi(t)dt\Big)\\
&\le \phi\Big(\int_0^{\delta(\cup_{D\in C_{FG}} D\{x,y\})} \varphi(t)dt\Big).
\end{aligned}$$

Hence, (2.11) holds. That is, the conditions of Theorem 2.3 are satisfied. It follows from Theorem 2.3 that F and G have a unique common stationary point $0 \in X$ and the sequence $\{F^nG^nx\}_{n\in\mathbb{N}}$ converges to $\{0\}$ for all $x \in X$.

Now we claim that Theorems 1.1, 1.2 and 1.3 in the first section, Theorem 3.1 in [14] and Theorem 2 in [17] are useless in proving the existence of fixed points of F in X. Suppose that F satisfy the conditions of Theorem 1.1. That is, there exist $\varphi \in \Phi_1$ and a constant $c \in [0, 1)$ satisfying (1.1). It follows from

(1.1) and $\varphi \in \Phi_1$ that

$$\int_0^{\frac{1}{2}} \varphi(t)dt = \int_0^{d(F1,F\frac{3}{2})} \varphi(t)dt \le c \int_0^{d(1,\frac{3}{2})} \varphi(t)dt$$
$$= c \int_0^{\frac{1}{2}} \varphi(t)dt < \int_0^{\frac{1}{2}} \varphi(t)dt,$$

which is impossible. Suppose that F satisfy the conditions of Theorem 1.2. That is, there exist $\varphi \in \Phi_1$ and a function $\alpha \in \Theta$ satisfying (1.2) and (1.3). By virtue of (1.2), (1.3) and $(\varphi, \alpha) \in \Phi_1 \times \Theta$, we conclude that

$$m\Big(1, \frac{3}{2}\Big) = \max\Big\{d\Big(1, \frac{3}{2}\Big), d(1, F1), d\Big(\frac{3}{2}, F\frac{3}{2}\Big), \frac{d(1, F\frac{3}{2}) + d(\frac{3}{2}, F1)}{2}\Big\}$$
$$= \max\Big\{\frac{1}{2}, d\Big(1, \frac{1}{2}\Big), d\Big(\frac{3}{2}, 1\Big), \frac{d(1,1) + d(\frac{3}{2}, \frac{1}{2})}{2}\Big\} = \frac{1}{2}$$

and

$$\int_0^{\frac{1}{2}} \varphi(t)dt = \int_0^{d(F1,F\frac{3}{2})} \varphi(t)dt \le \alpha\Big(d\Big(1, \frac{3}{2}\Big)\Big) \int_0^{m(1,\frac{3}{2})} \varphi(t)dt$$
$$= \alpha\Big(\frac{1}{2}\Big) \int_0^{\frac{1}{2}} \varphi(t)dt < \int_0^{\frac{1}{2}} \varphi(t)dt,$$

which is a contradiction.

Suppose that F satisfy the conditions of Theorem 1.3. That is, there exist $\varphi \in \Phi_1$ and a function $\alpha \in \Theta$ satisfying (1.4). It follows from (1.4), $(\varphi, \alpha) \in \Phi_1 \times \Theta$ and $m(1, \frac{3}{2}) = \frac{1}{2}$ that

$$\int_0^{\frac{1}{2}} \varphi(t)dt = \int_0^{d(F1,F\frac{3}{2})} \varphi(t)dt \le \alpha\Big(m\Big(1, \frac{3}{2}\Big)\Big) \int_0^{m(1,\frac{3}{2})} \varphi(t)dt$$
$$= \alpha\Big(\frac{1}{2}\Big) \int_0^{\frac{1}{2}} \varphi(t)dt < \int_0^{\frac{1}{2}} \varphi(t)dt,$$

which is impossible.

Since Theorems 1.2 and 1.3 are generalizations of Theorem 3.1 in [14] and Theorem 2 in [17], respectively, it follows that Theorem 3.1 in [14] and Theorem 2 in [17] are unapplicable.

Remark 3.5. Theorem 2.5 extends Theorem 2 in [7] and Theorem 1 in [11]. The following example reveals that Theorem 2.5 is different from Theorems 1 and 2 in [3].

