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UNIVALENCE PRESERVING INTEGRAL OPERATOR DEFINED BY GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. Let $I_\mu(f_1, f_2, ...f_l, g_1, g_2, ...g_l)(z)$ be the integral operator defined by generalized hypergeometric functions where each of the functions f_m and g_m are, respectively, analytic functions in the open unit disk for all $m = 1, ..., l$. The object of this paper is to obtain several univalence conditions for this integral operator. Our main results contain some interesting corollaries as special cases.

1. Introduction

The study of hypergeometric functions plays a vital role in mathematics. Hypergeometric functions had been extensively studied (for example) by Euler, Gauss, Riemann and of course many others. They obtained many interesting results associated with this type of functions, these results could be attributed to the applications of the hypergeometric theory along with its beautiful structure. It is applicable in many subjects such as combinatorics, numerical analysis, dynamical analysis and mathematical physics. Basically,

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$q-$hypergeometric functions are a generalization of the classical hypergeometric functions, in the sense of taking the (formal) $limit\, q \to 1$, it will return to the classical hypergeometric setting. The $q-$hypergeometric level can generalize many results for the classical hypergeometric functions. The generalization $q-$Taylor's formula in fractional $q-$caluculs introduced by Purohit and Raina [17], where certain $q-$generating functions for $q-$hypergeometric functions are derived.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and normalized in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

$q-$hypergeometric function is a power series in one complex variable z with power series coefficients which depend, apart from q on r complex upper parameters $a_1, a_2, ..., a_r$ and s complex lower $b_1, b_2, ...b_s$ as follows (See Gasper and Rahman [8])

$$_r\Omega_s\left(a_1, ...a_r; b_1, ...b_s, q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, q)_k ... (a_r, q)_k}{(q, q)_k (b_1, q)_k ... (b_s, q)_k} \left[(-1)^k q^{\left(\begin{array}{c} k \\ 2 \end{array}\right)} \right]^{1+s-r} z^k, \tag{1.2}$$

with $\left(\begin{array}{c} k \\ 2 \end{array}\right) = \frac{k(k-1)}{2}$, where $q \neq 0$ when $r > s+1, (r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U})$, $\mathbb{N}$ denote the set of positive integers and $(a, q)_k$ is the $q-$shifted factorial defined by

$$(a, q)_k = \begin{cases} 1, & k = 0; \\ (1-a)(1-aq)(1-aq^2)...(1-aq^{k-1}), & k \in \mathbb{N}. \end{cases}$$

By using the ratio test, one recognize that, if $|q| < 1$, the series (1.2) converges absolutely (see Gasper and Rahman [8]) and Ghany [9]) for all z if $r \leq s$ and for $|z| < 1$ if $r = s + 1$. For brief survey on $q-$hypergeometric functions, one may refer to [2, 4, 10] also see [11, 12].

Now for $z \in \mathbb{U}, |q| < 1$, and $r = s + 1$, the $q-$hypergeometric function defined in (1.2) takes the form

$$_r\upsilon_s\left(a_1, ..., a_r; b_1, ..., b_s, q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, q)_k ... (a_r, q)_k}{(q, q)_k (b_1, q)_k ... (b_s, q)_k} z^k,$$

which converges absolutely in the open unit disk $\mathbb{U}$.

Corresponding to a function ${}_r\Lambda_s(a_i;b_j;q,z)$ defined by

$${}_r\Lambda_s(a_i;b_j;q,z) = z\,{}_r\upsilon_s(a_i;b_j;q,z) = z + \sum_{k=2}^{\infty} \frac{(a_1,q)_{k-1}...(a_r,q)_{k-1}}{(q,q)_{k-1}(b_1,q)_{k-1}...(b_s,q)_{k-1}} z^k,$$

where $i = 1,...,r,\ j = 1,...,s,\ a_i, b_j \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0,-1,-2,...\}$.

We will use the following operator which defined and studied by the authors (see [1]).

$$\begin{aligned}
\mathcal{M}^0_{r,s,\lambda}(a_i,b_j;q)f(z) =& f(z) * {}_r\Lambda_s(a_i,b_j;q;z),\\
\mathcal{M}^1_{r,s,\lambda}(a_i,b_j;q)f(z) =& (1-\lambda)f(z) * {}_r\Lambda_s(a_i,b_j;q;z) + \lambda z\left(f(z) * {}_r\Lambda_s(a_i,b_j;q;z)\right),\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z) &= \mathcal{M}^1_{r,s,\lambda}\left(\mathcal{M}^{n-1}_{r,s,\lambda}(f(z))\right)\\
&= z + \sum_{k=2}^{\infty} [1+(k-1)\lambda]^n\, \Upsilon_k a_k z^k,
\end{aligned} \tag{1.3}$$

where $*$ denotes the usual Hadamard product of analytic functions and

$$\Upsilon_k = \frac{(a_1,q)_{k-1}...(a_r,q)_{k-1}}{(q,q)_{k-1}(b_1,q)_{k-1}...(b_s,q)_{k-1}}. \tag{1.4}$$

In the following definitions, we introduce new subclasses of analytic functions defined by a linear operator $\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)$.

