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A TYPE OF THE EXPONENTIAL OF A MATRIX OVER DUAL QUATERNION IN CLIFFORD ANALYSIS

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Abstract. This paper proposes a form and use of exponential of a matrix over dual quaternions. Due to the property of the product for dual quaternions, we give the way of computing the exponential of a matrix with the exponential map from their Lie algebras into the dual quaternionic matrices and a form of an eigenvalue of the dual quaternionic exponential of a matrix.

1. Introduction

The set of quaternions, which is introduced by Hamilton [9] as

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_r \in \mathbb{R} \ (r = 0, 1, 2, 3)\},$$

where the units $i^2 = j^2 = k^2 = -1$ satisfying

$$ij = k = -ji, \ jk = i = -kj, \ ki = j = -ik.$$

Quaternion is non-commutative algebras. So, there are many results associated with quaternion matrices. Zhang [14] first presented matrices of quaternions and studied properties of non-commutativity of quaternion matrices. Baker [3] discussed the theory of eigenvalues of a quaternion matrix with a topological approach. Huang and So [10] studied on left eigenvalues of a quaternionic matrix and gave some differences between right and left eigenvalues of quaternion matrices. Also, Ablamowicz [1] proposed a way to compute exponential of matrices over real, complex numbers and quaternions by means of an isomorphism between matrix algebras and simple orthogonal Clifford

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algebras. Erdoğdu and Özdemir [5] examined matrix groups over the split quaternions and the exponential map and gave a method of finding exponential of a split quaternion matrix.

Clifford [4] introduced the set of dual quaternions. McAulay [11] gave the dual unit to generate the dual quaternion algebra. Study showed that dual quaternion algebra described the group of motions of three-dimensional space and developed the structure of three eight-dimensional algebras referred to as biquaternions. Based to rotations in 3D space which can be represented by quaternions of unit length, dual quaternions also represented rigid motions, theoretical kinematics, robotics and computer vision in 3D space (see [12, 13]). Kim [6] gave properties of functions of multidual complex variables and defined a corresponding inverse of these functions. Kim [7, 8] represented the extended regularity of dual quaternionic functions and their polar form in Clifford analysis.

Let A be a square matrix with real or complex entries. The matrix exponential of A, denoted by e^A, is a matrix function on square matrices which give a relation between any matrix Lie algebra and the corresponding Lie group. The matrix exponential e^A is given by the power series as follows:

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n. \tag{1.1}$$

The well-defined exponential of A is widely used to approximation theory, differential equations and mathematical physics for providing computational convenience. The series (1.1) is similar to the exponential with numbers, but their properties depend on the commutativity of the matrix product. If given two matrices A and B commute, then they satisfy properties of the exponential with numbers. However, the number system that we deal with has the non-commutativity for the product. Thus, we consider suitable formulas and calculations for the matrix exponential consisted by dual quaternions. We introduce calculating the exponential of a matrix with dual quaternion entries. We give several theorems which are useful to the calculation of the exponential of a dual quaternionic matrix.

2. Preliminaries

Consider the imaginary unit i, the dual unit j and $k := ij$ satisfying the following relations

$$i^2 = -1,\ j^2 = k^2 = 0,\ ij = -ji,$$

$$jk = kj = 0,\ ki = j = -ik.$$

For example, the following units satisfy the above relations

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; k = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

The set of dual quaternions can be expressed as

$$\mathbb{H}_d := \{q = x_0 + ix_1 + jx_2 + kx_3 \mid x_t \in \mathbb{R} \; (t = 0, 1, 2, 3)\}.$$

For any $q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}_d$, we give the real and vector part of q as follows:

$$S(q) = x_0$$

and

$$V(q) = ix_1 + jx_2 + kx_3,$$

respectively. The conjugate and modulus of q is defined by

$$q^* = S(q) - V(q) = x_0 - ix_1 - jx_2 - kx_3$$

and

$$\|q\| = \sqrt{qq^*} = \sqrt{x_0^2 + x_1^2},$$

respectively. For two dual quaternions $q = x_0 + ix_1 + jx_2 + kx_3$ and $p = y_0 + iy_1 + jy_2 + ky_3$ the sum is

$$q + p = S(q) + S(p) + V(q) + V(p)$$

and

$$qp = S(q)S(p) + S(q)V(p) + S(p)V(q) + < V(q), V(p) >_d + V(q) \wedge_d V(p),$$

where $<,>_d$ and $\wedge_d$ denote inner and vector product as

$$< q, p >_d = -x_1 y_1$$

and

$$q \wedge_d p = -j(x_1 y_3 - x_3 y_1) + k(x_1 y_2 - x_2 y_1),$$

respectively.