Example 3.6. Let $X = \{1, 2, 5, 7, 9\}$ be endowed with the Euclidean metric $d = |\cdot|$. Define $f, g : X \to X$, $F, G : X \to B(X)$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ by

$$f1 = f2 = f5 = f7 = 2, \quad f9 = 1, \quad gx = x, \quad \forall x \in X,$$

$$F1 = F2 = F7 = \{2\}, \quad F5 = \{2, 7\}, \quad F9 = \{5\}, \quad G = F,$$

$$\varphi(t) = \begin{cases} 1, & \forall t \in [0, 5], \\ e^t, & \forall t \in (5, +\infty) \end{cases}$$

and

$$\phi(t_1, t_2, t_3, t_4, t_5) = \frac{6}{7} \max\{t_1, t_2, t_3, t_4, t_5\}, \quad \forall t_i \in \mathbb{R}^+, i \in \{1, 2, 3, 4, 5\}.$$

Take $p = 2$ and $q = 3$. Obviously, (X, d) is a bounded complete metric space, F and G are commuting, $f, g \in CC_F \cap CC_G$ and $(\varphi, \phi) \in \Phi_1 \times \Phi_3$. Put $x, y \in X$. In order to verify (2.14), we need to consider three possible cases as follows:

Case 1. $x \in \{1, 2, 5, 7\}$, $y \in X$. It is clear that

$$\begin{aligned} &\int_0^{\delta(F^2x, G^3y)} \varphi(t)dt = \int_0^{|2-2|} \varphi(t)dt = 0 \\ &\le \phi\bigg(\int_0^{\delta(fx, F^2x)} \varphi(t)dt, \int_0^{\delta(gy, G^3y)} \varphi(t)dt, \int_0^{\delta(fy, G^3y)} \varphi(t)dt, \\ &\quad \int_0^{\delta(gy, F^2x)} \varphi(t)dt, \int_0^{d(fx, gy)} \varphi(t)dt \bigg); \end{aligned}$$

Case 2. $x = 9$, $y \in \{1, 2, 5, 7\}$. It follows that

$$\begin{aligned} &\int_0^{\delta(F^2 9, G^3y)} \varphi(t)dt = \int_0^{\delta(\{2,7\}, 2)} \varphi(t)dt = 5 \le \frac{6}{7}(5 + e^6 - e^5) \\ &= \frac{6}{7}\bigg(\int_0^5 dt + \int_5^6 e^t dt \bigg) = \frac{6}{7} \int_0^{\delta(1, \{2,7\})} \varphi(t)dt = \frac{6}{7} \int_0^{\delta(f9, F^2 9)} \varphi(t)dt \\ &\le \phi\bigg(\int_0^{\delta(f9, F^2 9)} \varphi(t)dt, \int_0^{\delta(gy, G^3y)} \varphi(t)dt, \int_0^{\delta(f9, G^3y)} \varphi(t)dt, \\ &\quad \int_0^{\delta(gy, F^2 9)} \varphi(t)dt, \int_0^{d(f9, gy)} \varphi(t)dt \bigg); \end{aligned}$$

Case 3. $x = y = 9$. Notice that

$$\begin{aligned}\int_0^{\delta(F^29,G^39)} \varphi(t)dt &= \int^{\delta(\{2,7\},2)} = 5 \le \frac{6}{7}(5+e^8-e^5)\\ &= \frac{6}{7}\bigg(\int_0^5 1dt + \int_5^8 e^t dt\bigg) = \frac{6}{7}\int_0^{d(f9,g9)} \varphi(t)dt\\ &\le \phi\bigg(\int_0^{\delta(f9,F^29)} \varphi(t)dt, \int_0^{\delta(g9,G^39)} \varphi(t)dt, \int_0^{\delta(f9,G^39)} \varphi(t)dt,\\ &\qquad \int_0^{\delta(g9,F^29)} \varphi(t)dt, \int_0^{d(f9,g9)} \varphi(t)dt\bigg).\end{aligned}$$

Hence, (2.14) holds. That is, the conditions of Theorem 2.5 are fulfilled. It follows from Theorem 2.5 that f, g, F^2 and G^3 have a unique common fixed point $2 \in X$.

However, Theorems 1 and 2 in [3] cannot be used to prove the existence of common fixed points of f, g, F and G in X. Suppose that f, g, F and G satisfy the conditions of Theorem 1 in [3]. That is, there exist $\varphi \in \Phi_1$, $\alpha \in [0,1)$ and $a \ge 0, b \ge 0$ with $a+b<1$ satisfying

$$\begin{aligned}&\int_0^{H(Fx,Gy)} \varphi(t)dt\\ &\le \alpha \int_0^{\max\{d(fx,gy),d(fx,Fx),d(gy,Gy)\}} \varphi(t)dt \qquad (3.1)\\ &\quad + (1-\alpha)\bigg(a\int_0^{\frac{d(fx,Gy)}{2}} \varphi(t)dt + b\int_0^{\frac{d(gy,Fx)}{2}} \varphi(t)dt\bigg).\end{aligned}$$

It follows from (3.1) and $\varphi \in \Phi_1$ that

$$\begin{aligned}\int_0^5 \varphi(t)dtdt &= \int_0^{H(F5,G2)} \varphi(t)dt\\ &\le \alpha \int_0^{\max\{d(f5,g2),d(f5,F5),d(g2,G2)\}} \varphi(t)dt\\ &\quad + (1-\alpha)\bigg(a\int_0^{\frac{d(f5,G2)}{2}} \varphi(t)dt + b\int_0^{\frac{d(g2,F5)}{2}} \varphi(t)dt\bigg)\\ &= \alpha\int_0^0 \varphi(t)dt + (1-\alpha)\bigg(a\int_0^0 \varphi(t)dt + b\int_0^0 \varphi(t)dt\bigg)\\ &= 0,\end{aligned}$$

which is a contradiction.

