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AN ADDITIVE (U, β)-FUNCTIONAL EQUATION AND MODULE LINEAR MAPPINGS

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Abstract. Let A be a unital C^*-algebra. In this paper, we investigate the additive (u, β)-functional equation:

$$f(x) + u^* f(uy) = \beta^{-1} f(\beta(x + y))$$

for all unitary elements u in A and for a fixed nonzero complex number β.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (u, β)-functional equation in Banach modules.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [6, 7, 9, 13, 15-22] for more informations on functional equations.

We recall a fundamental result in fixed point theory.

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Theorem 1.1. ([2, 5]) *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $\alpha < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set*

$$Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\};$$

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 23]).

In Section 2, we solve the additive (u, β)-functional equation in vector spaces and prove the Hyers-Ulam stability of the additive (u, β)-functional equation in Banach spaces by using the fixed point method. In Section 3, we prove the Hyers-Ulam stability of the additive (u, β)-functional equation in Banach spaces by using the direct method.

Throughout this paper, assume that A is a unital C^*-algebra with unit e and $U(A)$ is the set of unital elements of A. Let X be a normed A-module and Y a Banach A-module. Let β be a fixed nonzero complex number.

2. Additive (u, β)-functional equation in Banach modules I

We solve the additive (u, β)-functional equation in A-modules.

Lemma 2.1. *Let* X *and* Y *be* A*-modules. If a mapping* $f : X \to Y$ *satisfies*

$$f(x) + u^* f(uy) = \beta^{-1} f(\beta(x + y)) \tag{2.1}$$

for all $x, y \in X$ *and all* $u \in U(A)$, *then* $f : X \to Y$ *is* A*-linear.*

Proof. Assume that $f : X \to Y$ satisfies (2.1). Letting $x = y = 0$ in (2.1), we get $(1 + u^*) f(0) = \beta^{-1} f(0)$ for all $u \in U(A)$. So $f(0) = 0$. Letting $u = e$, $y = -x$ in (2.1), we get $f(x) + f(-x) = 0$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $u = e$, $x = 0$ and replacing y by $x + y$ in (2.1), we get

$$f(x + y) = \beta^{-1} f(\beta(x + y))$$

for all $x, y \in X$. Letting $u = e$ in (2.1), we get $f(x) + f(y) = \beta^{-1} f(\beta(x + y))$ and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

Letting $z = 0$ and $y = -x$ in (2.1), we get $f(x) + u^* f(-ux) = 0$ and so $f(ux) = uf(x)$ for all $x \in X$ and all $u \in U(A)$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then $|\frac{a}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [12, Theorem 1], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. So

$$\begin{aligned} f(ax) &= f(\frac{M}{3} \cdot 3\frac{a}{M}x) \\ &= M \cdot f(\frac{1}{3} \cdot 3\frac{a}{M}x) \\ &= \frac{M}{3} f(3\frac{a}{M}x) \\ &= \frac{M}{3} f(u_1 x + u_2 x + u_3 x) \\ &= \frac{M}{3} (f(u_1 x) + f(u_2 x) + f(u_3 x)) \\ &= \frac{M}{3} (u_1 + u_2 + u_3) f(x) \\ &= \frac{M}{3} \cdot 3\frac{a}{M} f(x) \\ &= af(x) \end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-linear, as desired. □

Since $U(\mathbb{C}) = \mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, we get the following corollary.

Corollary 2.2. *Let X and Y be complex vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x) + \overline{\alpha} f(\alpha y) = \beta^{-1} f(\beta(x + y))$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$, then $f : X \to Y$ is $\mathbb{C}$-linear.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (2.1) in complex Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2} \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|f(x) + u^* f(uy) - \beta^{-1} f(\beta(x+y))\right\| \leq \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leq \frac{L}{2(1-L)}(\varphi(0,2x) + \varphi(x,x)) \tag{2.4}$$

for all $x \in X$.

Proof. Let $u = e$. Letting $y = x$ in (2.3), we get

$$\left\|2f(x) - \beta^{-1} f(2\beta x)\right\| \leq \varphi(x,x) \tag{2.5}$$

for all $x \in X$. Replacing y by $2x$ and letting $x = 0$ in (2.3), we get

$$\left\|f(2x) - \beta^{-1} f(2\beta x)\right\| \leq \varphi(0,2x) \tag{2.6}$$

for all $x \in X$. It follows from (2.5) and (2.6) that

$$\|f(2x) - 2f(x)\| \leq \varphi(0,2x) + \varphi(x,x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(\varphi(0,2x) + \varphi(x,x)),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [14]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$. Hence