Definition 1.1. Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^n_{r,s,\lambda}(\alpha)$ if and only if

$$\left|\frac{z\left(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)\right)'}{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)} - 1\right| < \alpha,\ 0 < \alpha \le 1,\ z \in \mathbb{U}, \tag{1.5}$$

where $\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)$ is the operator given by (1.3).

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in \mathcal{B}^n_{r,s,\lambda}(\eta,\beta)$ if and only if

$$\begin{aligned}
&\left|\left(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)\right)'\left(\frac{z}{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)}\right)^{\eta} - 1\right|\\
&< 1-\beta,\ 0 \le \beta < 1,\ \eta \ge 0,\ z \in \mathbb{U},
\end{aligned} \tag{1.6}$$

where $\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f(z)$ is the operator given by (1.3).

We present some examples by using specializing the values of $r, s, a_1, a_2...a_r$, $b_1, b_2, ...b_s$ and n.

Example 1.3. For $r=1, s=0, a_1=q$ and $n=0$ in Definition 1.1, then

$$\mathcal{S}^*_\alpha = \left|\frac{zf'(z)}{f(z)} - 1\right| < \alpha,\ 0<\alpha\le 1,\ z\in\mathbb{U},$$

where $\mathcal{S}^*_\alpha$ denote the class of analytic functions (see[6] and [7]).

Example 1.4. For $r=1, s=0, a_1=q$, and $n=0$ in Definition 1.2, then

$$\mathcal{B}(\eta,\beta) = \left|g'(z)\left(\frac{z}{g(z)}\right)^\eta - 1\right| < 1-\beta,$$

$z\in\mathbb{U},\ 0\le\beta<1,\ \eta\ge 0,$

where $\mathcal{B}(\eta,\beta)$ denote the class of analytic functions which has been studied by Frasin and Jahangiri [5].

Using the operator defined by (1.3), we introduce the following new general integral operator:

Definition 1.5. For $l\in\mathbb{N}\cup\{0\}, f_m(z), g_m(z)\in\mathcal{A}$ and $\delta_m,\gamma_m,\mu\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. We define the integral operator $I_\mu(f_1,f_2,...f_l,g_1,g_2,...g_l)(z):\mathcal{A}^n\to\mathcal{A}^n$ by

$$\begin{aligned} &I_\mu(f_1,f_2,...f_l,g_1,g_2,...g_l)(z)\\ &=\left\{\mu\int_0^z t^{\mu-1}\prod_{m=1}^{l}\left(\frac{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f_m(t)}{t}\right)^{\delta_m}\left(e^{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(t)}\right)^{\gamma_m}dt\right\}^{\frac{1}{\mu}}. \end{aligned} \tag{1.7}$$

It should be remarked that the integral operator $I_\mu(f_1,f_2,...f_n,g_1,g_2,...g_l)(z)$ is a generalization of many other operators considered earlier, for example:

- For $n=0, r=1, s=0, a_1=q$, and $\mu=1$, where $\delta_m,\gamma_m\in\mathbb{C}$, the integral operator

  $$F_\beta(f_1,f_2,...f_l,g_1,g_2,...g_l)(z) = \int_0^z\prod_{m=1}^{l}\left(\frac{f_m(t)}{t}\right)^{\delta_m}\left(e^{g_m(t)}\right)^{\gamma_m}dt$$

  investigated by Stanciu and Breaz [18].
- For $n=0, r=1, s=0, a_1=q, \mu=1$ and $l=1$, where $\delta,\gamma\in\mathbb{C}$, we have the integral operator (see [18])

  $$F(f_1,f_2,...f_l,g_1,g_2,...g_l)(z) = \int_0^z\left(\frac{f(t)}{t}\right)^{\delta}\left(e^{g(t)}\right)^{\gamma}dt$$

- For $n = 0, r = 1, s = 0, a_1 = q$ and $\gamma_m = 0$,where $\delta_m \in \mathbb{C}$ and $\mu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, the integral operator

$$F_\mu(f_1, f_2, ...f_l)(z) = \left\{\mu \int_0^z t^{\mu-1} \prod_{m=1}^{l} \left(\frac{f_m(t)}{t}\right)^{\delta_m} dt\right\}^{\frac{1}{\mu}}$$

introduced by Breaz and Breaz [3].
- For $n = 0, r = 1, s = 0, a_1 = q, \gamma_m = 0$ and $\mu = 1$, where $\delta_m \in \mathbb{C}$, we obtain the integral operator

$$F_n(z) = \int_0^z \prod_{m=1}^{l} \left(\frac{f_m(t)}{t}\right)^{\delta_m} dt$$

studied by Breaz and Breaz [3].