Suppose that f, g, F and G satisfy the conditions of Theorem 2 in [3]. That is, there exists $(\varphi, \phi) \in \Phi_1 \times \Phi_2$ satisfying

$$\begin{aligned}&\int_0^{H(Fx,Gy)} \varphi(t)dt \\ &\leq \phi\Big(\int_0^{\max\{d(fx,gy),d(fx,Fx),d(gy,Gy),\frac{d(fx,Gy)}{2},\frac{d(gy,Fx)}{2}\}} \varphi(t)dt\Big).\end{aligned} \tag{3.2}$$

By virtue of (3.2) and $(\varphi, \phi) \in \Phi_1 \times \Phi_2$, we conclude that

$$\begin{aligned}\int_0^5 \varphi(t)dt &= \int_0^{H(F5,G2)} \varphi(t)dt \\ &\leq \phi\Big(\int_0^{\max\{d(f5,g2),d(f5,F5),d(g2,G2),\frac{d(f5,G2)}{2},\frac{d(g2,F5)}{2}\}} \varphi(t)dt\Big) \\ &= \phi\Big(\int_0^0 \varphi(t)dt\Big) = \phi(0) = 0,\end{aligned}$$

which is impossible.

References

[1] I. Altun and D. Türkoğlu, *Some fixed point theorems for mappings satisfying contractive condition of integral type on d-complete topological spaces*, Fasc. Math., **42** (2009), 5–15.

[2] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **9** (2002), 531–536.

[3] S. Chauhan, M. Imdad, E. Karapina and B. Fisher, *An integral type fixed point theorem for multi-valued mappings employing strongly tangential property*, J. Egyptian Math. Soc., **22** (2014), 258–264.

[4] D. Dey, A. Ganguly and M. Saha, *Fixed point theorems for mappings under general contractive condition of integral type*, Bull. Math. Anal. Appl., **3** (2011), 27–34.

[5] B. Fisher, *Some theorems on fixed points*, Studia Sci. Math. Hungar., **12** (1977), 159–160.

[6] B. Fisher, *Quasi-contraction metric spaces*, Proc. Amer. Math. Soc., **75** (1979), 321–325.

[7] B. Fisher, *Set-valued mappings on bounded metric space*, Indian J. Pure Appl. Math., **11** (1980), 8–12.

[8] B. Fisher, *Common fixed points of mappings and set-valued mappings*, Rostock. Math. Kolloq., **12** (1981), 305–311.

[9] U.C. Gairola and A.S. Rawat, *A fixed point theorem for integral type inequality*, Int. J. Math. Anal., **2** (2008), 709–712.

[10] J. Jachymski, *Remarks on contractive conditions of integral type*, Nonlinear Anal., **71** (2009), 1073–1081.

[11] Z. Liu, *Common fixed points of multivalued mappings*, Rostock. Math. Kolloq., **48** (1995), 53–58.

[12] Z. Liu and S.M. Kang, *Commom stationary points of multivalued mappings on bounded metric spaces*, Int. J. Math. Math. Sci., **24** (2000), 773–779.

[13] Z. Liu, X. Li and S.M. Kang, *Existence and approximations of fixed points for contractive mappings of integral type*, Fixed Point Theory Appl., **2014** (2014), Article ID 138, 20 pages.
[14] Z. Liu, X. Li, S.M. Kang and S.Y. Cho, *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*, Fixed Point Theory Appl., **2011** (2011), Article ID 64, 18 pages.
[15] S. Moradi and M. Omid, *A fixed-point theorem for integral type inequality depending on another function*, Int. J. Math. Anal., **4** (2010), 1491–1499.
[16] K. Rao, M.M. Ali and A.S. Babu, *Common fixed points for d-maps satisfying integral type condition*, J. Nonlinear Sci. Appl., **3** (2010), 294–301.
[17] B.E. Rhoands, *Two fixed-point theorems for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **63** (2003), 4007–4013.
[18] M. Sahaa and D. Deyb, *Fixed point theorems for A-contraction mappings of integral type*, J. Nonlinear Sci. Appl., **5** (2012), 84–92.
[19] S.P. Singh and B.A. Meade, *On common fixed point theorems*, Bull. Austral. Math. Soc., **16** (1997), 49–53.
[20] M. Stojaković, L. Gajić, T. Došenović and B. Carić, *Fixed point of multivalued integral type of contraction mappings*, Fixed Point Theory Appl., **2015** (2015), Articla ID 146, 10 pages.