Now, we give a dual quaternionic matrix, denoted by $A = (a_{st})$ with $a_{st} \in \mathbb{H}_d$, which is a matrix with dual quaternion entries. The set of $m \times n$ dual quaternion matrices is denoted by $M_{m\times n}(\mathbb{H}_d)$. The dual quaternionic matrix is a ring with ordinary matrix addition and multiplication. Specially, for $A \in M_{m\times n}(\mathbb{H}_d)$ and $q \in \mathbb{H}_d$ the right and left scalar multiplication as follows:

$$Aq = (a_{st} q)$$

and

$$qA = (q a_{st}).$$

For given $A = (a_{st}) \in M_{m\times n}(\mathbb{H}_d)$, the conjugate transpose of A is defined by

$$A^* = (a_{ts}^*),$$

where each a^*_{ts} is the conjugate of a_{ts}. For example, for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ its conjugate is $A^* = \begin{pmatrix} a^*_{11} & a^*_{21} \\ a^*_{12} & a^*_{22} \end{pmatrix}$. Then we have $trace(AA^*) = \sum_{s,t=1}^{2} |a_{st}|^2$.

Proposition 2.1. *For two square dual quaternion matrix A and B, they satisfy*

$$(AB)^* = B^*A^*.$$

Any square dual quaternion matrix $A \in M_{n\times n}(\mathbb{H}_d)$ can be also written by complex matrices which have the following form

$$A = A_1 + A_2 j,$$

where A_1 and A_2 are $n \times n$ complex matrices. For example,

$$\begin{aligned} A &= \begin{pmatrix} x_0 + ix_1 + jx_2 + kx_3 & y_0 + iy_1 + jy_2 + ky_3 \\ u_0 + iu_1 + ju_2 + ku_3 & v_0 + iv_1 + jv_2 + kv_3 \end{pmatrix} \\ &= \begin{pmatrix} z_1 + z_2 j & w_1 + w_2 j \\ \zeta_1 + \zeta_2 j & \eta_1 + \eta_2 j \end{pmatrix} \\ &= \begin{pmatrix} z_1 & w_1 \\ \zeta_1 & \eta_1 \end{pmatrix} + \begin{pmatrix} z_2 & w_2 \\ \zeta_2 & \eta_2 \end{pmatrix} j \\ &= A_1 + A_2 j. \end{aligned}$$

We define the complex adjoint matrix of A as follows:

$$\chi(A) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1^* \end{pmatrix}.$$

For the complex adjoint matrix of A a q-determinant of A is defined as

$$|A|_q = |\chi(A)| = A_1 A_1^*$$

and the set of left and right eigenvalues of A is given as follows:

$$\Lambda_l(A) = \{\lambda \in \mathbb{H}_d \mid A\nu = \lambda\nu\}$$

and

$$\Lambda_r(A) = \{\lambda \in \mathbb{H}_d \mid A\nu = \nu\lambda\}$$

for some $\nu \neq 0$, respectively.

3. Exponential of a dual quaternion matrix

The exponential map between matrices by the power series for any dual quaternionic square matrix A of order n is defined as

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n.$$

Definition 3.1. A matrix sequence $\{A_k\}_{k=1}^{\infty}$ is said to converge if $[A_k]_{ij}$ converges as $k \to \infty$ for i and j, where $[A_k]_{ij}$ denote the element on row i and column j of A and by convention $A^0 := I$ $n \times n$ identity matrix.

Definition 3.2. A matrix series is said to converge if the corresponding matrix sequence of partial sums converges.

For any $n \times n$ matrix A, by referring [2], we give

$$\|A\| := \max_{1 \le i \le n} \sum_{j=1}^{n} |A_{ij}|.$$

Then we have

$$\begin{aligned} \|AB\| &= \max_{1 \le i \le n} \sum_{j=1}^{n} |(AB)_{ij}| \\ &= \max_{1 \le i \le n} \sum_{j=1}^{n} |\sum_{k=1}^{n} A_{ik} B_{kj}| \\ &\le \max_{1 \le i \le n} \sum_{j=1}^{n} \sum_{k=1}^{n} |A_{ik}||B_{kj}| \\ &= \max_{1 \le i \le n} \sum_{k=1}^{n} |A_{ik}| \sum_{j=1}^{n} |B_{kj}| \\ &\le \max_{1 \le i \le n} \sum_{k=1}^{n} |A_{ik}| \|B\| \\ &= \|A\|\|B\|. \end{aligned}$$

By induction, we obtain $\|A^n\| \le \|A\|^n$. Thus, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|A^n\| \le \sum_{n=0}^{\infty} \frac{1}{n!} \|A\|^n = e^{\|A\|} < \infty.$$

Since $|A_{ij}^n| \le \|A^n\|$ for all i and j, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} |A_{ij}^n| \le e^{\|A\|} < \infty.$$

Remark 3.3. The power series for any dual quaternionic square matrix A of order n

$$\sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

converges to a dual quaternionic square matrix with the same order.