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \\
&\leq 2\varepsilon\left(\varphi(0,x) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)\right) \\
&\leq 2\varepsilon\frac{L}{2}(\varphi(0,2x) + \varphi(x,x)) \\
&= L\varepsilon(\varphi(0,2x) + \varphi(x,x))
\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\begin{aligned}\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\| &\le \varphi(0,x) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le \frac{L}{2}(\varphi(0,2x) + \varphi(x,x))\end{aligned}$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$;
(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;
(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$. It follows from (2.2) and (2.3) that

$$\begin{aligned}&\left\|H(x) + u^*H(uy) - \beta^{-1}H(\beta(x+y))\right\| \\ &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + u^* f\left(\frac{uy}{2^n}\right) - \beta^{-1} f\left(\beta\left(\frac{x+y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in X$ and all $u \in U(A)$. So

$$H(x) + u^*H(uy) - \beta^{-1}H(\beta(x+y)) = 0$$

for all $x, y \in X$ and all $u \in U(A)$. By Lemma 2.1, the mapping $H : X \to Y$ is A-linear, as desired. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\left\| f(x) + u^* f(uy) - \beta^{-1} f(\beta(x+y)) \right\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \frac{2^r + 2}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Corollary 2.5. *Let X be a complex normed space and Y a complex Banach space. Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{2}\varphi(x, y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\| f(x) + \overline{\alpha} f(\alpha y) - \beta^{-1} f(\beta(x+y)) \right\| \le \varphi(x, y) \tag{2.10}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \frac{L}{2(1-L)}(\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$.

Theorem 2.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique A-linear mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \frac{1}{2(1-L)}(\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi\,(0,2x)+\varphi\,(x,x))$$

for all $x\in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{2}g\,(2x)$$

for all $x\in X$. The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.7. *Let $r<1$ and θ be positive real numbers, and let $f:X\to Y$ be a mapping satisfying* (2.9)*. Then there exists a unique A-linear mapping $H:X\to Y$ such that*

$$\|f(x)-H(x)\|\le\frac{2+2^r}{2-2^r}\theta\|x\|^r$$

for all $x\in X$.

Proof. The proof follows from Theorem 2.6 by taking $\varphi(x,y)=\theta(\|x\|^r+\|y\|^r)$ for all $x,y\in X$. Then we can choose $L=2^{r-1}$ and we get desired result. □

Corollary 2.8. *Let X be a complex normed space and Y a complex Banach space. Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\,(x,y)\le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (2.10)*. Then there exists a unique $\mathbb{C}$-linear mapping $H:X\to Y$ such that*

$$\|f(x)-H(x)\|\le\frac{1}{2(1-L)}(\varphi\,(0,2x)+\varphi\,(x,x))$$

for all $x\in X$.

3. Additive (u,β)-functional equation in Banach modules II

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive (u,β)-functional equation (2.1) in Banach modules.

Theorem 3.1. *Let $\varphi:X^2\to[0,\infty)$ be a function and let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and*

$$\Psi(x,y):=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right)\quad<\quad\infty,$$

$$\left\|f(x)+u^*f(uy)-\beta^{-1}f(\beta(x+y))\right\|\quad\le\quad\varphi(x,y)\qquad(3.1)$$

for all $x, y \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{2}(\Psi(0, 2x) + \Psi(x, x)) \tag{3.2}$$

for all $x \in X$.

Proof. Let $u = e$. It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(0, x\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} \left(2^j \varphi\left(0, \frac{x}{2^j}\right) + 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right) \end{aligned} \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned} \|H(x) - T(x)\| &= \left\|2^q H\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|2^q H\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq 2^q \Psi\left(0, \frac{2x}{2^q}\right) + 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9)*. Then there exists a unique A-linear mapping $H : X \to Y$ such that*

$$\|f(x) - H(x)\| \leq \frac{2^r + 2}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

Corollary 3.3. *Let* X *be a complex normed space and* Y *a complex Banach space. Let* $\varphi : X^2 \to [0, \infty)$ *be a function and let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \quad < \quad \infty,$$

$$\left\|f(x) + \overline{\alpha} f(\alpha y) - \beta^{-1} f(\beta(x + y))\right\| \quad \leq \quad \varphi(x, y) \tag{3.4}$$

for all $x, y \in X$ *and all* $\alpha \in \mathbb{T}$. *Then there exists a unique* $\mathbb{C}$*-linear mapping* $H : X \to Y$ *such that*

$$\|f(x) - H(x)\| \leq \frac{1}{2}(\Psi(0, 2x) + \Psi(x, x))$$

for all $x \in X$.

Theorem 3.4. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function and let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$, (3.1) *and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. *Then there exists a unique* A*-linear mapping* $H : X \to Y$ *such that*

$$\|f(x) - H(x)\| \leq \frac{1}{2}(\Psi(0, 2x) + \Psi(x, x)) \tag{3.5}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2} f(2x)\right\| \leq (\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$. Hence we have

$$\begin{aligned} \left\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\right\| \quad & \leq \quad \sum_{j=l}^{m-1} \left\|\frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right)\right\| \\ & \leq \quad \sum_{j=l}^{m-1} \left(\frac{1}{2^{j+1}} \varphi(0, 2^{j+1} x) + \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x)\right) \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.5). The rest of the proof is similar to the proofs of Theorems 2.3 and 3.1. □

Corollary 3.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9)*. Then there exists a unique A-linear mapping $H : X \to Y$ such that*

$$\|f(x) - H(x)\| \le \frac{2+2^r}{2-2^r}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

Corollary 3.6. *Let X be a complex normed space and Y a complex Banach space. Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (3.4) *and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique $\mathbb{C}$-linear mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \frac{1}{2}(\Psi(0, 2x) + \Psi(x, x)) \tag{3.7}$$

for all $x \in X$.

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