The following results will be required in our investigation.

Lemma 1.6. ([14, 15]) Let $\mu \in \mathbb{C}$ with $Re(\mu) > 0$. If $f \in \mathcal{A}$ satisfies

$$\frac{1 - |z|^{2Re(\mu)}}{Re(\mu)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\mu(z) = \left\{\mu \int_0^z t^{\mu-1} f'(t) dt\right\}^{\frac{1}{\mu}}$$

is in the class $\mathcal{S}$.

Lemma 1.7. ([16]) Let $\mu \in \mathbb{C}$ with $Re(\mu) > 0, c \in \mathbb{C}$, with $|c| \leq 1, c \neq -1$. If $f \in \mathcal{A}$ satisfies

$$\left|c|z|^{2\mu} + (1 - |z|^{2\mu}) \frac{zf''(z)}{\mu f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\mu(z) = \left\{\mu \int_0^z t^{\mu-1} f'(t) dt\right\}^{\frac{1}{\mu}}$$

is in the class $\mathcal{S}$.

Lemma 1.8. (Generalized Schwarz Lemma, [13]) Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad (z \in \mathbb{U}_R).$$

Equality can hold only if

$$f(z) = e^{i\theta}\left(\frac{M}{R^m}\right) z^m,$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let the functions $f_m, g_m \in \mathcal{A}$, for all $m \in 1, 2, ..., l$ and suppose that*

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z)| \leq M_m, \ (z \in \mathbb{U})$$

with $M_m \geq 1$. If $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m), 0 < \alpha_m \leq 1$ and $g_m \in \mathcal{B}^n_{r,s,\lambda}(\eta_m, \beta_m), \eta_m \geq 0, 0 \leq \beta_m < 1$, then the integral operator $I_\mu(f_1, ..., f_l, g_1, ...g_l)(z)$ given by (1.7) is analytic and univalent in $\mathbb{U}$, where

$$Re(\mu) \geq \sum_{m=1}^{l} \left[|\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m)M_m^{\eta_m}\right]. \tag{2.1}$$

Proof. By setting

$$w(z) = \int_0^z \prod_{m=1}^{l} \left(\frac{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(t)}{t}\right)^{\delta_m} \left(e^{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(t)}\right)^{\gamma_m} dt$$

and

$$w'(z) = \prod_{m=1}^{l} \left(\frac{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z)}{z}\right)^{\delta_m} \left(e^{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)}\right)^{\gamma_m}.$$

Logarithmic derivative of $w'(z)$ yields

$$\frac{zw''(z)}{w'(z)} = \sum_{m=1}^{l} \left[\delta_m \left(\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z)} - 1\right) + \gamma_m z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z))'\right].$$

This implies that

$$\left|\frac{zw''(z)}{w'(z)}\right| \leq \sum_{m=1}^{l} \left[|\delta_m| \left|\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z)} - 1\right| + |\gamma_m| \left|z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z))'\right|\right],$$

which readily shows that

$$\frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\left|\frac{zw''(z)}{w'(z)}\right| \leq \frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\sum_{m=1}^{l}\left[|\delta_m|\left|\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f_m(z)}-1\right| + |\gamma_m|\left|z(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z))'\right|\right]$$

that is,

$$\begin{aligned}
&\frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\left|\frac{zw''(z)}{w'(z)}\right| \\
&\leq \frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\sum_{m=1}^{l}\left[|\delta_m|\left|\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)f_m(z)}-1\right|\right. \\
&\quad + |\gamma_m|\left|(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z))'\left(\frac{z}{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)}\right)^{\eta_m}\right| \\
&\quad \left.\times\left|\frac{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)}{z}\right|^{\eta_m}|z|\right]. \qquad (2.2)
\end{aligned}$$

Since $|\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)| \leq M_m, z \in \mathbb{U}$ using the General Schwarz Lemma for the functions $\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)$, we receive

$$|\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)| \leq M_m|z|, \quad z \in \mathbb{U}$$

for all $m \in 1,2,...,l$.