Proposition 3.4. *For two dual quaternionic square matrices* $A = A_1 + A_2 j$ *and* $B = B_1 + B_2 j$*, if* A_1 *and* B_1 *are commutative, then they satisfy the following equation*

$$\begin{cases} \chi(AB) = \chi(A)\chi(B), \\ \chi(BA) = \chi(B)\chi(A). \end{cases} \tag{3.1}$$

Proof. By the definition of $\chi()$, we have the following equations

$$\chi(A)\chi(B) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1^* \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ 0 & B_1^* \end{pmatrix} = \begin{pmatrix} A_1 B_1 & A_1 B_2 + A_2 B_1^* \\ 0 & A_1^* B_1^* \end{pmatrix}$$

and

$$\chi(AB) = \begin{pmatrix} A_1 B_1 & A_1 B_2 + A_2 B_1^* \\ 0 & (A_1 B_1)^* \end{pmatrix},$$

where

$$AB = (A_1 + A_2 j)(B_1 + B_2 j) = A_1 B_1 + (A_1 B_2 + A_2 B_1^*) j.$$

Since A_1 and B_1 are commutative, we have $(A_1 B_1)^* = A_1^* B_1^*$. □

Proposition 3.5. *For any dual quaternionic square matrix* A *it satisfies the following equation*

$$\chi(A + B) = \chi(A) + \chi(B). \tag{3.2}$$

Proof. From

$$\chi(A + B) = \begin{pmatrix} A_1 + B_1 & A_2 + B_2 \\ 0 & A_1^* + B_1^* \end{pmatrix}$$

and

$$\chi(A) + \chi(B) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1^* \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ 0 & B_1^* \end{pmatrix} = \begin{pmatrix} A_1 + B_1 & A_2 + B_2 \\ 0 & A_1^* + B_1^* \end{pmatrix}$$

we have the desired result. □

Proposition 3.6. *For any dual quaternionic square matrix* A *it satisfies the following equation*

$$(\chi(A))^n = \chi(A^n). \tag{3.3}$$

Proof. For $n = 1$, it is trivial. Suppose that for any positive integer $n = k$ the equation (3.3) is satisfied. Let $n = k + 1$.

$$(\chi(A))^{k+1} = (\chi(A))^k \chi(A) = \chi(A^k)\chi(A).$$

Since the product for dual quaternionic square matrices satisfies the linear property, that is, $\chi(A^k)\chi(A) = \chi(A^{k+1})$. Thus, the proof is composed. □

Theorem 3.7. *For any exponential of complex adjoint matrix of A and complex adjoint matrix of exponential of A, the following equation*

$$e^{\chi(A)} = \chi(e^A) \tag{3.4}$$

is satisfied.

Proof. For given $A \in M_{n\times n}(\mathbb{H}_d)$ and dual-quaternionic adjoint matrix $\chi(A)$ of A, by the definition of the exponential matrix, we can write

$$\begin{aligned} e^{\chi(A)} &= I + \chi(A) + \frac{1}{2!}(\chi(A))^2 + \frac{1}{3!}(\chi(A))^3 + \cdots \\ &= I + \chi(A) + \frac{1}{2!}\chi(A^2) + \frac{1}{3!}\chi(A^3) + \cdots \\ &= \chi(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots) \\ &= \chi(e^A). \end{aligned}$$

Thus, the equation (3.4) is satisfied. □

Theorem 3.8. *Let A and B be dual-quaternionic square matrices of the same order. If A and B can be commutative, then the equation*

$$e^{A+B} = e^A e^B \tag{3.5}$$

is satisfied.

Proof. For given $A, B \in M_{n\times n}(\mathbb{H}_d)$, by the properties of dual-quaternionic adjoint matrices, taking χ on both sides, we have

$$\chi(e^{A+B}) = e^{\chi(A+B)} = e^{\chi(A)+\chi(B)} = e^{\chi(A)}e^{\chi(B)}$$

and

$$\chi(e^A e^B) = \chi(e^A)\chi(e^B) = e^{\chi(A)}e^{\chi(B)}.$$

Thus, the equation (3.5) is satisfied. □

Remark 3.9. For a dual-quaternionic square matrix A and an eigenvalue λ of A, they satisfy

$$A^n\chi = \lambda^n\chi,$$

where χ is a non-zero dual quaternionic vector.

Theorem 3.10. *Let A be a dual-quaternionic square matrix and λ be an eigenvalue of A. Then the eigenvalue of the matrix e^A is e^λ.*

Proof. Since λ is an eigenvalue of $A \in M_{n\times n}(\mathbb{H}_d)$, the following equation is satisfied:

$$A\chi = \lambda\chi.$$

Then e^A is also a dual-quaternionic square matrix, so we have

$$e^A\chi = \chi + A\chi + \frac{A^2}{2!}\chi + \frac{A^3}{3!}\chi + \cdots .$$

By Remark 3.9, we have

$$e^A\chi = \chi + \lambda\chi + \frac{\lambda^2}{2!}\chi + \frac{\lambda^3}{3!}\chi + \cdots = e^\lambda\chi.$$

Thus, we obtain an eigenvalue e^λ of the matrix e^A with the same corresponding eigenvector χ as the result of [5]. □

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