Also, since $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m), 0 < \alpha_m \leq 1$, applying the relation (1.5) in the relation (2.2), we have

$$\begin{aligned}
&\frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\left|\frac{zw''(z)}{w'(z)}\right| \\
&\leq \frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\sum_{m=1}^{l}\left[|\delta_m|\alpha_m\right. \\
&\quad \left.+ |\gamma_m|\left(\left|(\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z))'\left(\frac{z}{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g_m(z)}\right)^{\eta_m}-1\right|+1\right)M_m^{\eta_m}\right].
\end{aligned}$$

According to the relation (1.6), we obtain

$$\frac{1-|z|^{2Re(\mu)}}{Re(\mu)}\left|\frac{zw''(z)}{w'(z)}\right| \leq \frac{1}{Re(\mu)}\sum_{m=1}^{l}\left[|\delta_m|\alpha_m+|\gamma_m|(2-\beta_m)M_m^{\eta_m}\right].$$

Thus using (8) and applying Lemma 1.6, we obtain that the integral operator $I_\mu(f_1, f_2, ...f_l, g_1, g_2, ...g_l)(z) \in \mathcal{S}$. This completes the proof. □

In Theorem 2.1, if we set $l = 1$, then we have the following corollary.

Corollary 2.2. *Let the functions $f, g \in \mathcal{A}$ and suppose that*

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g(z)| \leq M,\ (z \in \mathbb{U})$$

with $M \geq 1$. If $f \in \mathcal{S}^n_{r,s,\lambda}(\alpha)$, $0 < \alpha \leq 1$ and $g \in \mathcal{B}^n_{r,s,\lambda}(\eta, \beta), \eta \geq 0, 0 \leq \beta < 1$, then the integral operator

$$I_\mu(f, g)(z) = \left\{ \mu \int_0^z t^{\mu-1} \left(\frac{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f(t)}{t} \right)^\delta \left(e^{\mathcal{M}^n_{r,s,\lambda}(a_i,b_j;q)g(t)} \right)^\gamma dt \right\}^{\frac{1}{\mu}}, \tag{2.3}$$

is analytic and univalent in $\mathbb{U}$, where

$$Re(\mu) \geq |\delta|\alpha + |\gamma|(2 - \beta)M^\eta.$$

In Theorem 2.1, if we consider $M_1 = M_2 = \cdots = M_l = M$, then we get the following corollary.

Corollary 2.3. *Let $M \geq 1$, each of the functions $f_m, g_m \in \mathcal{A}$ for all $m \in 1, 2, ..., l$ satisfies $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m)$, $0 < \alpha_m \leq 1$ and $g_m \in \mathcal{B}^n_{r,s,\lambda}(\eta_m, \beta_m), \eta_m \geq 0$, $0 \leq \beta_m < 1$ with*

$$Re(\mu) \geq \sum_{m=1}^{l} \left[|\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m)M^{\eta_m} \right].$$

If

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z)| \leq M,$$

then the integral operator $I_\mu(f_1, f_2, ...f_l, g_1, g_2, ...g_l)(z)$ defined by (1.7) is analytic and univalent in $\mathbb{U}$.

In Corollary 2.3, we consider $M = 1$, then we have the following corollary.

Corollary 2.4. *Let each of the functions $f_m, g_m \in \mathcal{A}$ for all $m \in 1, 2, ..., l$ satisfies $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m), 0 < \alpha_m \leq 1$ and $g_m \in \mathcal{B}^n_{r,s,\lambda}(\eta_m, \beta_m), \eta_m \geq 0, 0 \leq \beta_m < 1$ with*

$$Re(\mu) \geq \sum_{m=1}^{l} \left[|\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m) \right].$$

If

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z)| \leq 1,$$

then the integral operator $I_\mu(f_1, f_2, ...f_l, g_1, g_2, ...g_l)(z)$ *defined by* (1.7) *is analytic and univalent in* $\mathbb{U}$.

Theorem 2.5. *Let the functions* $f_m, g_m \in \mathcal{A}$ *for all* $m \in 1, 2, ..., l$ *and suppose that*

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z)| \leq M_m, (z \in \mathbb{U})$$

with $M_m \geq 1$. *If* $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m), 0 < \alpha_m \leq 1$ *and* $g_m \in \mathcal{B}^n_{r,s,\lambda}(\eta_m, \beta_m), \eta_m \geq 0, 0 \leq \beta_m < 1$, *then the integral operator* $I_\mu(f_1, ..., f_l, g_1, ...g_l)(z)$ *given by* (1.7) *is analytic and univalent in* $\mathbb{U}$, *where*

$$1 - |c| \geq \frac{1}{\mu} \sum_{m=1}^{l} \left[|\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m)M_m^{\eta_m}\right]. \tag{2.4}$$

Proof. From the proof of Theorem 2.1, we have

$$\frac{zw''(z)}{w'(z)} = \sum_{m=1}^{l} \left[\delta_m \left(\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z)} - 1\right) + \gamma_m z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z))'\right].$$

Then we find

$$\begin{aligned}
&\left|c|z|^{2\mu} + (1 - |z|^{2\mu})\frac{zw''(z)}{\mu w'(z)}\right| \\
&= \left|c|z|^{2\mu} + \frac{(1 - |z|^{2\mu})}{\mu} \sum_{m=1}^{l} \left[\delta_m \left(\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z)} - 1\right)\right.\right. \\
&\quad \left.\left. + \gamma_m z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z))'\right]\right| \\
&\leq |c| + \frac{1}{\mu} \sum_{m=1}^{l} \left[|\delta_m| \left|\frac{z(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z))'}{\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)f_m(z)} - 1\right|\right. \\
&\quad \left. + |\gamma_m||z||(\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z))'|\right].
\end{aligned} \tag{2.5}$$

From the result in (2.5) and using the proof of Theorem 2.1, we impose

$$\left|c|z|^{2\mu} + (1 - |z|^{2\mu})\frac{zw''(z)}{\mu w'(z)}\right| \leq |c| + \frac{1}{\mu} \sum_{m=1}^{l} |\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m)M_m^{\eta_m}.$$

According to the hypothesis (2.4) and by Lemma 1.7, we obtain that
$I_\mu(f_1, ..., f_l, g_1, ...g_l)(z) \in \mathcal{S}$. □

In Theorem 2.5, we put $l = 1$. Then we have:

Corollary 2.6. *Let the functions $f, g \in \mathcal{A}$ and suppose that*

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g(z)| \leq M, (z \in \mathbb{U})$$

with $M \geq 1$. If $f \in \mathcal{S}^n_{r,s,\lambda}(\alpha), 0 < \alpha \leq 1$ and $g \in \mathcal{B}^n_{r,s,\lambda}(\eta, \beta, \eta \geq 0, 0 \leq \beta < 1$, then the integral operator $I_\mu(f, g)(z)$ given by (2.3) is analytic and univalent in $\mathbb{U}$, where

$$1 - |c| \geq \frac{1}{\mu}\left[|\delta|\alpha + |\gamma|(2 - \beta)M^\eta\right].$$

In Theorem 2.5, we consider $M_1 = M_2 = \cdots = M_l = M$, then we have the following corollary.

Corollary 2.7. *Let $M \geq 1$, each of the functions $f_m, g_m \in \mathcal{A}$ for all $m \in 1, 2, ..., l$ satisfies $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m), 0 < \alpha_m \leq 1$ and $g_m \in \mathcal{B}^n_{r,s,\lambda}(\eta_m, \beta_m), \eta_m \geq 0, 0 \leq \beta_m < 1$ with*

$$1 - |c| \geq \frac{1}{\mu}\sum_{m=1}^{l}\left[|\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m)M^{\eta_m}\right].$$

If

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z)| \leq M,$$

then the integral operator $I_\mu(f_1, f_2, ...f_l, g_1, g_2, ...g_l)(z)$ defined by (1.7) is analytic and univalent in $\mathbb{U}$.

In Corollary 2.7, we consider $M = 1$, the we have:

Corollary 2.8. *Each of the functions $f_m, g_m \in \mathcal{A}$ for all $m \in 1, 2, ..., l$ satisfies $f_m \in \mathcal{S}^n_{r,s,\lambda}(\alpha_m), 0 < \alpha_m \leq 1$ and $g_m \in \mathcal{B}^n_{r,s,\lambda}(\eta_m, \beta_m), \eta_m \geq 0,\ 0 \leq \beta_m < 1$ with*

$$1 - |c| \geq \frac{1}{\mu}\sum_{m=1}^{l}\left[|\delta_m|\alpha_m + |\gamma_m|(2 - \beta_m)\right].$$

If

$$|\mathcal{M}^n_{r,s,\lambda}(a_i, b_j; q)g_m(z)| \leq 1,$$

then the integral operator $I_\mu(f_1, f_2, ...f_l, g_1, g_2, ...g_l)(z)$ defined by (1.7) is analytic and univalent in $\mathbb{U}$.

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